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Langevin Description of Markovian Integro-Differential Master Equations*

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For a given master equation of a discontinuous irreversible Markov process, we present the derivation of stochastically equivalent Langevin equations in which the noise is either multiplicative white generalized Poisson noise or a spectrum of multiplicative white Poisson noise. In order to achieve this goal, we introduce two new stochastic integrals of the Ito type, which provide the corresponding interpretation of the Langevin equations. The relationship with other definitions for stochastic integrals is discussed. The results are elucidated by two examples of integro-master equations describing nonlinear relaxation.

1. Introduction

Stochastic differential equations (or Langevin equations) with random perturbations have been revealed to be a useful tool for the study of statistical processes in physics, engineering and many other fields [1–7]. In particular the concept of stochastic differential equations (SDE) for continuous Markovian processes (Fokker-Planck processes) has found wide application in the statistical theory of irreversible processes [1–10]. A common feature of the latter approach is that the noise which enters the Langevin equation is often written down in an *ad hoc* manner rather than derived from first principles. It has been assumed, on the basis of phenomenological arguments, to be Gaussian white noise. There have been put forward exact generalized Langevin equations employing the projector method, which are either of the "linear"** Mori type [8] or explicitly nonlinear in the system variables [9]. However, it is very difficult to determine microscopically the stochastic structure of the in general multiplicative noise in those exact equations.

Another approach to describe irreversible processes is based on the concept of master equations [1, 4, 5, 10]. For the study of the dynamics of discontinuous Markovian processes (i.e. the sample paths are no longer continuous functions) as they occur in a stochastic treatment of chemical kinetics [5, 10], quantum optics, spin relaxation or relaxation processes described by linear Boltzmann equations, the master equation approach presents the usual concept.

One of the challenges of this work is to extend the method of a Langevin description to discontinuous processes so that the latter description is stochastically equivalent to a *given* master equation of the integro-differential type. This challenge has been addressed in some previous works [11–16, 18]. In [12, 18] we have considered a Langevin equation description of a discontinuous process composed of multiplicative white Gaussian noise (continuous component) and white generalized Poisson noise. In recent works [14, 15] dealing with the derivation of master equations of non-Markovian Langevin equations, we have studied the limiting case of a Langevin equation composed of both multiplicative white Gaussian noise and multiplicative white generalized Poisson noise. Using the usual rules of calculus (Stratonovich interpretation), the resulting master equation has been shown to contain a fluctuation-induced drift

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** The term "linear" is misleading. Actually, these equations hold only for one special process (in most cases the stationary equilibrium process)

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of *complicated* structure. While this latter work gives a rather complete answer to the question of the corresponding master equation, the converse problem of finding for a given master equation a stochastically equivalent Langevin description remains unanswered. Another interesting attempt at solving the above problem is in [16]: Considering a multi-Poissonian process with a *parameter dependent* measure, the author derives the corresponding master equation and also presents an attempt to recast the master equation in Langevin form. However, he does *not* specify the definition of the stochastic integral and consequently does not properly state the properties of the random force. Thus, his Langevin equation is of no use for determining the statistical properties of the solutions. In other words, the corresponding master equation is not uniquely defined.

The outline of this paper is as follows: In Sect 2 we elaborate on the structure of the master equation. In Sect. 3 we present the statistical properties of two noise sources, the white Poisson noise and the white generalized Poisson noise which, as we will show, are quite useful for the description of a discontinuous process. In Sect. 4 we define two new stochastic integrals with respect to white Poisson noise and white generalized Poisson noise and study their properties. Throughout this paper we prefer to point out the essential ideas and to derive key relations, not always with complete mathematical rigor. The main results are contained in Sect. 5, where we give a well defined Langevin description, either in terms of multiplicative generalized Poisson increments or in terms of multiplicative Poisson increments, which is stochastically equivalent to a given master equation of the integro-differential type. In Sect. 6 we illustrate the results and methods with two examples of integro-master equations describing nonlinear relaxation. We extract explicitly the structure of the multiplicative noise entering the Langevin description. Some conclusions and a brief discussion on the use of different definitions for the stochastic integral are given in Sect. 7.

2. Structure of the Master Equation

In order not to complicate the main ideas, we restrict the following discussions to a one-dimensional process $x(t)$. Let us consider a system described by a Markovian process $x(t)$ obeying the master equation

$$\dot{p}(xt)=\int \Gamma(x,y;t)\,p(yt)\,dy. \tag{2.1}$$

In general the kernel in (2.1) may contain a contribution proportional to the distributions $\delta'(x-y)$ and $\delta''(x-y)$ which account for a continuous component of the process $x(t)$. For $x=y$ the kernel in (2.1) contains a negative δ-contribution representing the loss of weight of state x during dt due to all transitions from x to different values y. Hence we write for $\Gamma(x,y;t)$

$$\Gamma(x,y;t)=W(x,y;t)-\lambda(yt)\,\delta(x-y) \tag{2.2}$$

with

$$W(x,y;t)\geqq 0 \tag{2.3}$$

and

$$\lambda(xt)=\int W(y,x;t)\,dy<\infty \tag{2.4}$$

denoting the total jump frequency. Writing the stochastic kernel as

$$\Gamma(x,y;t)=\int \Gamma(z,y;t)\,\delta(z-x)\,dz \tag{2.5}$$

and formally expanding the δ-function in (2.5) at $z=y$, we obtain the Kramers-Moyal expansion of (2.1) with operator $\Gamma(t)$ given by

$$\Gamma(t)=\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\left(\frac{\partial}{\partial x}\right)^n a_n(t,x). \tag{2.6}$$

The Kramers-Moyal moment $a_n(t,x)$ is given by

$$a_n(t,x)=\int (z-x)^n\,W(z,x;t)\,dz$$
$$n=1,2,\ldots. \tag{2.7}$$

For what follows, we call the operator $\Gamma(t)$ in (2.1) or (2.6) the *forward generator* of the Markov process $x(t)$. Of main importance for the following is the concept of the conditional expectation: The conditional expectation $\langle f(x(t))|y,s\rangle$ of a function $f(x)$ is defined as the statistical mean taken over the subset of sample paths passing through state y at former time s. The rate of change of this quantity with respect to the later time t obeys the equation [17, 18]

$$\frac{\partial}{\partial t}\langle f(t)|ys\rangle=\langle\Gamma^+(t)\,f(t)|ys\rangle,\quad t>s. \tag{2.8}$$

By letting t approach s, $t\downarrow s$, we obtain from (2.8)

$$\lim_{t\downarrow s}\frac{\langle f(t)|ys\rangle-f(y)}{t-s}=(\Gamma^+(t)\,f)(y). \tag{2.9}$$

The transposed operator $\Gamma^+(t)$ is called the *backward generator* of $x(t)$. It has from (2.6) the Kramers-Moyal expansion

$$\Gamma^+(t)=\sum_{n=1}^{\infty}\frac{1}{n!}\,a_n(t,x)\left(\frac{\partial}{\partial x}\right)^n. \tag{2.10}$$

The backward operator has the property that it describes the time evolution of the conditional probability $R(xt|ys)$ of (2.1) with respect to the former time s [17, 18]

$$\frac{\partial R(xt|ys)}{\partial s} = -\int \Gamma^{+}(y, z; s)\, R(xt|zs)\, dz. \tag{2.11}$$

The problem stated in the introduction can now be formulated explicitly: Given the Markov process $x(t)$ possessing the backward generator in (2.10), we look for a stochastic differential equation description

$$dx = v(t, x)\, dt + \mathscr{F}(t, x, d\eta) \tag{2.12}$$

with $v(xt)$ denoting a regular drift* and $\mathscr{F}$ a linear functional in some noise η such that the stochastic realizations (ω) of (2.12), $x(t, x_0, \omega)$, $x(0) = x_0$, are stochastically equivalent to the process $x(t)$ with master equation given in (2.1). The second part in (2.12) represents the irregular part of the displacement. The stochastic integral with respect to the noise $d\eta$ can in general not be defined uniquely. Henceforth, the structure in (2.12) must be accomplished with a specific definition for the stochastic integral. A continuous component of the irregular part of $x(t)$ is well known to be represented by the term (see e.g. [7, 18, 21])

$$b(t, x)\, dw \tag{2.13}$$

where $dw = w(t+dt) - w(t)$ is the increment of the Wiener process. In the following section we elaborate on the structure of noise sources which are adequate for the representation of the discontinuous part of the random perturbation in (2.12).

3. Poisson Increments and White Generalized Poisson Noise

Consider a Poisson counting process $n(t; \lambda) \equiv n(t)$ with probability

$$P(n(t) = n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!} \tag{3.1}$$

and a characteristic function given by

$$\langle \exp i\omega n(t) \rangle = \exp\{(e^{i\omega} - 1)\, \lambda t\}. \tag{3.2}$$

The sample paths of the Poisson process $y(t, z=1)$ with a jump width $z=1$ are given by

$$y(t, z=1) = \sum_{i=1}^{n(t)} \theta(t - t_i). \tag{3.3}$$

Here $\{t_i\}$ is the set of the arrival times of the Poisson counting process. The stochastic properties of $y(t, z=1)$ are of course given by those of $n(t)$. For example, we have

$$\langle [y(t_1, z=1) - y(t_2, z=1)][y(t_3, z=1) - y(t_4, z=1)] \rangle = \begin{cases} \lambda^2 (t_1 - t_2)(t_3 - t_4), & t_1 > t_2 > t_3 > t_4 \\ \lambda^2 (t_1 - t_2)(t_3 - t_4) + \lambda (t_3 - t_4), & t_1 > t_3 > t_2 > t_4. \end{cases} \tag{3.4}$$

Next we consider the Poisson increment

$$dy(t, z=1) = [y(t+\varepsilon, z=1) - y(t, z=1)], \quad \varepsilon > 0. \tag{3.5}$$

Clearly we have for the probability

$$P(dy(t, z=1) = k) = e^{-\lambda\varepsilon} \frac{(\lambda\varepsilon)^k}{k!} \tag{3.6}$$

and $\langle dy(t, z=1) \rangle = \lambda\varepsilon$. In what follows we consider the fluctuation process $\eta(\varepsilon, z=1)$

$$\eta(t, \varepsilon, z=1) = dy(t, z=1) - \lambda\varepsilon; \quad \langle \eta \rangle = 0. \tag{3.7}$$

The correlation function of $\eta(t, \varepsilon, z=1)$ is from (3.4) given by

$$\langle \eta(t, \varepsilon, z=1)\, \eta(s, \varepsilon, z=1) \rangle = \begin{cases} 0, & |t-s| > \varepsilon \\ \lambda\varepsilon - \lambda |t-s|, & |t-s| \leqq \varepsilon. \end{cases} \tag{3.8}$$

The white Poisson process $\xi(t)$ is then defined by

$$\xi(t) = \lim_{\varepsilon \to 0} \frac{\eta(t, \varepsilon, z=1)}{\varepsilon} = \sum_i \delta(t - t_i) - \lambda. \tag{3.9}$$

From (3.7), (3.8) we find

$$\langle \xi(t) \rangle = 0 \tag{3.10a}$$

$$\langle \xi(t)\, \xi(s) \rangle = \lambda\, \delta(t-s). \tag{3.10b}$$

The properties in (3.10) are analogous to those of white Gaussian noise. In this context it should be stressed that (3.10) alone does not guarantee that the solutions of a SDE with a noise satisfying (3.10) correspond to realizations of a Markov process. A complete characterization of the noise has to include knowledge of all higher cumulants as well. By using the cumulant generating functional $\psi_t[v]$ (see Equation (A.11), (A.16) in [15])

$$\psi_t[v] = \ln \left\langle \exp i \int_0^t ds\, v(s)\, \xi(s) \right\rangle = \lambda \int_0^t ds (e^{iv(s)} - 1) \tag{3.11}$$

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* It is important to note that the drift $v(xt)$ is in general *not* identical to the drift in a deterministic equation. The latter is usually proportional to a thermodynamic force [19], whereas the drift in (2.12) may contain in general nonlinear effects of the noise η

we find for the cumulants, $\langle\ \rangle_c$, of $\xi(t)$

$$\langle \xi(t_1)\,\xi(t_2)\ldots\xi(t_n)\rangle_c = \lambda\,\delta(t_1-t_2)\,\delta(t_2-t_3)\ldots\delta(t_{n-1}-t_n), \quad n\geqq 2. \tag{3.12}$$

In order to represent sample functions of a process with variable jump lengths, we have to generalize the above structure for the discontinuous noise. One possibility for such a generalization is obtained by allowing the jump width z_i at Poisson arrival time t_i to be a random variable. Let $\{z_1, \ldots, z_n, \ldots\}$ be a set of independent random variables with a common probability ζ. The *generalized* Poisson process $y(t)$ is then given by

$$y(t) = \sum_{j=1}^{n(t)} z_j \theta(t-t_j). \tag{3.13}$$

Its characteristic function can be calculated as follows: Noting that

$$\left\langle \exp i\omega \sum_{j=1}^{m} z_j \right\rangle = \phi^m(\omega) \tag{3.14}$$

where $\phi(\omega) = \langle e^{i\omega z_1}\rangle$ we obtain by observing that $z_0 = 0$ and $n(t)$ independent of the jump variables [18]

$$\begin{aligned}\langle \exp i\omega y(t)\rangle &= \left\langle\!\left\langle \exp i\omega \sum_{j=0}^{n(t)} z_j | n(t)=m \right\rangle\!\right\rangle \\ &= \sum_{m=0}^{\infty} \phi^m(\omega)\, P(n(t)=m) \\ &= \exp\{\lambda t[\phi(\omega)-1]\}.\end{aligned} \tag{3.15}$$

Thus we obtain for the statistical mean of $y(t)$

$$\langle y(t)\rangle = \langle z\rangle \lambda t. \tag{3.16}$$

Denoting the generalized Poisson process where $\langle z\rangle = 0$ by $\mu(t)$ we can form the generalized white Poisson process $\xi_{\mathrm{GP}}(t)$

$$\xi_{\mathrm{GP}}(t) = \frac{d\mu(t)}{dt} = \sum_{i=1}^{n(t)} z_i \delta(t-t_i). \tag{3.17}$$

Its stochastic properties can be deduced from its cumulant generating functional given in [15]. For example we have

$$\langle \xi_{\mathrm{GP}}(t)\rangle = 0 \tag{3.18a}$$

$$\langle \xi_{\mathrm{GP}}(t)\,\xi_{\mathrm{GP}}(s)\rangle = \lambda \langle z^2\rangle\,\delta(t-s) \tag{3.18b}$$

$$\langle \xi_{\mathrm{GP}}(t_1)\ldots\xi_{\mathrm{GP}}(t_n)\rangle_c = \lambda\langle z^n\rangle\,\delta(t_1-t_2)\ldots\delta(t_{n-1}-t_n). \tag{3.18c}$$

Note that for $\lambda\to\infty$, $\langle z^n\rangle\to 0$, $n=2,3\ldots$ such that $\lambda\langle z^2\rangle\to 1$, $\lambda\langle z^n\rangle\to 0$, $n=3,4\ldots$ we obtain from $\xi_{\mathrm{GP}}(t)$ the Gaussian white noise.

Another possibility to represent a jump process $y(t)$ with *independent* increments of variable size can be obtained in the following way: Let A denote and interval $A\subset(-\infty, 0^-)\cup(0^+, \infty)$ and $n(t, A)$ the *number* of points $s\varepsilon[0, \infty)$ for which $y(s^+)-y(s^-)\in A$. The counting process $n(t, A)$ has independent increments because $n(t, A)-n(s, A)$ with $t\geqq s$ is completely determined by the increments of $y(r)-y(s)$, $r\in[s, t]$. We further assume that the probability $P([n(t, A) - n(s, A)] = k)$ depends only on the length $t-s(n(t, A)$ time-homogeneous). Under the above assumptions the process $n(t, A)$ is shown to be a time-homogeneous Poisson process [20]. The parameter λ is given by $\langle n(t, A)\rangle/t$. For the following we denote the quantity λ by

$$\lambda \equiv \Pi(A) = \int_A p(u)\,du, \quad p(u)\geqq 0. \tag{3.19}$$

The set of functions $\Pi(A)$* is a measure on $(-\infty, \infty)$: Let $A = \bigcup_{i=1}^{\infty} a_i$ where the intervals $\{a_i\}$ are pairwise disjoint. Then the process $n(t, A)$ equals $\sum_i n(t, a_i)$ and as a consequence

$$\Pi(A) = \sum_i \langle n(t, a_i)\rangle/t = \sum_{i=1}^{\infty} \Pi(a_i) \tag{3.20}$$

with $\Pi(a_i)\geqq 0$ because $n(t, a_i)\geqq 0$. The independent increment process $y(t, A)$ with jump widths $z_i\varepsilon A$ is given by

$$y(t, A) = \sum_{i=1}^{n(t,A)} z_i \theta(t-t_i) \tag{3.21}$$

with $z_i = y(t_i^+) - y(t_i^-)\,\varepsilon A$. In other words, the process $y(t, A)$ is the *sum* (not number) of the jumps of the process $y(t)$ that occur up to time t and fall in the set A. It may be looked upon as a restricted generalized Poisson process with a random variable z_i varying in the interval A. For the special choice $A=\{1\}$, we obtain the process $y(t, z=1)$ in (3.3). As a generalization of the Poisson increment $dy(t, z=1)$ we study the properties of the process $n(\varepsilon, A)$:

$$n(\varepsilon, A) = n(t+\varepsilon, A) - n(t, A), \tag{3.22}$$

which is a *Poisson counting process* with parameter $\lambda = \Pi(A)$. Noting that

$$\langle n(\varepsilon, A)\rangle = |\varepsilon|\,\Pi(A) \tag{3.23}$$

we consider the process $\eta(\varepsilon, A)$ of vanishing mean

$$\eta(\varepsilon, A) = n(\varepsilon, A) - |\varepsilon|\,\Pi(A). \tag{3.24}$$

* The density $p(u)$ may in general contain δ-functions

By virtue of the characteristic function in (3.2), we recover the following statistical properties:

$$\langle \eta(\varepsilon, A)^2 \rangle = |\varepsilon| \Pi(A) \tag{3.25a}$$

$$\langle \eta(\varepsilon, A)^3 \rangle = |\varepsilon| \Pi(A) \tag{3.25b}$$

$$\langle \eta(\varepsilon, A)^4 \rangle = |\varepsilon| \Pi(A)(1 + 3|\varepsilon| \Pi(A)). \tag{3.25c}$$

A very important property of the process $\eta(t, A)$ is that in the limes of mean square (l.i.m.)

$$\begin{aligned}\int_{t_0}^{t} \eta(dt, A)\, \eta(dt, B) &= \lim_{|\Delta t| \to 0} \sum_j \eta(\Delta t, A)\, \eta(\Delta t, B), \qquad t > t_0 \\ &= \text{l.i.m.}\, n(t - t_0, A \cap B) \\ &= \text{l.i.m.}\, \eta(t - t_0, A \cap B) + (t - t_0)\, \Pi(A \cap B).\end{aligned} \tag{3.26}$$

To prove this relation, we set

$$x = \eta(\Delta t, A)\, \eta(\Delta t, B) - n(\Delta t, A \cap B). \tag{3.27}$$

By noting that $\eta(\Delta t, A)$ and $\eta(\Delta t, B)$ are *independent* for $A \cap B = \emptyset$, we find from (3.25a) $\langle x \rangle = 0$. Forming the variance $\sigma(x)$

$$\sigma(x) = \langle x^2 \rangle - \langle x \rangle^2 \tag{3.28}$$

we find

$$\begin{aligned}\sigma(x) &= \sigma[(\eta(\Delta t, A \cap \bar{B}) + \eta(\Delta t, A \cap B)) \\ &\quad \cdot (\eta(\Delta t, B \cap \bar{A}) + \eta(\Delta t, A \cap B)) - n(\Delta t, A \cap B)] \\ &= \sigma[\eta(\Delta t, A \cap \bar{B} \cup B \cap \bar{A})\, \eta(\Delta t, A \cap B) \\ &\quad + \eta(\Delta t, A \cap \bar{B})\, \eta(\Delta t, B \cap \bar{A}) + \eta^2(\Delta t, A \cap B) \\ &\quad - n(\Delta t, A \cap B)] \\ &\leqq 3\{\langle \eta(\Delta t, A \cap \bar{B} \cup B \cap \bar{A})\, \eta(\Delta t, A \cap B) \rangle^2 \\ &\quad + \langle \eta(\Delta t, A \cap \bar{B})\, \eta(\Delta t, B \cap \bar{A}) \rangle^2 \\ &\quad + \langle \eta^2(\Delta t, A \cap B) - n(\Delta t, A \cap B) \rangle^2\} \\ &= 3|\Delta t|^2 [\Pi(A \cap \bar{B} \cup B \cap \bar{A})\, \Pi(A \cap B) \\ &\quad + \Pi(A \cap \bar{B})\, \Pi(B \cap \bar{A}) + 2(\Pi(A \cap B))^2].\end{aligned} \tag{3.29}$$

In the last step we made extensive use of the properties in (3.25). Due to the term $|\Delta t|^2$ in front of (3.29) the integral in (3.26) equals in the limit of mean square the random variable $n(t - t_0, A \cap B)$! Equation (3.26) may be written in shorthand form

$$\eta(\Delta t, A)^2 = \int_{\Delta t} \eta(dt, A)^2 = \text{l.i.m.}\, \eta(\Delta t, A) + \Delta t\, \Pi(A) \tag{3.30}$$

but it should be stressed that the integration over a nonzero interval is essential. Equation (3.30) has with $\eta(dt, A)$ substituted by the increment $dw = w(t + dt) - w(t)$ of the Wiener process $w(t)$ its well known analogue [18]

$$(dw)^2 = dt \quad \text{with probability 1.} \tag{3.31}$$

As in the theory of SDE for Fokker-Planck processes [7, 18, 21], it is this fact (Eq. 3.30) that in an SDE of the form

$$\int_0^T \varphi(x(t))\, \eta(dt, A) \tag{3.32}$$

the final result does depend on the choice of the point $x(t_i')$ with $t_i < t_i' < t_{i+1}$! Consequently, a stochastic integral of the type in (3.32) cannot be defined uniquely. In the next section we will elaborate on a convenient definition for the stochastic integral with respect to the noise sources discussed in this section.

4. Stochastic Integrals

4.1. Stochastic Integrals with Respect to Generalized Poisson Noise $d\mu(t)$

To start, let us first consider a process $y(t)$ with independent increments. By considering the jump length as a random variable, we can represent $y(t)$ by a generalized Poisson process (3.13)

$$y(t) = \langle y(t) \rangle + \mu(t), \qquad y(0) = y_0. \tag{4.1}$$

(4.1) can be recast in difference form

$$dy(t) = \left(\frac{d}{dt} \langle y(t) \rangle\right) dt + d\mu(t). \tag{4.2}$$

With $0 = t_0 < t_1 < \ldots < t_n = t$ denoting a partition of the time interval $[0, t]$, we have

$$\begin{aligned}y(t) &= \int_0^t dy(s) = y_0 + \int_0^t \left(\frac{d}{ds} \langle y(s) \rangle\right) ds \\ &\quad + \lim_{\delta \to 0} \sum_i \mu(t_{i+1}) - \mu(t_i)\end{aligned} \tag{4.3}$$

where $\delta = \max_i (t_{i+1} - t_i)$. More generally, we may study the integral

$$\int_0^t f(\mu(s))\, d\mu(s) \tag{4.4}$$

with $f(t)$ a random function satisfying $\int \langle f^2(t) \rangle dt < \infty$. Then we *define* the integral in (4.4) by

$$\int_0^t f(\mu(s))\, d\mu(s) = \lim_{\delta \to 0} \sum_i f(\mu(t_i))(\mu(t_{i+1}) - \mu(t_i)). \tag{4.5}$$

With this definition we find the following properties:

$$1) \quad \left\langle \int_0^T f(\mu(t))\, d\mu(t) \right\rangle = 0 \tag{4.6a}$$

2) $\left\langle\left[\int_0^T f(\mu(t))\,d\mu(t)\right]^2\right\rangle$

$$=\lim_{\delta\to 0}\sum_i \langle f^2(t_i)\rangle\,\lambda\langle z_i^2\rangle(t_{i+1}-t_i)$$

$$=\lambda\langle z^2\rangle\int_0^T\langle f^2(t)\rangle\,dt. \qquad (4.6\text{b})$$

Hereby, we made extensive use of the fact that $f(\mu(t_i))$ and $[\mu(t_{i+1})-\mu(t_i)]$ are independent random variables. In (4.6b) we made use of (3.18b) and assumed that the jump variables $\{z_i\}$ possess a common probability ζ.

Let us now consider a general discontinuous process $x(t)$ with no continuous component which in general does not possess independent increments. For the following, we denote the local increment of vanishing mean by $\gamma(t,x(t),h)$

$$\gamma(t,x(t),h)=x(t+h)-x(t)-(\langle x(t+h)\rangle-\langle x(t)\rangle). \qquad (4.7)$$

As in the theory of the representation of Fokker-Planck processes [1, 7, 18, 20, 21], we represent the increment in (4.7) by the following stochastic integral

$$\gamma(t,x(t),h)=\int_h c(t,x(t))\,d\mu(t) \qquad (4.8\text{a})$$

$$=\lim_{\delta\to 0}\sum_i c(t,x(t_i))[\mu(t_{i+1})\mu(t_i)] \qquad (4.8\text{b})$$

with $c(t,x)$ denoting some function and $\mu(t)$ a "standard" generalized Poisson process (3.13). In (3.13) we have assumed that the probability ζ for the jump length as well as the parameter λ for the Poisson counting process do not depend on the state $x(t_i)$ at Poisson arrival time t_i. If we allow for the measure of $d\mu(t)$ to depend on time t and state x, we may alternatively write for (4.8)

$$\gamma(t,x(t),h)=\int_h d\mu_{t,x}$$

$$=\lim_{\delta\to 0}\sum_i[\mu_{t_i,x(t_i)}(t_{i+1})-\mu_{t_i,x(t_i)}(t_i)]. \qquad (4.9)$$

Here the choice of the time point in (4.9) for the parameter values (t,x) of the measures ζ, $n(t;\lambda)$ of $\mu(t)$ is important; only with this choice do the jump variables that occur in the infinitesimal time interval $[t_i,t_{i+1}]$ be independent of the parameter dependent Poisson counting process $n(t)$ with $\lambda=\lambda_{t,x(t_i)}$. In this context the function $c(t,x)$ in (4.8) can be looked upon as the mapping, $c(t,x)$, of the parameter dependent measure to the fixed "standard" measures ζ, $n(t;\lambda)$ of the generalized Poisson process in (4.8).

4.2. *Stochastic Integrals with Respect to $\eta(dt,du)$*

Instead of describing the sample paths in terms of a generalized Poisson process [see (4.8)], we can just as well represent the realization of a discontinuous process $x(t)$ with no continuous component as a superposition of processes with jumps of a fixed length. First, let us again consider a process with independent increments $y(t,A)$ (3.21). In terms of the Poisson counting process $n(t,du)$, we can represent $y(t,A)$ in the following form:

$$y(t,A)=\int_A u\,n(t,du)=\lim_{\varepsilon\to 0}\sum_i u_i n(t,\Delta u_i) \qquad (4.10)$$

where

$$\varepsilon=\max_i|\Delta u_i|=\max_i|u_{i+1}-u_i|. \qquad (4.11)$$

By noting that $A=\bigcup_{i=1}^{\infty}\Delta u_i$ and $\{\Delta u_i\}$ pairwise *disjoint* sets we consequently obtain

$$|y(t,A)-\sum_i u_i n(t,\Delta u_i)|$$

$$\leqq\sum_i|y(t,\Delta u_i)-u_i n(t,\Delta u_i)|$$

$$\leqq\varepsilon\sum_i n(t,\Delta u_i)=\varepsilon n(t,A). \qquad (4.12)$$

For example, the generalized Poisson process in (3.13) has the representation

$$y(t)=\int_{-\infty}^{\infty}u\,n(t,du),\qquad d\pi(u)=\lambda\zeta(u)\,du. \qquad (4.13)$$

As before, we look for a representation of the increment $\gamma(t,x(t),h)$ of *vanishing mean* of a general discontinuous process with no continuous component. In a way analogous to that constructed in (4.9), we can write

$$\gamma(t,x(t),h)=\int_h\left(\int u\,\eta_{t,x}(dt,du)\right). \qquad (4.14)$$

The integral in (4.14) is hereby defined by

$$\int_0^T\int u\,\eta_{t,x}(dt,du)$$

$$=\lim_{\delta\to 0}\sum_i\int u\,\eta_{t_i,x(t_i)}(t_{i+1}-t_i,du). \qquad (4.15)$$

Because the dependence of the measure η on the parameters (t,x) causes a certain inconvenience, we go with the help of measure transforming mapping $f(t,x,u)$ to a representation of the form

$$\int_0^T\int u\,\eta_{t,x}(dt,du)=\int_0^T\int f(t,x(t),u)\,\eta(dt,du)$$

$$=\lim_{\delta\to 0}\sum_i\int f(t_i,x(t_i),u)\,\eta(t_{i+1}-t_i,du). \qquad (4.16)$$

As in the theory of SDE for Fokker-Planck processes, where the methods and properties of the stochastic calculus were developed long before physicists recognized its usefulness, the stochastic integral of the type in (4.16) was first considered by Ito [22]. The theory has been further developed by Gikhman and Skorokhod [23]. But as often occurs in such cases, these quoted texts are more or less in incomprehensible form for anyone not armed with a solid background in the theory of stochastic processes from the measure-theoretic point of view. We therefore prefer to follow a more pedestrian approach with emphasis on the physical essential ideas and relations. Let us summarize some important features of the stochastic integral in (4.16):

$$1)\ \left\langle \int_{t_1}^{t_2}\int f(t, x(t), u)\,\eta(dt, du)\right\rangle = 0 \qquad t_2 > t_1 \tag{4.17a}$$

$$2)\ \left\langle \left[\int_{t_1}^{t_2}\int f(t, x(t), u)\,\eta(dt, du)\right]^2\right\rangle = \int_{t_1}^{t_2}\int \langle f^2(t, x(t), u)\rangle\, p(u)\, du\, dt \tag{4.17b}$$

$$3)\ \lim_{dt\to 0}\frac{1}{(dt)^2}\langle \gamma^n(t, x(t), dt)\,\gamma^m(s, x(s), ds)\rangle = \int f^{n+m}(t, x, u)\, p(u)\, du\, \delta(t-s) \tag{4.17c}$$

where $x(t)=x$.
The properties in (1), (2), (3) above follow from the fact that $f(t, x(t), u)$ within the definition of the stochastic integral in (4.16) is *independent* of $\eta(dt, du)$ (Ito definition, [23]). By use of a Stratonovich-like definition for (4.16), the relationship between the two stochastic integrals reveals a complicated fluctuation induced drift [15] as well as additional stochastic terms (see Appendix A).
If a process $x(t)$ satisfies the Ito-SDE

$$dx = a(t, x)\, dt + \int f(t, x(t), u)\,\eta(dt, du) \tag{4.18}$$

the total differential of the process $g(t, x(t))$ satisfies [23]:

$$\begin{aligned}4)\ dg(t, x) = {}& [\dot g(t, x) + g'(t, x)\, a(t, x) \\ & - g'(t, x)\int f(t, x, u)\, p(u)\, du]\, dt \\ & + \int \{g[t, x(t) + f(t, x(t), u)] \\ & - g(t, x(t))\}\, n(dt, du).\end{aligned} \tag{4.19}$$

5. Langevin Description of Integro-Master Equations for Markovian Processes

With the results derived in the previous sections, we are now prepared sufficiently to study the SDE-description for a given master equation (2.1).

5.1. Langevin Description in Terms of Generalized Poisson Noise Increments

Let us introduce the probability $\zeta_{t,y}$ and $\rho_{t,y}$

$$\zeta_{t,y}(z = x-y) = W(x, y; t)/\lambda_{t,y} \geqq 0,$$

$$\rho_{t,y}(z = x-y) = \frac{1}{2\pi}\int e^{-i\omega z} \cdot \left[1 + \frac{1}{\lambda_{t,y}}\sum_{j=2}^{\infty}\frac{(i\omega)^j}{j!}a_j(t, y)\right] d\omega \tag{5.1}$$

with

$$\int \zeta_{t,y}(z)\, dz = \int W(x, y; t)\, dx/\lambda_{t,y} = 1, \qquad \langle z\rangle_\rho = 0 \tag{5.2}$$

and

$$\lambda_{t,y} \equiv \lambda(yt) = \int W(x, y; t)\, dx \tag{5.3}$$

$\zeta_{t,y}$ gives the probability for a jump of length z when the value of the process $x(t)$ *prior* to the jump was y. In the following we will show that a discontinuous irreversible process $x(t)$ with *no* continuous component obeying (2.1) is represented stochastically equivalent by the (*Ito*)-*SDE*

$$dx = dy_{t,x} = a(t, x)\, dt + d\mu_{t,x}. \tag{5.4}$$

Hereby $a(t, x)$ is equal to the first Kramers-Moyal moment [the drift-term in the master equation (2.1)]

$$a(t, x) = a_1(t, x) = \int (z-x)\, W(z, x; t)\, dz \tag{5.5}$$

and $y_{t,x}(s)$, $\mu_{t,x}(s)$ are generalized Poisson processes with parameter dependent Poisson counting process $n(dt; \lambda(t, x))$ and a parameter dependent jump probability $\zeta_{t,x}$ and $\rho_{t,x}$ respectively.
If we denote by $\phi(\omega, t)$ the characteristic function of the process $x(t)$ of (5.4), we obtain [12]

$$\frac{\partial}{\partial t}\phi(\omega, t) = \left\langle \exp i\omega x(t)\frac{\langle \exp(i\omega dx(t)) - 1|x(t)\rangle}{dt}\right\rangle. \tag{5.6}$$

For the expectation $\langle \exp i\omega dx(t)|x(t)\rangle$ we obtain from (5.4)

$$\langle \exp i\omega dx(t)|x(t)\rangle = \langle \exp i\omega dy_{t,x}|x(t)\rangle. \tag{5.7}$$

Further, the probability, P, of two and more jumps is of order $o(dt)$ such that

$$\begin{aligned}\langle \exp i\omega dy_{t,x}|x(t)\rangle &= P\ (\text{no jumps}) + \langle \exp i\omega z\rangle_{t,x} P\ (\text{only one jump in } dt) \\ &= 1 - \lambda_{t,x} dt + \{\langle \exp i\omega z\rangle \lambda\}_{t,x} dt + o(dt).\end{aligned} \tag{5.8}$$

Hereby, we make explicit use of the definition of the stochastic integral in (4.9), where the probability for a jump to occur in the infinitesimal interval $[t+dt, t]$ is independent of the jump variable z. Thus we have

$$\frac{\partial}{\partial t}\phi(\omega,t)=\langle(\exp i\omega x(t))[-\lambda_{t,x(t)} + \{\langle\exp i\omega z\rangle\lambda\}_{t,x(t)}]\rangle. \tag{5.9}$$

Observing (5.2) we find by use of a Fourier inversion the master equation result in (2.1)

$$\dot{p}(xt)=-\lambda(t,x)\,p(x,t)+\int W(x,y;t)\,p(y,t)\,dy. \tag{5.10}$$

If the total jump frequency $\lambda(t,x)$ becomes *independent* of the parameters (t,x), the SDE description in (5.4) allows a simple solution method for the conditional probability $R(xt|x_0 0)$ of the master equation in (2.1): In this case the jump variables z are independent of the time-homogeneous Poisson counting process $n(t;\lambda(t,x)=\lambda)$ at all times! The solution for $R(xt|x_0 0)$ is consequently given by the expression

$$R(xt|x_0 0)=\sum_{j=0}^{\infty}P_j(t)\,r_j(x;x_0) \tag{5.11a}$$

with

$$P_j(t)=\frac{(\lambda t)^j}{j!}\exp-\lambda t. \tag{5.11b}$$

The term $r_j(x;x_0)$ is the probability of finding the jump $x_0\to x$ *after exactly j jumps have occurred.* Clearly we have

$$r_0(x;x_0)=\delta(x-x_0) \tag{5.11c}$$

$$r_1(x;x_0)=\zeta_{x_0}(x)=W(x,x_0)/\lambda \tag{5.11d}$$

and because the random variables of the jumps are independent of the Poisson arrival times

$$r_j(x;x_0)=\int\zeta_{x'}(z=x-x')\,r_{j-1}(x';x_0)\,dx' \tag{5.11e}$$

$j=1,\ldots.$

In general, the solution to (5.10) is obtained by setting

$$R(xt|x_0 0)=\sum_{i=0}^{\infty}v_i(x,x_0;t) \tag{5.12}$$

where the sequence of functions $\{v_i(x,x_0;t)\}$ obeys the system of equations

$$\dot{v}_0=-\lambda(t,x)\,v_0;\qquad v_0(x,x_0;0)=\delta(x-x_0)$$
$$\dot{v}_i=-\lambda(t,x)\,v_i+\int W(x,x';t)\,v_{i-1}(x',x_0;t)\,dx';$$
$$v_i(x,x_0;0)=0. \tag{5.13}$$

If the total jump frequency depends only on the parameter t, i.e., $\lambda(t,x)\equiv\lambda(t)$, we obtain for the solution of (5.13)

$$v_0(x,x_0;t)=\exp-\int_0^t\lambda(s)\,ds\,\delta(x-x_0)$$
$$v_i(x,x_0;t)=\exp-\int_0^t\lambda(s)\,ds\int_0^t\left(\exp\int_0^s\lambda(u)\,du\right)\cdot\int W(x,x';s)\,v_{i-1}(x',x_0,s)\,dx'ds. \tag{5.14}$$

5.2. Langevin Description in Terms of $\eta(dt,du)$

Given a Markovian irreversible process $x(t)$ with no continuous component and backward generator $\Gamma^+(t)$ of the form (2.10),

$$\Gamma^+(t)=\sum_{n=1}^{\infty}\frac{a_n(t,x)}{n!}\left(\frac{\partial}{\partial x}\right)^n. \tag{5.15}$$

We look in this subsection for a stochastically equivalent (Ito)-SDE in which the structure of the noise is extracted explicitly in the form of *multiplicative noise:*

$$dx=a_1(t,x)\,dt+\int f(t,x(t),u)\,\eta(dt,du). \tag{5.16}$$

In what follows we show that the SDE in (5.16), interpreted in the sense of (4.16), describes the process $x(t)$ in (5.15) if the function $f(t,x,u)$ in (5.16) obeys the set of equations

$$\int f^n(t,x,u)\,p(u)\,du=a_n(t,x),\qquad n=2,3,\ldots. \tag{5.17}$$

For the measure $p(u)\,du$ in (5.17) we require a finite total jump frequency

$$\int p(u)\,du=\lambda<\infty. \tag{5.18}$$

In order to prove the proposition in (5.16–5.18), let us calculate the backward generator of the (Ito)-SDE in (5.16) by using the property in (2.9). Considering (5.16), where with $h>0$, $x(t)=y$,

$$x(t+h)=y+\int_t^{t+h}a_1(s,x(s))\,ds + \int_t^{t+h}\int f(s,x(s),u)\,\eta(ds,du) \tag{5.19}$$

we introduce the process $\delta\bar{x}(t,y,h)$

$$\delta\bar{x}(t,y,h)=\int_t^{t+h}a_1(s,y)\,ds + \int_t^{t+h}\int f(s,y,u)\,\eta(ds,du). \tag{5.20}$$

The difference $g(x(t+h))-g(y+\delta\bar{x}(t,y,h))$ will be denoted by $z(h)$. With these definitions we can write for (2.9)

$$\lim_{h\to 0}\frac{1}{h}\{\langle g(x(t+h))\rangle - g(y)\}$$

$$=\lim_{h\to 0}\frac{1}{h}\{\langle g(y+\delta\bar{x}(t,y,h))\rangle - g(y)+\langle z(h)\rangle\}. \quad (5.21)$$

By use of the (generalized Ito) rule in (4.19), we obtain

$$\langle g(y+\delta\bar{x}(t,y,h)\rangle - g(y)$$
$$=\Big\langle \int_t^{t+h} [g'(y+\delta\bar{x}(t,y,s-t)\,a_1(s,y))$$
$$+\int \{g(y+\delta\bar{x}(t,y,s-t)+f(s,y,u))$$
$$-g(y+\delta\bar{x}(t,y,s-t))-g'(y+\delta\bar{x}(t,y,s-t))$$
$$\cdot f(s,y,u)\}\,p(u)\,du]\,ds\Big\rangle. \quad (5.22)$$

Thus we have in the limit $h\to 0$ and noting the convergence of $\delta\bar{x}\to 0$ for $h\to 0$

$$(\Gamma^+(t)\,g)(y)=a_1(t,y)\,g'(y)-\lambda g(y)$$
$$-v(t,y)\,g'(y)+\int g(y+f(t,y,u))\,p(u)\,du$$
$$+\lim_{h\to 0}\langle z(h)\rangle/h \quad (5.23)$$

with

$$v(t,y)=\int f(t,y,u)\,p(u)\,du. \quad (5.24)$$

The expression for $\langle z(h)\rangle$ is complicated. However, it has been shown in [24] that $\langle z(h)\rangle$ obeys the inequality

$$\langle |z(h)|\rangle \leqq C(1+|y|)\,h^{3/2} \quad (5.25)$$

so that the last term in (5.23) approaches zero. By virtue of (2.11), we obtain for the backward equation of the conditional probability $R(xt|ys)$, $t>s$, of $x(t)$*

$$\frac{\partial}{\partial s}R(xt|ys)$$
$$=-(a_1(s,y)-v(s,y))\frac{\partial}{\partial y}R(xt|ys)+\lambda R(xt|ys)$$
$$-\int R(x,t|y+f(s,y,u),s)\,p(u)\,du. \quad (5.26)$$

The Kramers-Moyal expansion of $\Gamma^+(t)$ is consequently given by (5.15) with $a_n(yt)$, $n=2,3,\dots$ given by (5.17). This proves the proposition.
A possible continuous component of the Markov process $x(t)$ can simply be included in the SDE (5.16) by adding the (Ito)-SDE of a Fokker-Planck process $x_c(t)$

$$dx_c(t)=a_c(t,x)\,dt+b_c(t,x)\,dw. \quad (5.27)$$

* (5.26) is the appropriate relation in order to solve the "moment problem" in (5.17)

The Langevin equation in (5.16) may alternatively be written as

$$\dot{x}=a_1(t,x)+\xi(t,x) \quad (5.28)$$

with $\xi(t,x)$ a "generalized process"

$$\xi(t,x)=\lim_{dt\to 0}\int f(t,x(t),u)\,\eta(dt,du)/dt. \quad (5.29)$$

The statistical mean and correlation function of this random force over $\eta(dt,du)$ is then, by virtue of (4.17a) and (4.17c) with (5.17), given by

$$\langle \xi(t,x)\rangle = 0 \quad (5.30\text{a})$$

$$\langle \xi(t,x)\,\xi(s,x)\rangle = a_2(t,x)\,\delta(t-s). \quad (5.30\text{b})$$

By noting the cumulant properties of the white Poisson process, we find for the cumulant averages of $\xi(tx)$ the result

$$\langle \xi(t_1x)\,\xi(t_2x)\dots\xi(t_nx)\rangle_c$$
$$=a_n(t_1x)\,\delta(t_1-t_2)\dots\delta(t_{n-1}-t_n), \quad n\geqq 2. \quad (5.30\text{c})$$

It is important to point out that the properties of the fluctuating force $\xi(t,x)$ hold *only* within the (Ito)-definition of the stochastic integral in (4.16)!
Finally, we would like to mention that the (Ito)-Langevin description of (5.16) with a parameter dependent measure $\eta_{t,x}(dt,du)$ is given by

$$dx=a_1(tx)\,dt+\int u\eta_{t,x}(dt,du) \quad (5.31)$$

where

$$a_n(tx)=\int u^n p_{t,x}(u)\,du \qquad n=2,3,\dots. \quad (5.32)$$

Thus remembering that u refers to the jump length $u=y-x$, we obtain for $p_{t,x}$ the explicit result

$$p_{t,x}(y-x)=W(y,x;t). \quad (5.33)$$

6. Examples

In this section we elucidate the results and concepts with some examples. As a first example, we consider an integro-master equation describing nonlinear relaxation with backward equation [$x(t)$ time-homogeneous]

$$\frac{\partial}{\partial t}R(x|y;t)=a(y)\frac{\partial}{\partial y}R(x|y;t)$$
$$-\lambda S y\frac{\partial}{\partial y}R(x|y;t)-\lambda R(x|y;t)+\lambda R(x|y+Sy;t). \quad (6.1)$$

By use of (2.8) we find for the transposed equation the master equation

$$\dot{p}(x,t) = -\frac{\partial}{\partial x}\{(a(x)-\lambda S x)\, p(x,t)\}$$
$$-\lambda p(x,t) + \frac{\lambda}{1+S}\, p\left(\frac{x}{1+S}, t\right). \tag{6.2}$$

From (6.1) we immediately read off the corresponding (*Ito*)-*SDE* with multiplicative noise

$$\dot{x} = a(x) + x\,\xi(t) \tag{6.3}$$

where $\xi(t)$ is a white Poisson process (3.9) with a jump length $z=S$. The function $f(t,x,u)$ in (5.16) is consequently given by

$$f(t,x,u) = xu \tag{6.4}$$

and the measure $p(u)$

$$p(u) = \lambda\,\delta(S-u). \tag{6.5}$$

Thus we obtain for the Kramers-Moyal moments $a_n(x)$ from (5.17)

$$a_n(x) = \lambda (xS)^n \qquad n = 2, 3, \ldots \tag{6.6}$$

which is consistent with the Kramers-Moyal expansion of (6.1). It is interesting to compare the (Ito)-SDE in (6.3) with the corresponding Stratonovich definition (see Appendix A). The resulting master equation can be found by use of the results in [14, 15]. A simple derivation can be obtained by noting that (6.3) is recast in the form

$$\dot{x} = a(x) - \lambda S x + x\, y(t) \tag{6.7}$$

with $y(t)$ a white Poisson process of *nonvanishing* mean

$$y(t) = \sum_i S\,\delta(t-t_i) \tag{6.8}$$

and using the fact that the Stratonovich integral, with respect to the noise $y(t)$, can be calculated as if the functions involved were smooth [15]. Thus we find for the change in x due to one impulse from (6.8)

$$x_f = x_i e^S. \tag{6.9}$$

Now, recalling that the probability for one impulse in dt is λdt and that a particle in $(x, x+dx)$ *after* an impulse was in $(e^{-S}x, e^{-S}x+dx)$ *before* we find with $dx_i = e^{-S}dx$ the master equation (see also [25])

$$\dot{p}(x,t) = -\frac{\partial}{\partial x}\{(a(x)-\lambda S x)\, p(x,t)\}$$
$$-\lambda p(x,t) + \lambda e^{-S} p(x e^{-S}, t). \tag{6.10}$$

By expanding the master equation in (6.2) up to order S^2, we find

$$\dot{p}(xt) = -\frac{\partial}{\partial x}\{a(x)\, p(xt)\}$$
$$+\lambda S^2\left(p + 2xp' + \frac{x^2}{2}p''\right) + O(S^3) \tag{6.11}$$

whereas in the case of (6.10) we have

$$\dot{p}(xt) = -\frac{\partial}{\partial x}\{a(x)\, p(xt)\}$$
$$+\lambda S^2\left(\tfrac{1}{2}p + \tfrac{3}{2}xp' + \frac{x^2}{2}p''\right) + O(S^3). \tag{6.12}$$

In the diffusion limit $S\to 0$, $\lambda\to\infty$, $\lambda S^2 \to 1$ we obtain from (6.11) the expected result

$$\dot{p}(xt) = -\frac{\partial}{\partial x}\{a(x)\, p(xt)\} + \tfrac{1}{2}\frac{\partial^2}{\partial x^2}\{x^2 p(xt)\}. \tag{6.13}$$

From (6.12) we find the well-known fluctuation induced drift by Gaussian white noise

$$\dot{p}(xt) = -\frac{\partial}{\partial x}\{(a(x) + \tfrac{1}{2}x)\, p(xt)\} + \tfrac{1}{2}\frac{\partial^2}{\partial x^2}\{x^2 p(xt)\} \tag{6.14}$$

As a second example, we study the nonlinear model for Brownian motion introduced in [26]: Assuming that there is a finite probability that the particle will have undergone no collision for all subsequent time t, we have a master equation of the integro-type [26]

$$\dot{p}(vt) = -\lambda p(vt) + \lambda\left(\frac{\beta}{\pi}\right)^{1/2} \int \exp - \beta(v-\gamma v')^2 p(v't)\,dv'. \tag{6.15}$$

Here v denotes the stochastic variable of the velocity and β is the Boltzmann factor. The stationary solution of (6.15) reads

$$p_s(v) = \sqrt{\frac{\beta(1-\gamma^2)}{\pi}}\,\exp - \beta(1-\gamma^2)\, v^2, \qquad 0<\gamma<1. \tag{6.16}$$

Because the total jump frequency is independent of the parameters (t, v), we can immediately make use of the results in (5.11). The process $v(t)$ obeying (6.15) can consequently be represented by an (Ito)-SDE of type (5.4)

$$dv(t) = dy_v(t) \tag{6.17}$$

with jump probability ζ_{v_0} given by

$$\zeta_{v_0}(z = v - v_0) = \left(\frac{\beta}{\pi}\right)^{1/2} \exp - \beta(v - \gamma v_0)^2. \tag{6.18}$$

For the probability $r_j(v; v_0)$ in (5.11), we find explicity

$$r_j(v; v_0) = \left(\frac{\beta}{\pi c_j}\right)^{1/2} \exp -\frac{\beta}{c_j}(v - \gamma^j v_0)^2 \tag{6.19}$$

with

$$c_j = (\gamma^{2j} - 1)/(\gamma^2 - 1), \quad j = 1, \ldots . \tag{6.20}$$

Thus, the conditional probability of the master equation in (6.15) has the solution

$$R(v|v_0; t) = \exp -\lambda t \left\{\delta(v - v_0) + \sum_{n=1}^{\infty} \frac{(\lambda t)^n}{n!} r_n(v; v_0)\right\} \tag{6.21}$$

in agreement with [26] derived there by involved Fourier-Laplace methods.

7. Conclusions

In this paper we have considered the representation of Markov processes obeying an integro-differential master equation in terms of a stochastically equivalent Langevin equation. Such a representation is interesting in itself, but it also provides an understanding of the structure of the involved noise. In this context it should be mentioned that many of the statistical properties of the process can often be obtained more simply by addressing oneself to the SDE (e.g., see the solution of the master equation in Sect. 5.1).

Throughout this paper we have used an Ito-like definition for the various Langevin equations. The question of which definition is most appropriate for the description of a physical system may depend on the special problem. For a *given* physically correct master equation, this question is in principle a matter of taste. However, for the modeling of irreversible discontinuous processes, the author tends to favor the concept of an Ito-definition: The fluctuations are in general always present. Thus, *prior* to any *statistical* modeling, the drift $a_1(tx)$ in the master equation, which describes the average motion, should be known at least approximately. Because of the complicated structure of the fluctuation induced drift by using the Stratonovich interpretation [15] (see also Appendix A), knowledge of the "Stratonovich drift" in the SDE *alone* is of no use for a stochastic modeling. The situation is of course different if one starts from a completely deterministic system and "adds" noise explicitly. In this case the desire might be that the difference between the solutions with smooth noise [non-Markovian process $x(t)$] and the ones with idealized white Poisson and white Gaussian noise is small, thus favoring a Stratonovich interpretation.

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Appendix A

The "Stratonovich integral" (S) with respect to noise $\eta(dt, du)$ is defined by

$$\underset{S}{\int_{t_0}^{t}}\int f(s, x(s), u)\,\eta(ds, du) = \lim_{\delta \to 0} \sum_j \int f(t_j, \tfrac{1}{2}x(t_j) + \tfrac{1}{2}x(t_{j+1}), u) \cdot (\eta(t_{j+1}, du) - \eta(t_j, du)). \tag{A.1}$$

Using a Taylor expansion about $x(t_j)$ and the properties of $\eta(dt, du)$, we can calculate the relation between the Stratonovich integral and the Ito-integral of (4.18): The first correction is obtained by

$$\underset{S}{\int_{t_0}^{t}}\int f(s, x(s), u)\,\eta(ds, du) = \underset{\text{Ito}}{\int_{t_0}^{t}}\int f(s, x(s), u)\,\eta(ds, du) + \lim_{\delta \to 0} \sum_j \int f(t_j, x(t_j), u) \frac{\partial f}{\partial x} [\eta(\Delta t, du) - \tfrac{1}{2}\eta(\Delta t, du)]\,\eta(\Delta t, du) + O(\Delta t) \tag{A.2}$$

$$= \underset{\text{Ito}}{\int_{t_0}^{t}}\int \left(f_t(s, x(s), u) + \tfrac{1}{2} f(s, x(s), u) \frac{\partial f}{\partial x}\right) \eta(ds, du) + \tfrac{1}{2}\int_{t_0}^{t} \left\{\int \left(f(s, x, u) \frac{\partial f}{\partial x}\right) p(u)\,du\right\} ds + O(t - t_0). \tag{A.3}$$

In the step from (A.2) to (A.3), we have used the result of (3.30). We see from (A.3) that by use of the Stratonovich definition for (4.18), not only the drift term is modified by a fluctuation induced drift but *also* the higher Kramers-Moyal diffusion moments (see also Eq. (3.33) in [15])!

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