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Quasiparticle states in disordered superfluids

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A model for disordered superfluids and superconductors is considered in terms of the Bogoliubov-de Gennes equation with a random order parameter field. Two characteristic cases are distinguished: model I with a real order parameter (time reversal invariant system) and model II with a complex order parameter (broken time reversal invariance). The fluctuations of the order parameter close the gap in both models, and we investigate the states at the center of the filled gap. The two models have distinctive properties in terms of the quasiparticle states due to different symmetries. Model II exhibits only localized quasiparticle states at the band center. In contrast, the fluctuations of the real order parameter of model I can be described by a nonlinear sigma model which leads to a transition from localized to extended states for dimensions $d>2$.

Noninteracting quasiparticles in a superfluid or a superconductor can be described in a first order approximation by the Bogoliubov-de Gennes (BG) equation [1–3]. This is an equation of motion for the two-component quasiparticle field $\Psi(r,t)$:

$$-i\frac{\partial}{\partial t}\Psi=[(-\nabla^2+\mu)\sigma_3+\Delta_1\sigma_1-\Delta_2\sigma_2]\,\Psi. \tag{1}$$

$\{\sigma_j\}$ are the Pauli matrices and Δ_1 (Δ_2) the real and imaginary part of the order parameter Δ, respectively. Starting from a microscopic theory of the superfluid the order parameter can be evaluated by solving a self-consistent (BCS) equation [1]. Since this equation is nonlinear, solutions are only known for homogeneous systems where Δ is uniform or periodic in space. In inhomogeneous systems, however, it might be very difficult to find solutions even in finite systems where we can apply numerical methods [4]. In general, if there are random terms (e.g. random potentials [1, 2] or random couplings) in the microscopic theory, the order parameter will also be random in space. Although its distribution will be determined by the distribution of the microscopic randomness through the BCS equation, we can apply the universality hypothesis that qualitative properties will not be affected by the details of the specific distribution. On the other hand, it might be very difficult to determine the microscopic randomness from an experiment with real materials, whereas it is easy to measure the disorder of the order parameter. Therefore, it seems to be more physical to start with an effective model which accounts for the observable disorder of the order parameter field Δ.

Starting from the universality hypothesis we can evaluate qualitative properties as the gap structure or the localization properties of the quasiparticle states in the average system using a specific distribution for the order parameter. This approach applies to a number of physical situations. For instance, there are inhomogeneities in superfluid ^{3}He due to nucleation of ^{3}He-A in the ^{3}He-B-phase. Another example is ^{4}He with a disordered vortex structure [5] or high T_c superconductors in a magnetic field with a frozen inhomogeneous vortex structure [6], where the order parameter vanishes inside the vortex core. These systems have in common that the order parameter has a very short coherence length such that it can fluctuate on short scales. From this point of view the BG equation is a phenomenological approach to a number of inhomogeneous superfluid systems.

It is convenient to define the model, which is given by the BG Eq. (1), on a hypercubic lattice Λ. The lattice constant is of the order of the coherence length of the order parameter. Then ∇^2 in (1) is the lattice Laplacian. The BG equation has been studied extensively in the literature for a homogeneous order parameter field Δ and a random potential $\mu(r)$ [1, 2] or for a deterministic space dependent $\Delta(r)$ in the case of normal-superconducting interfaces [2, 7]. Numerical simulations have been performed recently [8, 9] in the case of a random $\Delta(r)$ in a homogeneous potential μ. Two different models were distinguished: one with a real order parameter field and another one where the order parameter is complex with a random phase. It was found in the simulations that the quasiparticles are localized in one dimension for both models [8]. The situation is different in two dimensions [9], however, where it turned out that only the model with real Δ leads to localization but not the random phase model. Therefore, one suspects that the models belong to different universality classes which exhibit qualitatively different properties. The purpose of this article is to discuss the different effects of a real and a complex order parameter in terms of symmetries and symmetry breaking, and the consequences for the existence of localized or extended states. In contrast to the simulations of Refs. 8, 9 we will not study the case of a random phase but randomly independent real and imaginary parts of the order parameter (model II). This situation could be realized in systems where the time reversal symmetry is broken by magnetic disorder (e.g. in the vortex phase of superconductors) or in vortex systems of ^{4}He, since the phase of the order parameter is changing by a multiple of 2π if one goes around a vortex. In contrast to model II we will also consider a model where the phase fluctuations of the order parameter are neglected (model I). In this case the remaining global phase can be gauged away. The independent fluctuations of the real and imaginary part in model II will lead to new results which deviate from the random phase model.

As it is well known [8, 9] model I is invariant under time reversal transformation whereas model II is not. This can easily be seen in the BG equation: if Ψ is a solution of (1) then $\Psi'=i\sigma_2\Psi$ is a solution of the time reversed equation:

$$i\frac{\partial}{\partial t}\Psi'=[(-\nabla^2+\mu)\,\sigma_3+\Delta_1\,\sigma_1+\Delta_2\,\sigma_2]\,\Psi'. \tag{2}$$

We notice that the order parameter field transforms under time reversion as $\Delta\to\Delta^*$. On the other hand, there is a global gauge transformation $\Psi\to U\Psi$ which transforms the phase of the order parameter Δ. The unitary transformation U can be written as $U=\alpha_0\sigma_0+\alpha_3\sigma_3$ with $\alpha_0=[\exp(i\varphi_1)+\exp(i\varphi_2)]/2$ and $\alpha_3=[\exp(i\varphi_1)-\exp(i\varphi_2)]/2$. Then Δ transforms as $\Delta\to\exp[i(\varphi_1-\varphi_2)]\,\Delta$. This phase transformation means in terms of model II that we apply a global transformation inside the random ensemble or, in other words, only the random ensemble of model II is invariant under this global gauge transformation.

Let us now consider the Green's function which corresponds to (1). This is a $2|\Lambda|\times 2|\Lambda|$ matrix ($|\Lambda|$ is the volume of the lattice Λ)

$$\mathbf{G}(\xi_m)=[(-\nabla^2+\mu)\,\sigma_s+\Delta_1\,\sigma_1-\Delta_2\,\sigma_2+i\xi_m\,\sigma_0]^{-1}, \tag{3}$$

where we have obtained the Matsubara frequency ξ_m after a Fourier transformation with respect to t. This Green's function can also be considered as a result of a BCS approximation of a many-particle system subject to Cooper pairing [1]. The order parameter of the latter is indeed Δ. The Matsubara frequency can then be identified with the inverse temperature β of the thermodynamical system [1] as $\xi_m=\pi m/\beta$.

The time reversal transformation reads in terms of the Green's function

$$s\sigma_2\,\mathbf{G}(r,r';\xi_m)\,s\sigma_2=[\mathbf{G}(r,r';\xi_m^*)]^* \qquad (s=i). \tag{4}$$

ξ_m is here regarded as a complex parameter. In particular, (4) implies for a real order parameter (i.e. $\Delta_2=0$)

$$s\sigma_2\,\mathbf{G}(r,r';\xi_m)\,s\sigma_2=\mathbf{G}(r,r';-\xi_m). \tag{5}$$

Thus, only the Matsubara frequency changes the sign under this transformation. (4) implies that the diagonal elements of $\mathbf{G}(r\ r';\ \xi_m)$ are complex conjugate to each other (with a minus sign); i.e.,

$$|\mathbf{G}_{ii}(r,r';\xi_m)|^2=-\mathbf{G}_{ii}(r,r';\xi_m)\,\mathbf{G}_{jj}(r,r';\xi_m^*) \qquad (i\neq j). \tag{6}$$

The expression on the l.h.s. can be used as a localization criterion: a quasiparticle state is localized with a localization length ξ if this quantity decays exponentially on the length scale ξ.

Up to here the time reversal transformation was only a discrete transformation. For the real order parameter, however, one can generalize this transformation for the Green's function to a *continuous* transformation $T=c\sigma_0+s\sigma_2$ as

$$T^{-1}\,\mathbf{G}(r,r';\xi_m\,\sigma_0)\,T^{-1}=\mathbf{G}(r,r';\xi_m\,T^2) \tag{7}$$

with the constraint $c^2-s^2=1$. This is apparently a continuous generalization of (5) where the Matsubara frequency plays the role of a symmetry breaking field as before. Therefore, both models are subject to continuous symmetries. However, there is a fundamental difference between these symmetries: the symmetry transformation T of model I is non-compact (due to the hyperbolic constraint $c^2-s^2=1$) whereas the unitary symmetry transformation of model II is compact. This situation can be compared with that of a single particle in a random potential [10]. Then the order parameter Δ vanishes and (1) describes the dynamics of the particle and the conjugate hole in a random potential μ. The continuous generalization of the time reversal transformation is $c\sigma_0+s_1\,\sigma_1+s_2\,\sigma_2$ with the constraint $c^2-s_1^2-s_2^2=1$ instead of $c\sigma_0+s_1\,\sigma_1$ in (7). Due to this analogy we expect similar properties for model I as we found for the particle in a random potential. For instance, there are only localized states in $d\leq 2$ dimension, a result which was indeed found in the simulation [9]. The compact symmetry of model II, on the other hand, might be responsible for a qualitatively different behavior. This will be discussed in the following for model I and II by means of two effective models which take the symmetries into account.

The first characteristic quantity of interest is the average density of states (DOS). For both models the DOS can be obtained by an analytic continuation $\xi_m\to -iE+\varepsilon$. The DOS then reads

$$\rho_\Lambda(E)=-\frac{1}{\pi|\Lambda|}\lim_{\varepsilon\downarrow 0}\operatorname{Im}\sum_{r\in\Lambda}\mathbf{G}_{11}(r,r;-iE+\varepsilon). \tag{8}$$

A consequence of property (4) is a symmetric DOS

$$\rho_\Lambda(E)=\rho_\Lambda(-E). \tag{9}$$

In a pure system (i.e. Δ uniform) there is gap of width $2|\Delta|$. This gap is affected by disorder (i.e. if Δ is random), since the bands are broadened and, therefore, the gap is reduced or even completely filled for an appropriate distribution of Δ. In general, the fluctuations of Δ are not restricted (e.g. they are Gaussian) and we expect realizations of the order parameter field with no gap [11]. This raises the question if the weight of gapless realizations of $\Delta(r)$ is such that $\langle\rho(E=0)\rangle_\Delta>0$. Indeed, the latter was found in the case of a real Δ for a sufficiently broad distribution [12] (i.e. for any Gaussian). A positive DOS at $E=0$ is related to a breaking of the symmetry in (4) or (7) because the imaginary part of $\mathbf{G}$ in the limit $\varepsilon\to 0$ is an order parameter for the symmetry similar to the magnetization in a ferromagnet. On the other hand, $\xi_m(=iE-\varepsilon)$ is the symmetry breaking field similar to the external magnetic field in the ferromagnet. If the imaginary part of $\mathbf{G}$ is discontinuous at $\xi_m=0$ (i.e. there is a step) the symmetry under the transformation (4) or (7) is spontaneously broken. Thus, the symmetry is spontaneously broken if the gap is filled in the average system due to the DOS in (8). Now we shall discuss the possibility of spontaneous symmetry breaking and its consequences. To this end we consider the matrix element $\mathbf{G}_{ij}(r,r;\xi_m)$ of the Green's function with random distributed $\Delta(r)$. Instead of evaluating the average of the matrix element $\langle\mathbf{G}(r,r;\xi_m)\rangle_\Delta$ directly, we introduce a random matrix field Q_r such that this expectation value can be expressed by the expectation value of the matrix field:

$$\langle\mathbf{G}_{ij}(r,r;\xi_m)\rangle_{1\Delta}\propto\langle Q_r^{ij}\rangle_Q. \tag{10a}$$

Furthermore, a correlation function of matrix elements of the Green's function can also be expressed by a corresponding correlation function of the matrix field:

$$\langle\mathbf{G}_{ii}(r,r';\xi_m)\,\mathbf{G}_{jj}(r,r';\xi_m)\rangle_\Delta\propto\langle Q_r^{ij}\,Q_{r'}^{ji}\rangle_Q. \tag{10b}$$

The correspondence between the distribution of the Green's function and the distribution of the matrix field Q_r has been discussed in Ref. 10 for a particle in a random potential by means of the replica trick. Later it was suggested independently in Ref. 13 and Ref. 15 that this correspondence can also be constructed using a combination of complex and Grassmann matrix elements for the field Q_r (sometimes incorrectly called "supersymmetric approach"). Using the second method for model I with Gaussian distributed Δ_1, the correspondence can be expressed formally [13–15] by relating the 2×2 matrix $\mathbf{G}(r,r;\varepsilon)$ with a 4×4 matrix as

$$\mathbf{G}(r,r;\varepsilon)\leftrightarrow\mathbf{Q}_r=\begin{pmatrix}Q_r & \Theta_r\\ \bar{\Theta}_r & iP_r\end{pmatrix}, \tag{11}$$

where the diagonal elements are complex fields and the off-diagonal elements are elements of a Grassmann algebra. The extension to the 4×4 matrix takes care of the fact that the representation of the "distribution density" $\exp(-S_1)$ in

$$\langle\ldots\rangle_Q=\int\ldots\exp(-S_1)\prod_{r\in\Delta}d\mathbf{Q}_r \tag{12}$$

is normalized; i.e.,

$$\int\exp(-S_1)\prod d\mathbf{Q}_r=1. \tag{13}$$

The integration goes over a Hermitean graded matrix field [16]. An appropriate distribution can also be constructed for model II, where we need two independent Hermitean matrix fields $\mathbf{Q}_1$, $\mathbf{Q}_2$ which are related to the two independent components Δ_1, Δ_2 of the complex order parameter field. Details of the construction method can be found in the literature. The notation follows that of Ref. 17, where also paths of integration for the matrix field are discussed.

With $\langle\Delta_j\rangle=0$ and $\langle\Delta_1^2\rangle=\langle\Delta_2^2\rangle=4g/v$ the action S_v is found for $v=1,2$ (i.e., model I and model II, respectively) as

$$\begin{aligned}S_v=&\frac{v}{2g}\sum_r\mathrm{Tr}\,g_4(\mathbf{Q}_{1,r}^2+(v-1)\,\mathbf{Q}_{2,r}^2)\\&+\log\det g_{4\Delta}(\mathbf{H}_0+i\varepsilon\gamma_0-2\gamma_1^{1/2}\mathbf{Q}_1\,\gamma_1^{1/2}\\&-2(v-1)\,\gamma_2^{1/2}\,\mathbf{Q}_2\,\gamma_2^{1/2})\end{aligned} \tag{14}$$

with

$$\mathbf{H}_0=\begin{pmatrix}(-\nabla^2+\mu)\,\sigma_3 & 0\\ 0 & (-\nabla^2+\mu)\,\sigma_3\end{pmatrix} \tag{15}$$

and

$$\gamma_j=\begin{pmatrix}\sigma_j & 0\\ 0 & \sigma_j\end{pmatrix}. \tag{16}$$

detg and Trg are the determinant and the trace operators with respect to the 4×4 graded matrix field $\mathbf{Q}_{j,r}$, where the index refers to the size of the matrix [13–16]. With this notation, the Q-field scales with $1/g$ in the expectation values of (10).

The integration over $Q_{j,r}$ and $P_{j,r}$ can be performed in saddle point (s.p.) approximation, whereas the Grassmann elements Θ are treated as fluctuations. The s.p. equations read

$$\delta\Big[\frac{v}{2g}\mathrm{Tr}(Q_1^2+(v-1)\,Q_2^2)+\log\det(H_0+i\varepsilon\sigma_0-2\sigma_1^{1/2}\,Q_1\,\sigma_1^{1/2}-2(v-1)\,\sigma_2^{1/2}\,Q_2\,\sigma_2^{1/2})\Big]=0 \tag{17a}$$

$$\delta\Big[-\frac{v}{2g}\mathrm{Tr}(P_1^2+(v-1)\,P_2^2)+\log\det(H_0+i\varepsilon\sigma_0-2\sigma_1^{1/2}\,iP_1\,\sigma_1^{1/2}-2i(v-1)\,\sigma_2^{1/2}\,P_2\,\sigma_2^{1/2})\Big]=0. \tag{17b}$$

These equations are identical if we set $P_j=-iQ_j$. From (17) we obtain for the uniform s.p. solution

$$\sigma_1^{1/2}\,Q_1\,\sigma_1^{1/2}+(v-1)\,\sigma_2^{1/2}\,Q_2\,\sigma_2^{1/2}=\begin{pmatrix}q_{11} & 0\\ 0 & q_{22}\end{pmatrix} \tag{18a}$$

and the vanishing commutator for $v=2$

$$[\sigma_3,(\sigma_1^{1/2}\,Q_1\,\sigma_1^{1/2}-\sigma_2^{1/2}\,Q_2\,\sigma_2^{1/2})]=0 \tag{18b}$$

with the s.p. conditions

$$q_{11}=\frac{2g}{(2\pi)^d}\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}\frac{1}{2q_{11}^*-\mu+i\varepsilon+h(k)}\,\mathrm{d}k_1\dots\mathrm{d}k_d \tag{19}$$

$$q_{22}=-q_{11}^* \tag{20}$$

and $\mathrm{sign}(\mathrm{Im}\,q_{11})=-\mathrm{sign}\,\varepsilon$. $h(k)$ is the Fourier transform of the d-dimensional lattice Laplacian ∇^2

$$h(k)=-\frac{1}{d}\sum_{j=1}^{d}\cos(k_j)\qquad(-\pi\leqq k_j<\pi). \tag{21}$$

A s.p. solution with $q_1\neq 0$ breaks only the continuous symmetry of model I, whereas it does not break the symmetry under the global gauge transformation U of model II. Nevertheless, it breaks the discrete time reversal symmetry of the random ensemble of model II which leads to a non-zero DOS at $E=0$. The latter can be evaluated from the average Green's function

$$\langle \mathbf{G}(r,r;\varepsilon)\rangle_A\approx\frac{1}{g}[q_0\,\sigma_3-i q_1\,\sigma_0], \tag{22}$$

where $q_0=-2\,\mathrm{Re}(q_{11})$ and $q_1=-2\,\mathrm{Im}(q_{11})$. The calculation simplifies essentially when we set $\mu=0$. Then the real part of the s.p. Eq. (19) yields $q_0=0$. The imaginary part can be evaluated using the density

$$\rho'(\kappa)=\frac{1}{\pi}\lim_{\varepsilon\downarrow 0}\mathrm{Im}\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}\left[-\frac{1}{d}\sum_{j=1}^{d}\cos k_j-\kappa-i\varepsilon\right]^{-1}\cdot\frac{\mathrm{d}k_1}{2\pi}\dots\frac{\mathrm{d}k_d}{2\pi}. \tag{23}$$

Thus we obtain a nonzero density of states at $E=0$, since

$$q_1=q_1\,4g\int_{-1}^{1}\frac{1}{\kappa^2+q_1^2}\,\rho'(\kappa)\,\mathrm{d}\kappa\approx 2g\pi. \tag{24}$$

The approximation is here related to a constant approximation of the density ρ' on the interval $[-1,1]$. This calculation can be extended to $\mu\neq 0$, where we get $q_0\neq 0$. In particular, $q_0\propto g$ such that the shift of the chemical potential is small for weak disorder ($g\sim 0$). Then we obtain again $q_1\neq 0$ if μ is inside the band of ∇^2 not too close to the band edges.

In order to investigate localization properties of the quasiparticle states at $E=0$ we shall evaluate the correlation function of the **Q**-field on large scales. This means that we must also include fluctuations around the s.p. solution. For this purpose it is important to discuss the effect of symmetry breaking again. Because the continuous symmetry of model I is spontaneously broken, we expect massless modes which will dominate the long range properties. On the other hand, since the continuous symmetry of Model II is *not* broken, there is no reason to expect massless modes. The mass terms can be calculated from the Gaussian fluctuations δq, δp, ψ and $\bar\psi$, where $\delta q_{1+4(\nu-1)}\approx Q_\nu^{11}$, $\delta q_{2+4(\nu-1)}+i\delta q_{3+4(\nu-1)}\approx Q_\nu^{12}$, $\delta q_{2+4(\nu-1)}-i\delta q_{3+4(\nu-1)}\approx Q_\nu^{21}$, $q_{4+4(\nu-1)}\approx Q_\nu^{22}$, and with the corresponding expressions for P_ν^{ij}, Θ_ν^{ij}. The action S_ν reads in terms of the Gaussian fluctuations

$$S_\nu\approx\int\sum_{\mu,\mu'=1}^{4\nu}(\mathbf{I}_\nu(k))_{\mu,\mu'}[\delta q_\mu(k)\,\delta q_{\mu'}(-k)+\delta p_\mu(k)\,\delta p_{\mu'}(-k)+2\psi_\mu(k)\,\bar\psi_{\mu'}(-k)]\,\mathrm{d}^d k \tag{25}$$

with the stability matrices

$$\mathbf{I}_1(k)=\frac{1}{2g}(K-2gB_1^T A(k)\,B_1),$$

$$\mathbf{I}_2(k)=\frac{1}{g}\begin{pmatrix}K-gB_1^T A(k)\,B_1 & -gB_1^T A(k)\,B_2\\ -gB_2^T A(k)\,B_1 & K-gB_2^T A(k)\,B_2\end{pmatrix}, \tag{26}$$

where

$$K=\begin{pmatrix}1&0&0&0\\0&2&0&0\\0&0&2&0\\0&0&0&1\end{pmatrix},$$

$$A(k)=\begin{pmatrix}A_1(k)&0&0&0\\0&A_2(k)&0&0\\0&0&A_2(k)&0\\0&0&0&A_1(k)^*\end{pmatrix}$$

and

$$B_1=\tfrac{1}{2}\begin{pmatrix}i&2&0&-i\\1&0&0&1\\0&0&2i&0\\-i&2&0&i\end{pmatrix},$$

$$B_2=\tfrac{1}{2}\begin{pmatrix}i&0&-2&-i\\0&2i&0&0\\-1&0&0&-1\\-i&0&-2&i\end{pmatrix}. \tag{28}$$

Here we have

$$A_1(k)=\frac{2}{(2\pi)^d}\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}a(k')\,a(k'-k)\,\mathrm{d}k_1'\dots\mathrm{d}k_d' \tag{29a}$$

and

$$A_2(k)=-\frac{4}{(2\pi)^d}\mathrm{Re}\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}a(k')\,a(k'-k)^*\,\mathrm{d}k_1'\dots\mathrm{d}k_d' \tag{29b}$$

with

$$a(k)=[-h(k)+\mu+q_0+i(\varepsilon+q_1)]^{-1}. \tag{29c}$$

In particular, we obtain from (29a) with $\varepsilon\to 0$

$$A_1(k=0)=-1/2g+\frac{4}{(2\pi)^d}\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}\left\{\frac{-h(k)+\mu+q_0}{[-h(k)+\mu+q_0]^2+q_1^2}\right\}^2\mathrm{d}k_1\dots\mathrm{d}k_d-\frac{4iq_1}{(2\pi)^d}\int_{-\pi}^{\pi}\dots\int_{-\pi}^{\pi}\frac{-h(k)+\mu+q_0}{\{[-h(k)+\mu+q_0]^2+q_1^2\}^2}\,\mathrm{d}k_1\dots\mathrm{d}k_d. \tag{30}$$

Furthermore, we have $A_2(k=0)=-1/g$ from the s.p. Eq. (19). The non-Hermitean part of $\mathbf{I}_v$ is irrelevant for the stability matrix (it leads to oscillations). Considering now only the Hermitean part we can evaluate the smallest eigenvalue of $\mathbf{I}_v(k=0)$ which dominates the behavior of the quasiparticle states. Using again $\mu=0$, we obtain due to $A_1(k=0)=A_1(k=0)^*$ in this case for the smallest eigenvalue of the Hermitean part of $\mathbf{I}_2(k=0)$

$$4[1/g+2A_1(k=0)]=\frac{32}{(2\pi)^d}\int\frac{\kappa^2}{[\kappa^2+q_1^2]^2}\rho'(\kappa)\,d\kappa\sim 4/g; \tag{31}$$

i.e., the Gaussian fluctuations for model II are massive as expected. Taking into account that the correlation function (Eq. (10b)) scales with g^{-2}, the correlation length and, therefore, the localization length of the quasiparticle are proportional to g^{-1}.

For model I we find that the modes δq_3, δp_3, ψ_3 are massless if $\varepsilon\to 0$. This massless part reads in terms of the Gaussian action for small k

$$S_1\approx\tfrac{1}{2}\int\left[a q_1\varepsilon+b\sum_{j=1}^{d}k_j^2\right][\delta q_3(k)\,\delta q_3(-k)+\delta p_3(k)\,\delta p_3(-k)+2\psi_3(k)\,\bar\psi_3(-k)]\,d^d k \tag{32}$$

with coefficients $a, b>0$. This means that the quasiparticles are diffusing. However, the diffusion coefficient b/aq_1, obtained in Gaussian approximation, is not correct: due to the massless fluctuations we need a renormalization from higher order than Gaussian terms. This might change the result of the Gaussian approximation drastically. For instance, the diffusion coefficient can be renormalized to zero. The effect of those higher order terms will be given in the following effective theory of the massless modes of model I.

$\varepsilon\sigma_0$ has fixed the symmetry breaking s.p. solution (18). However, any other choice of the symmetry breaking term εT^2, with ε arbitrarily small, may be used. Therefore, the global symmetry of the model under the transformation T generates a s.p. manifold if $\varepsilon=0$:

$$T\begin{pmatrix}q_{11} & 0\\ 0 & q_{22}\end{pmatrix}T=-\tfrac{1}{2}[iq_0\,T^2+q_1\,\sigma_3]. \tag{33}$$

Since $\varepsilon\to 0$, it does not cost much energy to restore this global symmetry in the symmetry broken case ($\varepsilon\neq 0$). This fact is reflected by the massless mode in (32). The massless mode is very sensitive to the interaction of the fluctuations. Consequently, the Gaussian fluctuations are not sufficient as an approximation, and we must take into account the fluctuations along the invariant s.p. manifold. The fluctuation field can be expressed by the symmetry transformation as

$$\gamma_1^{1/2}\,\mathbf{Q}_r\,\gamma_1^{1/2}-\frac{i}{2}\varepsilon\gamma_0=\delta\mathbf{Q}_r\equiv\mathbf{T}_r^2, \tag{34}$$

where $\mathbf{T}$ is the graded 4×4 generalization of the transformation [16] T. In particular, we have

$$\mathbf{T}\begin{pmatrix}Q & 0\\ 0 & 0\end{pmatrix}\mathbf{T}=\begin{pmatrix}TQT & 0\\ 0 & 0\end{pmatrix}. \tag{35}$$

From the global symmetry of S_1 follows that only non-local terms contribute to the action. Thus the logarithmic term in (14) can be expanded up to second order in $\delta\mathbf{Q}$:

$$S_1\approx\frac{1}{g'^2}\,\mathrm{Tr}\,g_{4A}[(\gamma_3\,\nabla\delta\mathbf{Q})^2]+i\varepsilon\,\mathrm{Tr}\,g_{4A}(\gamma_1\,\delta\mathbf{Q}), \tag{36}$$

where g' is a renormalized g. Equation (36) is a nonlinear sigma model similar to that found for the particle in a random potential [10, 13], except for different field $\delta\mathbf{Q}$ defined in (34). This difference reflects the different symmetries related to both problems. Nevertheless, in a renormalization group calculation we found no indication for extended states in dimensions $d\leqq 2$ analogous to the particle in the random potential but only for $d>2$.

In conclusion, we have found that there is a non-zero density of quasiparticles at the band center $E=0$ in both models. It means that the gap is closed due to the random fluctuations of the order parameter. Although we have only considered the case of an order parameter distribution with mean zero, the filling of the gap can be found also for other distributions [12]. The nature of states in the filled gap depends on the symmetry of the BG equation: The quasiparticle states of model I may undergo a transition from localized to delocalized states if $d>2$ with decreasing strength of disorder. The quasiparticle states at the band center of model II, on the other hand, are always localized in any dimension. Thus, the fluctuations of the real *and* the imaginary part of the order parameter means always strong disorder. In terms of symmetries, this implies the absence of a spontaneous breakdown of a non-compact symmetry. Our results demonstrate that a spontaneously broken non-compact symetry seems to be crucial for a localization-delocalization transition. The transition can be understood in terms of a renormalization group calculation for the massless modes. This has not been presented in this article because it is closely related to earlier work [10, 13].

The complete destruction of the gap by impurities in the superfluid has serious consequences for the stability of these systems. In a superconductor, for example, quasiparticles near the Fermi surface lead to metallic transport due to dissipation unless the quasiparticle are localized and the temperature is low. Therefore, the localization of the low energy quasiparticles is important for the stability of the superconducting state in a disordered material. The effect of impurities is particularly interesting in type II superconductors with short coherence length (e.g. high T_c superconductors) in a magnetic field. If the magnetic field is strong enough the superconductor will be in the mixed phase with magnetic vortices. Due to the short coherence length the vortices will be frozen in a disordered configuration ("vortex glass"). As we mentioned in the beginning, this system can be described by model II. The frozen (or pinned) vortices cannot contribute to the transport in the superconductor. According to our calculation, the low energy quasiparticles are localized by the disorder caused by the vortices.

Consequently, neither the vortices nor the low energy quasiparticles lead to dissipation. Thus the superconductor is stable.

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