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On the Number of Integral Ideals in Galois Extensions

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Abstract. If a_k denotes the number of integral ideals with norm k, in any finite Galois extension of the rationals, we study sums of the form $\sum_{k \leqslant x} a_k^l \; (l = 2, 3, \ldots)$, along with the integral means of the 2ϱ-th power (ϱ real, $\varrho \geqslant 1$) of the absolute value of the corresponding Dedekind zeta-function. The two averages are related if $\varrho = n^{l-1}/2$, where n is the degree of the Galois extension.

§ 1. Let K be an algebraic number field of finite degree over the rationals $\mathbb{Q}$. If a_k denotes the number of integral ideals in K with norm k, then the Dedekind zeta-function ζ_K of the field K is defined by

$$\zeta_K(s) = \sum_{k=1}^{\infty} a_k k^{-s}, \quad s = \sigma + i t,$$

for $\sigma > 1$. The object of this note is the proof of the following

Theorem 1. *If K is a Galois extension of $\mathbb{Q}$ of degree $n > 1$, then for every $\varepsilon > 0$ and any integer $l \geqslant 2$, we have*

$$\sum_{k \leqslant x} a_k^l = x P_K(\log x) + O(x^{1-2n^{-l}+\varepsilon}), \quad \text{as } x \to \infty,$$

where P_K denotes a suitable polynomial of degree $n^{l-1} - 1$.

The case $l = 2$ of the above sum was first considered in [2], where it was shown that

$$\sum_{k \leqslant x} a_k^2 \sim c x (\log x)^{n-1}, \text{ as } x \to \infty,$$

for a suitable constant $c = c(K)$.

If $l = 2$, and K is a quadratic field, the theorem yields the error-term $O(x^{1/2+\varepsilon})$. If, in addition, $D = -4$, where D is the discriminant of K, then a_k denotes the number of integral solutions of $k = x^2 + y^2$,

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solutions which differ only in order or sign not being counted as distinct. In that case, S. RAMANUJAN [5] gave the formula with the error-term $O(x^{3/5+\varepsilon})$, and a proof of it was later published by B. M. WILSON [8]. It is classical, on the other hand, that

$$\sum_{k\leqslant x} a_k = c\,x + O(x^{1-2/(n+1)}).$$

The proof of Theorem 1 is based on an estimate (Lemma 2) of the mean-value of $|\zeta_K(s)|^{2\varrho}$, for any real $\varrho \geqslant 1$, in a half-plane that includes a part of the critical strip. Such an estimate is first obtained in the case in which ϱ is an integer by means of the approximate functional equation for ζ_K, and then proved in general with the help of a two-variable convexity theorem due to R. M. GABRIEL [3]. When combined with the well-known method of F. CARLSON [1], it yields an asymptotic result on the mean-value of $|\zeta_K(s)|^{2\varrho}$ in a suitable half-plane (Theorem 2).

§ **2.** The connexion between the sums considered in Theorem 1 and the Dedekind zeta-function is given by the following

Lemma 1. *Let l denote an integer $\geqslant 2$. If K is any Galois extension of $\mathbb{Q}$ of degree $n > 1$, and*

$$D_l(s) = \sum_{k=1}^{\infty} a_k^l\, k^{-s}, \quad \sigma > 1,$$

then

$$D_l(s) = \zeta_K^{n^{l-1}}(s)\, U_l(s),$$

where $U_l(s)$ denotes a Dirichlet series, which is absolutely convergent for $\sigma > \frac{1}{2}$.

Proof. This has been proved in the case $l = 2$ in [2, pp. 56—58], and the argument in the general case is not essentially different. We give it here only for the sake of completeness.

It is known that a_k is multiplicative, and $a_k \ll k^{\varepsilon}$, for every $\varepsilon > 0$ [2, Lemma 9]. Hence we have

$$D_l(s)\,\zeta_K^{-n^{l-1}}(s) = \prod_p U_{l,p}(s), \quad \sigma > 1,$$

where the product runs over all rational primes p, and

$$U_{l,p}(s) = \Big(\sum_{m=0}^{\infty} a_{p^m}^l\, p^{-ms}\Big) \Big/ \Big(\sum_{m=0}^{\infty} a_{p^m} p^{-ms}\Big)^{n^{l-1}}, \quad \sigma > 0.$$

It is plain that for every $\varepsilon > 0$,

$$U_{l,p}(s) = 1 + O(p^{\varepsilon - 2\sigma})$$

uniformly for $\sigma \geqslant \frac{1}{2}$ and all primes with $a_p = 0$ or $a_p = n$.

Thus if a_p takes no other values, except possibly for finitely many primes p, then the product $\prod_p U_{l,p}(s)$ converges absolutely for $\sigma > \frac{1}{2}$, and the lemma follows. To show that this is the case, let (p) denote the principal ideal in K generated by p, with the factorization

$$(p) = \mathfrak{P}_1^{e_1} \dots \mathfrak{P}_\nu^{e_\nu},$$

where $\mathfrak{P}_\varkappa$ are distinct prime ideals in K with norm $p^{f_\varkappa}$, $\varkappa = 1, 2, \dots, \nu$. Then the integers $e_\varkappa, f_\varkappa$ satisfy the relation

$$\sum_{\varkappa=1}^{\nu} e_\varkappa f_\varkappa = n.$$

Suppose now that p is unramified in K, so that $e_1 = e_2 = \dots = e_\nu = 1$. Since K is Galois, all $\mathfrak{P}_\varkappa$ are conjugate, so that $f_1 = f_2 \dots = f_\nu = f$, say, and the above relation yields $f\nu = n$, whence

$$a_{p^m} = \begin{cases} 0, & \text{if } 0 < m < f, \\ n/f, & \text{if } m = f. \end{cases}$$

Since the number of primes p ramified in K is finite, the lemma follows.

Lemma 2. *If K is any algebraic number field of degree $n > 1$, ζ_K the associated Dedekind zeta-function, ϱ any real number, and $\varrho \geqslant 1$, then*

$$\int_0^T |\zeta_K(\sigma + i t)|^{2\varrho} \, dt \ll T,$$

for $1 - 1/\varrho n < \sigma < 1$.

Proof. If D denotes the discriminant of K, with r_1 real and $2r_2$ imaginary conjugates, then $\zeta_K(s)$ satisfies the approximate functional equation (cf. [2, Equation (65)]) given by

$$\zeta_K(s) = \sum_{k \leqslant x} a_k k^{-s} + B^{2s-1} \frac{\Delta(1-s)}{\Delta(s)} \sum_{k \leqslant x} a_k k^{s-1} + \\ + O(|t|^{(n/2)(1-1/n-\sigma)} \log |t|), \tag{1}$$

for $0 \leqslant \sigma \leqslant 1$, where

$x = |D|^{1/2}(|t|/2\pi)^{n/2}$, $B = 2^{r_2}\pi^{n/2}|D|^{-1/2}$, and $\Delta(s) = \Gamma^{r_1}(s/2)\Gamma^{r_2}(s)$.

Let us first assume that ϱ is an integer, say $\varrho = j$, where $j = 1, 2, \ldots$. Then it follows from (1), by Stirling's formula and the inequality between the arithmetic and geometric means, that

$$\int_0^T |\zeta_K(\sigma + it)|^{2j}\, dt \ll I(T, \sigma) + T^{(1-2\sigma)nj} I(T, 1-\sigma) + 1, \qquad (2)$$

for $1 - 1/jn < \sigma < 1$, where

$$I(T, \sigma) = \int_0^T \Big|\sum_{k \leqslant x} a_k k^{-\sigma - it}\Big|^{2j} dt.$$

Now

$$I(T, \sigma) = \sum_{k_1, \ldots, k_j = 1}^{\infty} \frac{a_{k_1} a_{k_2} \ldots a_{k_{2j}}}{(k_1 \ldots k_{2j})^{\sigma}} \int_{T'}^{T} \left(\frac{k_1 \ldots k_j}{k_{j+1} \ldots k_{2j}}\right)^{it} dt, \qquad (3)$$

where

$$T' = \min\big(T, \max_{1 \leqslant \nu \leqslant 2j} 2\pi (k_\nu/|D|^{1/2})^{2/n}\big),$$

so that the integral in (3) vanishes, unless $k = k_1 \ldots k_j \ll T^{nj/2}$ and $l = k_{j+1} \ldots k_{2j} \ll T^{nj/2}$. Further, if $k > l$, it is of the order

$$\ll \frac{1}{\log(k/l)} = \frac{1}{\log(1 + (k-l)/l)} < \frac{k}{k-l} \leqslant 1 + \frac{(kl)^{1/2}}{k-l},$$

while $a_k \ll k^{\varepsilon}$, for every $\varepsilon > 0$, by [2, Lemma 9]. Considering separately the sums which correspond to $k = l$ and $k \neq l$, we obtain

$$I(T, \sigma) \ll T \sum_{k \ll T^{nj/2}} k^{\varepsilon - 2\sigma} + \sum_{\substack{k, l \ll T^{nj/2} \\ k \neq l}} (kl)^{(\varepsilon/2) - \sigma}\left(1 + \frac{(kl)^{1/2}}{|k-l|}\right).$$

The first sum on the right-hand side is

$$\ll \begin{cases} T, & \text{if } \sigma > \tfrac{1}{2}, \\ T^{1 + (nj/2)(1 + \varepsilon - 2\sigma)}, & \text{if } \sigma < \tfrac{1}{2}, \end{cases}$$

for a small enough $\varepsilon > 0$, while the second sum gives

$$T^{nj(1+\varepsilon-\sigma)} + \sum_{0 < m \ll T^{nj/2}} (1/m) \sum_{0 < k \ll T^{nj/2}} (k(k+m))^{(1/2) + (\varepsilon/2) - \sigma} \ll$$

$$\ll T^{nj(1+\varepsilon-\sigma)}, \quad \text{if } 0 < \sigma < 1.$$

Hence we obtain

$$I(T,\sigma) \ll \begin{cases} T + T^{nj(1+\varepsilon-\sigma)}, & \text{if } \frac{1}{2} < \sigma < 1, \\ T^{nj(1+\varepsilon-\sigma)}, & \text{if } \sigma < \frac{1}{2}. \end{cases}$$

It follows that

$$I(T,\sigma) \ll T, \quad \text{if } \sigma > 1 - 1/jn. \tag{4}$$

Similarly we obtain

$$T^{(1-2\sigma)nj} I(T, 1-\sigma) \ll T^{nj(1+\varepsilon-\sigma)}, \quad \text{if } \tfrac{1}{2} < \sigma < 1,$$

from which we have

$$T^{(1-2\sigma)nj} I(T, 1-\sigma) \ll T, \quad \text{if } \sigma > 1 - 1/jn. \tag{5}$$

The lemma now follows from (2)—(5), if ϱ is a positive integer. If it is not, we use a two-variable convexity theorem due to R. M. GABRIEL [3], which implies that for $\alpha < \sigma < \beta < 1$, we have

$$\left(\int_0^T |\zeta_K(\sigma + i t)|^{1/(q\lambda+q'\mu)} dt\right)^{q\lambda+q'\mu} \ll \tag{6}$$
$$\ll \left(\int_0^T |\zeta_K(\alpha + i t)|^{1/\lambda} dt\right)^{q\lambda} \cdot \left(\int_0^T |\zeta_K(\beta + i t)|^{1/\mu} dt\right)^{q'\mu},$$

where $\lambda > 0$, $\mu > 0$, and

$$q = \frac{\beta - \sigma}{\beta - \alpha}, \quad q' = \frac{\sigma - \alpha}{\beta - \alpha}.$$

If ϱ is not an integer, so that $\varrho > 1$, let j denote the positive integer which satisfies the condition: $j < \varrho < j + 1$, so that $j \geqslant 1$.

We shall apply the convexity theorem with

$$\lambda = \frac{1}{2j}, \ \mu = \frac{1}{2(j+1)}; \ \alpha = \sigma + \frac{1}{\varrho n} - \frac{1}{jn}; \ \beta = \sigma + \frac{1}{\varrho n} - \frac{1}{(j+1)n}, \tag{7}$$

so that $\alpha < \sigma < \beta$, since $j < \varrho < j + 1$. Further we have

$$\beta - \alpha = \frac{1}{nj(j+1)}, \quad \beta - \sigma = \frac{1}{\varrho n} - \frac{1}{(j+1)n}, \quad \sigma - \alpha = \frac{1}{jn} - \frac{1}{\varrho n},$$

so that

$$q = \frac{\beta - \sigma}{\beta - \alpha} = \frac{j(j+1) - j\varrho}{\varrho}, \quad q' = \frac{\sigma - \alpha}{\beta - \alpha} = \frac{\sigma(j+1) - j(j+1)}{\varrho} \tag{8}$$

and

$$q\lambda + q'\mu = \frac{(j+1) - \varrho}{2\varrho} + \frac{\varrho - j}{2\varrho} = \frac{1}{2\varrho}.$$

The convexity theorem as stated in (6) now yields the inequality

$$\int_0^T |\zeta_K(\sigma+it)|^{2\varrho}\,dt \ll \Big(\int_0^T |\zeta_K(\alpha+it)|^{2j}\,dt\Big)^{\lambda'} \cdot \Big(\int_0^T |\zeta_K(\beta+it)|^{2(j+1)}\,dt\Big)^{\mu'}, \quad (9)$$

where $\lambda' = 2\varrho\lambda q$, $\mu' = 2\varrho\mu q'$, and $\lambda' + \mu' = 1$.

If σ lies in the range

$$1 - \frac{1}{\varrho n} < \sigma < 1 - \frac{1}{\varrho n} + \frac{1}{(j+1)n}, \quad (10)$$

then we have $0 < \alpha < \sigma < \beta < 1$ on the one hand, and

$$\alpha > 1 - \frac{1}{jn}, \quad \beta > 1 - \frac{1}{(j+1)n}$$

on the other. The lemma now follows from (9) and its already proved validity in the case in which ϱ is an integer $\geqslant 1$, provided that σ lies in the range given by (10). If, however, σ lies in the range

$$1 - \frac{1}{\varrho n} + \frac{1}{(j+1)n} \leqslant \sigma < 1, \quad (11)$$

then $\sigma > 1 - 1/(j+1)n$, since $2\varrho > 2j \geqslant j+1$, and by Hölder's inequality, together with the first part of the proof, we obtain

$$\int_0^T |\zeta_K(\sigma+it)|^{2\varrho}\,dt \ll \Big(\int_0^T |\zeta_K(\sigma+it)|^{2(j+1)}\,dt\Big)^{\varrho/(j+1)} \cdot T^{1-\varrho/(j+1)}$$

which completes the proof of the lemma.

§ 3. To prove Theorem 1, we introduce an auxiliary C^∞ function φ_u on $(0, \infty)$, for $u \geqslant 2$, as follows:

$$\varphi_u(y) = \begin{cases} 1, & \text{for } 0 < y \leqslant 1, \\ 0, & \text{for } y \geqslant 1 + 1/u; \end{cases}$$

its derivatives satisfy the condition

$$\varphi_u^{(r)}(y) \ll u^r, \quad r = 0, 1, 2, \ldots,$$

where the implicit constants depend only on r.

If we consider the Mellin transform

$$M_u(s) = \int_0^\infty \varphi_u(y)\, y^{s-1}\,dy, \quad \sigma > 0,$$

then

$$M_u^{(r)}(1) = \int_0^1 (\log y)^r \, dy + O\Big(\int_1^{1+1/u} (\log y)^r \, dy\Big)$$
$$= (-1)^r \Gamma(r+1) + O(u^{-r-1}), \text{ as } u \to \infty, \tag{12}$$

for $r = 0, 1, 2, \ldots$. On repeated integration by parts, we also have, for $r = 1, 2, \ldots$,

$$M_u(s) = \frac{(-1)^r}{s(s+1)\ldots(s+r-1)} \int_0^\infty \varphi_u^{(r)}(y) \, y^{s+r-1} \, dy \ll \frac{1}{|s|}\left(\frac{u}{|s|}\right)^r, \tag{13}$$

uniformly for $\frac{1}{2} \leqslant \sigma \leqslant 2$, and $u \geqslant 2$.

By Mellin's inversion formula, and Lemma 1, we have

$$\sum_{k=1}^{\infty} a_k^l \, \varphi_u(k/x) = (1/2\pi i) \int_{(2)} D_l(s) \, M_u(s) \, x^s \, ds = \tag{14}$$
$$= (1/2\pi i) \int_{(2)} \zeta_K^{n^{l-1}}(s) \, U_l(s) \, M_u(s) \, x^s \, ds,$$

where $\int_{(\sigma_0)}$ denotes integration along the line $\sigma = \sigma_0$ in the direction of increasing imaginary part. By the definition of φ_u, and since $a_k^l \ll k^\varepsilon$, we have

$$\sum_{k=1}^{\infty} a_k^l \, \varphi_u(k/x) = \sum_{k \leqslant x} a_k^l + \sum_{x < k < x(1+1/u)} a_k^l \, \varphi_u(k/x) = \sum_{k \leqslant x} a_k^l + O(x^{1+\varepsilon}/u). \tag{15}$$

The integrand in (14) is regular for $\sigma > \frac{1}{2}$ except for a pole of order n^{l-1} at $s = 1$. If we denote its residue by Res, we deduce from (12) that, uniformly for $x \geqslant 1$ and $u \geqslant 2$, we have

$$\text{Res} = x \, P_K(\log x) + O((x/u)(\log x)^{n^{l-1}-1}), \tag{16}$$

where P_K is a polynomial of degree $n^{l-1} - 1$, whose coefficients do not depend on φ_u. Hence by Cauchy's theorem, together with (13) and the estimate

$$|\zeta_K(\sigma + i t)| \ll |t|^{(n(1-\sigma)/2)+\varepsilon}$$

for $0 \leqslant \sigma \leqslant 1$, we obtain from (14)

$$\sum_{k=1}^{\infty} a_k^l \, \varphi_u(k/x) = \text{Res} + (1/2\pi i) \int_{(\sigma_0)} \zeta_K^{n^{l-1}}(s) \, U_l(s) \cdot M_u(s) \cdot x^s \, ds, \tag{17}$$

where σ_0 is such that

$$1 > \sigma_0 = 1 - 2/n^l + \delta > 1 - 2/n^l.$$

Now, for $\sigma = \sigma_0$, we have $U_l(s) = O(1)$, and

$$|M_u(s)| \ll (|s|^{-1}), \quad \text{if } |t| < u,$$

while

$$|M_u(s)| \ll |s|^{-1}(u|s|^{-1}), \quad \text{if } |t| \geq u,$$

because of (13). Hence

$$\int_{(\sigma_0)} |\zeta_K^{n^l-1}(\sigma_0 + it) \cdot U_l(s) \cdot M_u(s) \cdot x^s|\, ds \ll$$

$$\ll \left(\int_{|t|<u} |\zeta_K^{n^l-1}(\sigma_0 + it)| \cdot \frac{dt}{1+|t|} + \int_{|t|\geq u} |\zeta_K^{n^l-1}(\sigma_0 + it)| \cdot \frac{u}{t^2}\, dt \right)$$

$$\ll x^{\sigma_0} \cdot \log u, \tag{18}$$

if we integrate the last two integrals by parts, and use Lemma 2. Thus (17) and (16) yield the relation

$$\sum a_k^l \varphi_u(k/x) - x P_K(\log x) \ll (x/u)(\log x)^{n^l-1-1} + x^{\sigma_0}(\log u).$$

If we combine this with (15), and choose $u = x^{1-\sigma_0}$, we obtain Theorem 1.

Remark. If $n = 2, l = 2$, Theorem 1 follows from the known mean-value Theorem [2]

$$(1/T)\int_0^T |\zeta_K(\sigma + it)|^2\, dt = O(1), \quad \text{for} \quad \sigma > \tfrac{1}{2}.$$

§ **4.** If K is *any* algebraic number field of degree $n > 1$, and j is any positive integer, we have

$$\zeta_K^j(s) = \sum_{k=1}^{\infty} a_j(k) k^{-s} = \prod_{\mathfrak{P}} (1 - N\mathfrak{P}^{-s})^{-j}, \quad \text{for} \quad \sigma > 1, \tag{19}$$

where $a_j(k) = \sum_{k_1 k_2 \ldots k_j = k} a_{k_1} a_{k_2} \ldots a_{k_j}$, and the product runs over all prime ideals $\mathfrak{P}$ in K. Here $N\mathfrak{P}$ denotes the norm of $\mathfrak{P}$.

By Lemma 2 we have

$$\int_1^T |\zeta_K(\sigma + it)|^{2j}\, dt \ll T, \quad \text{for } 1 - 1/jn < \sigma < 1, \tag{20}$$

and because of the absolute convergence of $\sum_{k=1}^{\infty} a_k k^{-s}$ for $\sigma > 1$, this holds also for $\sigma > 1$. By a theorem of F. CARLSON [1, (a)] on general Dirichlet series, it follows that (20) holds for $\sigma > 1 - 1/jn$.

Since $\{\zeta_K(s)\}^j$ is regular except for a pole at $s = 1$, and is of finite order in t, by another theorem of CARLSON [1, (b)] we have

$$\lim_{T\to\infty} (1/T) \int_1^T |\zeta_K(\sigma + i t)|^{2j} dt = \sum_{k=1}^{\infty} a_j^2(k) \cdot k^{-2\sigma}, \quad \sigma > 1 - 1/jn. \tag{21}$$

This result can, in fact, be upheld for any real $\varrho \geqslant 1$ in place of the integer j.

If ϱ is any real number, with $\varrho > 0$, then define $\zeta_K^{\varrho}(s) = \exp\{\varrho \log \zeta_K(s)\}$, where $\log \zeta_K(s)$ is uniquely defined by the requirement

$$\log(1 - N\mathfrak{P}^{-s})^{-1} = \sum_{k=1}^{\infty} (1/k(N\mathfrak{P})^{ks}), \quad \sigma > 1,$$

so that

$$\log \zeta_K(s) = \sum_{\mathfrak{P}} \sum_{k=1}^{\infty} (1/k(N\mathfrak{P})^{ks}),$$

the double series converging absolutely for $\sigma > 1$. Let

$$\zeta_K^{\varrho}(s) = \sum_{k=1}^{\infty} a_{\varrho}(k) k^{-s}, \quad \sigma > 1, \tag{22}$$

so that when ϱ is a positive integer j, we have (19).

Let

$$\Pi_M(s) = \prod_{N\mathfrak{P}<M} (1 - N\mathfrak{P}^{-s})^{-1},$$

where M is an integer, $M > 0$. Then for any real $\varrho > 0$, we have

$$\{\Pi_M(s)\}^{\varrho} = \prod_{N\mathfrak{P}<M} (1 - N\mathfrak{P}^{-s})^{-\varrho} = \sum_{k=1}^{\infty} a'_{\varrho}(k) k^{-s}, \tag{23}$$

say, the series converging *absolutely* for $\sigma > 0$, with $a'_{\varrho}(k) = a_{\varrho}(k)$ for $1 \leqslant k < M$, and $0 \leqslant a'_{\varrho}(k) \leqslant a_{\varrho}(k)$ for all $k \geqslant 1$. Hence

$$\lim_{T\to\infty} (1/T) \int_1^T |\Pi_M(\sigma + i t)|^{2\varrho} dt = \sum_{k=1}^{\infty} \{a'_{\varrho}(k)\}^2 k^{-2\sigma}, \quad \text{for} \quad \sigma > 0,$$

and

$$\lim_{M\to\infty} \lim_{T\to\infty} (1/T) \int_1^T |\Pi_M(\sigma + i t)|^{2\varrho} dt = \sum_{k=1}^{\infty} \{a_{\varrho}(k)\}^2 k^{-2\sigma}, \quad \text{for} \quad \sigma > \tfrac{1}{2}. \tag{24}$$

If ϱ is real, $\varrho \geqslant 1$, then, as in Lemma 2, we have

$$(1/T)\int_1^T |\zeta_K(\sigma+it) - \Pi_M(\sigma+it)|^{2\varrho}\,dt \ll$$

$$\ll \Big((1/T)\int_1^T |\zeta_K(\alpha+it) - \Pi_M(\alpha+it)|^{2j}\,dt\Big)^{\lambda'} \times \tag{25}$$

$$\times \Big((1/T)\int_1^T |\zeta_K(\beta+it) - \Pi_M(\beta+it)|^{2(j+1)}\,dt\Big)^{\mu'},$$

where $\sigma, \alpha, \beta, \lambda, \lambda', \mu, \mu'$ are as before, $\lambda' + \mu' = 1$, and $\sigma > 1 - 1/\varrho n$. The function $\{\zeta_K(s) - \Pi_M(s)\}^j$ is regular except for a pole at $s = 1$, and is of finite order in t. Further

$$(1/T)\int_1^T |\zeta_K(\alpha+it) - \Pi_M(\alpha+it)|^{2j}\,dt = O(1), \quad \sigma > 1 - 1/jn, \tag{26}$$

because of Lemma 2, and (24). Hence, by CARLSON's Theorem [1, (b)],

$$\lim_{T\to\infty} (1/T)\int_1^T |\zeta_K(\alpha+it) - \Pi_M(\alpha+it)|^{2j}\,dt = \sum_{k=1}^{\infty} a_{j,M}^2(k)\cdot k^{-2\alpha},$$

say, the series converging absolutely since $\alpha > 1 - 1/jn \geq \frac{1}{2}$. Further $a_{j,M}(k) = 0$ for $k < M$, and $0 \leq a_{j,M}(k) \leq a_j(k)$ for all $k \geq 1$. Hence

$$\lim_{M\to\infty} \lim_{T\to\infty} (1/T)\int_1^T |\zeta_K(\alpha+it) - \Pi_M(\alpha+it)|^{2j}\,dt = 0.$$

Similarly

$$\lim_{M\to\infty} \lim_{T\to\infty} (1/T)\int_1^T |\zeta_K(\beta+it) - \Pi_M(\beta+it)|^{2(j+1)}\,dt = 0,$$

since $\beta > 1 - 1/(j+1)n$. It follows that

$$\lim_{M\to\infty} \lim_{T\to\infty} (1/T)\int_1^T |\zeta_K(\sigma+it) - \Pi_M(\sigma+it)|^{2\varrho}\,dt = 0, \quad \text{for} \quad \sigma > 1 - 1/\varrho n. \tag{27}$$

Since

$$\Big(\int_1^T |\zeta_K(\sigma+it)|^{2\varrho}\,dt\Big)^{1/2\varrho} \leq \Big(\int_1^T |\Pi_M(\sigma+it)|^{2\varrho}\,dt\Big)^{1/2\varrho} +$$

$$+ \Big(\int_1^T |\zeta_K(\sigma+it) - \Pi_M(\sigma+it)|^{2\varrho}\Big)^{1/2\varrho},$$

and

$$\left(\int_1^T |\Pi_M(\sigma + i\,t)|^{2\varrho}\,dt\right)^{1/2\varrho} \leqslant \left(\int_1^T |\zeta_K(\sigma + i\,t)|^{2\varrho}\,dt\right)^{1/2\varrho} + \\ + \left(\int_1^T |\zeta_K(\sigma + i\,t) - \Pi_M(\sigma + i\,t)|^{2\varrho}\,dt\right)^{1/2\varrho},$$

we obtain from (24) and (27) the following

Theorem 2. *If ϱ is any real number, $\varrho \geqslant 1$, then*

$$\lim_{T\to\infty} (1/T) \int_1^T |\zeta_K(\sigma + i\,t)|^{2\varrho}\,dt = \sum_{k=1}^{\infty} \{a_\varrho(k)\}^2 k^{-2\sigma}, \quad \sigma > 1 - 1/\varrho\, n,$$

where K is any algebraic number field of degree $n > 1$, and ζ_K the associated Dedekind zeta-function.

We may remark, in conclusion, that in the case of the Riemann zeta-function $\zeta(s)$, stronger results than Lemma 2 are known, which yield in most cases sharper asymptotic estimates for sums of the form $\sum_{k\leqslant x} d_m^l(k)$, where $d_m(k)$ is the number of ways of expressing k as a product of m factors, and l is any integer $\geqslant 2$. See [6, §§ 7.9, 7.19, and Ch. XII] and [4].

Mention may be made also of some related results announced in [7].

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