

Describing Functions: Atomic Decompositions Versus Frames

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Abstract. The theory of frames and non-orthogonal series expansions with respect to coherent states is extended to a general class of spaces, the so-called coorbit spaces. Special cases include wavelet expansions for the Besov-Triebel-Lizorkin spaces, Gabor-type expansions for modulation spaces, and sampling theorems for wavelet and Gabor transforms.

1. Introduction

In [FG1,2,3] a systematic theory of series expansions with respect to coherent states was developed. Given an integrable, irreducible, unitary representation π of a locally compact group \mathcal{G} acting on a Hilbert space \mathcal{H} , a suitable $g \in \mathcal{H}$ and a sufficiently dense subsequence $(x_i)_{i \in I}$ in \mathcal{G} , we constructed series expansions of the form

$$f = \sum_{i \in I} \lambda_i \pi(x_i) g$$

Here the expanding functions $\pi(x_i)g \in \mathcal{H}$ are all of a very simple form, namely, they lie in the orbit of a single element under the representation π . In mathematics such expansions are usually referred to as *atomic decompositions*, in the terminology of physicists they are discrete expansions with respect to *coherent states* [KS].

The main objective of [FG2,3] was to show that such expansions are not limited to the Hilbert space \mathcal{H} , but that they can be constructed for a wide class of Banach spaces, the so-called coorbit spaces. In general, the collection $\{\pi(x_i)g, i \in I\}$ is not linearly independent, hence the coefficients in such a non-orthogonal expansion are not uniquely determined. However, one can construct a map from functions f into

a Banach space of sequences (λ_i) on I , such that the coefficients depend linearly and continuously on f , and such that the norm of f is equivalent to the sequence space norm of the coefficients.

A wealth of examples is obtained by specific choices of a group and a representation: non-orthogonal wavelet expansions for Besov-Triebel-Lizorkin spaces on R^n , the Gabor-type expansions for modulation spaces, and atomic decompositions for Banach spaces of analytic functions. To see how such decompositions arise as special cases from the general theory, we refer to [FG1] and Sections 3 and 5 of this article. For direct approaches to what seemed to be mutually disjoint theories until the discovery of the group theoretic approach, see [FJ1,2,3], [F2], [CR], [R], [JPR], [RT], [L], among others.

In this paper we consider the following question which is related to the moment problem: given a discrete set of coherent states $\{\pi(x_i)g, i \in I\}$, under what conditions is a function f completely determined by the moments or coefficients $\langle \pi(x_i)g, f \rangle$ and how could f be reconstructed from these coefficients? This is an abstract formulation of a problem that occurs frequently in applications, notably in signal analysis, image processing, and data compression. If we think of the representation coefficient $x \rightarrow \langle \pi(x)g, f \rangle$ as a signal transform of f with g fixed [GMP], the question is (a) to what extent the continuous information $\langle \pi(x)g, f \rangle$ can be compressed into the discrete information $\{\langle \pi(x_i)g, f \rangle, i \in I\}$, and (b) how the original signal f can be recovered from the discrete set of values $\langle \pi(x_i)g, f \rangle, i \in I$.

To be more specific, let us look at the case of the Hilbert space \mathcal{H} and take it for granted that every element $f \in \mathcal{H}$ has a stable, non-orthogonal expansion with respect to the coherent states $\pi(x_i)g, i \in I$. Then there exist functionals $e_i \in \mathcal{H}$ and two constants $A, B > 0$ such that for every $f \in \mathcal{H}$

$$f = \sum_i \langle e_i, f \rangle \pi(x_i)g$$

and

$$A \|f\|_{\mathcal{H}} \leq \left(\sum_i |\langle e_i, f \rangle|^2 \right)^{1/2} \leq B \|f\|_{\mathcal{H}} \quad (1)$$

By duality one also obtains $f = \sum_i \langle \pi(x_i)g, f \rangle e_i$ and

$$B^{-1} \|f\|_{\mathcal{H}} \leq \left(\sum_i |\langle \pi(x_i)g, f \rangle|^2 \right)^{1/2} \leq A^{-1} \|f\|_{\mathcal{H}}.$$

Thus, question (a) above has an easy answer in Hilbert space.

A set $\{e_i, i \in I\}$ that satisfies (1) is called a *frame* for \mathcal{H} [DS, Y]. This is a much weaker notion than that of a basis, but quite useful in many contexts.

The concept of a frame for a Banach space is readily defined:

Definition: A family $\{e_i, i \in I\}$ in the dual B' of a Banach space B is a *Banach frame* for B , if

- (i) there is an associated Banach space B_d of sequences on I , such that the coefficient mapping $f \mapsto \langle e_i, f \rangle_{i \in I}$ is continuous from B into B_d ,
- (ii) the norms $\|f|B\|$ and $\|\langle e_i, f \rangle_{i \in I}|B_d\|$ are equivalent, and
- (iii) there exists a bounded, linear reconstruction operator S from B_d onto B , such that $S(\langle e_i, f \rangle) = f$.

In this paper, we lay the foundations for a theory of coherent Banach frames, i.e. of the form $\{\pi(x_i)g, i \in I\}$, and we construct coherent Banach frames for the coorbit spaces. Their existence is by no means evident. It is a remarkable fact that in Hilbert space the norm equivalence (1) alone guarantees an efficient method for the reconstruction of f and that condition (iii) is redundant in this case (cf. [DS], [Y] for the reconstruction). For Banach spaces, however, conditions (ii) and (iii) are independent, and to find the reconstruction operator S poses additional difficulties.

Because of many applications in signal processing the construction of special Hilbert frames (wavelet frames and Gabor frames) has been a subject of intensive investigation in the past years, see [DGM], [D1], [HW]. The only example of Banach frames is the “ ϕ -transform” [FJ1,2,3] which is used to characterize distributions in Besov-Triebel-Lizorkin spaces. We are not aware of any other attempts to describe functions in this way.

The construction of Banach frames also gives new insights for coherent Hilbert frames:

(a) The discrete set $\{x_i, i \in I\}$ in \mathcal{G} need not be a lattice, but can be distributed irregularly in \mathcal{G} .

(b) Stability results for frames are an easy consequence of the general theory (cf. Section 6).

(c) The coefficient sequence $\langle \pi(x_i)g, f \rangle_{i \in I}$ is a sampling of the representation coefficient $\langle \pi(x)g, f \rangle$. The construction of a coherent frame yields a sampling theorem for representation coefficients of integrable representations which is similar to the famous Shannon-Whittaker-Kotelnikov sampling theorem for band-limited functions.

The paper is organized as follows: the basic assumptions, facts on square-integrable representations, and some technical statements that are required from [FG2,3] are collected in Section 2. Some knowledge of [FG2,3] would be helpful, but we have tried to keep the exposition of the paper self-contained. Section 3 reviews the notion of coorbit spaces and provides a variety of examples. Section 4 is devoted to a detailed analysis of convolution operators on locally compact groups and their approximation by linear combination of translates of one factor. Using the local oscillation of a function, such an analysis can be carried out much simpler than in [FG2], and one obtains sharper estimates. Section 5 contains the main results of this paper, namely the construction of coherent Banach frames for coorbit spaces. We have tried to keep the exposition accessible through a variety of relevant examples. In Section 6 the stability of Banach frames is discussed. Some complementary results show that the assumptions in the main theorems are also necessary. They demonstrate that the general framework of [FG1,2,3] and of this paper is optimal.

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2. Prerequisites

This section contains the terminology and lists a few technical statements which were proved in [FG2, §3] and [FG3, §7]. Furthermore, it explains the basic assumptions on function spaces, weights and representations that are made throughout the paper.

2.1. Throughout this paper, \mathcal{G} will always denote a σ -compact group, therefore all coverings and index sets under consideration will be countable. λ or dx denote the left Haar measure of \mathcal{G} , Δ the Haar modulus, $e \in \mathcal{G}$ the identity element.

The following operations on functions F on \mathcal{G} will be used:

left translation by $x \in \mathcal{G}$: $L_x F(y) = F(x^{-1}y)$,

right translation $R_x F(y) = F(yx)$ and

the involution $F^\vee(y) = \overline{F(y^{-1})}$;

$\langle H, F \rangle = \int_{\mathcal{G}} \overline{H(x)} F(x) dx$ whenever the integral is defined.

2.2. Function spaces on \mathcal{G} . By Y we shall always mean a Banach space of functions on \mathcal{G} with norm $\|\cdot\|_Y$ which satisfies the following properties:

(i) Y is continuously embedded into $L^1_{loc}(\mathcal{G})$, the locally integrable functions on \mathcal{G} . (2.1)

(ii) Y is a Banach lattice, i.e. if $|F(x)| \leq |G(x)|$ a.e. and $G \in Y$, then

$$F \in Y \text{ and } \|F|Y\| \leq \|G|Y\| \quad (2.2)$$

(iii) Y is invariant under left and right translations. Set $u(x) = \|\|L_x|Y\|\|$ and $v(x) = \Delta(x^{-1})\|\|R_{x^{-1}}|Y\|\|$, the operator norms of translations on Y , then we require that

$$(iv) \quad L^1_u * Y \subseteq Y \quad (2.3)$$

$$\text{and} \quad Y * L^1_v \subseteq Y \quad (2.4)$$

are satisfied i.e. if $F \in Y$, $G \in L^1_v$ then

$$F * G \in Y \quad \text{and} \quad \|F * G|Y\| \leq \|F|Y\| \|G|L^1_v\|.$$

Examples: weighted L^p -spaces on \mathcal{G} , certain mixed norm spaces on \mathcal{G} , tent spaces etc.

In the sequel we shall always consider pairs (Y, w) where w is a weight function on \mathcal{G} such that (for a constant $C > 0$)

$$\text{and} \quad w(x) \geq C \max \{u(x), u(x^{-1}), v(x), v(x^{-1}) \Delta(x^{-1})\} \quad (2.5)$$

$$w(x) = w(x^{-1}) \Delta(x^{-1})$$

In particular, $w(x) \geq 1$, $\|f|L^1_w\| = \|f^\vee|L^1_w\|$ and

$$Y * L^1_w \subseteq Y \quad (2.6)$$

2.3. Sequence spaces. Given the compact set Q with non-void interior, a (countable) family $X = (x_i)_{i \in I}$ in \mathcal{G} is said to be Q -dense if $\bigcup_{i \in I} x_i Q = \mathcal{G}$, and *separated* if for some compact neighbourhood V of e we have $x_i V \cap x_j V = \emptyset$, $i \neq j$, and *relatively separated* if X is a finite union of separated sets.

In the applications usually “lattices” are chosen for Q -dense and relatively separated families. The general theory, however, allows much more freedom and irregular “sampling” sets.

Examples: The standard lattice in the n -dimensional Heisenberg group $\mathbb{R}^n \times \mathbb{R}^n \times T$ is $(\alpha \bar{l}, \beta \bar{m}, 1)$, $\bar{l}, \bar{m} \in \mathbb{Z}^n$ ($\alpha, \beta > 0$); for the $ax + b$ -group $\mathbb{R} \times \mathbb{R}^+$ the lattices $(\alpha k \beta^n, \beta^n)$, $k, n \in \mathbb{Z}$ are used ($\alpha > 0, \beta > 1$).

Given a Q -dense, relatively separated family $X = (x_i)_{i \in I}$ in \mathcal{G} , the sequence space $Y_d(X)$ of Y is

$$Y_d(X) = \left\{ (\lambda_i)_{i \in I} : \sum_{i \in I} \lambda_i L_{x_i} c_Q \varepsilon Y \right\} \quad (2.7a)$$

with norm

$$\|(\lambda_i)_{i \in I}\|_{Y_d} = \left\| \sum_{i \in I} \lambda_i L_{x_i} c_Q \varepsilon Y \right\| \quad (2.7b)$$

(where c_Q is the characteristic function of Q). $Y_d(X)$ is independent of Q and X (cf. [FG2], Lemma 3.5), therefore we shall suppress the index X frequently.

Example: $L_m^p(\mathcal{G})_d = l_m^p$, where $m'(i) = m(x_i)$. Thus the correspondence is natural, but it may be more complicated for other function spaces.

2.4. Maximal functions and convolution relations. Choose $Q \subseteq \mathcal{G}$ compact with nonvoid interior and $e \in Q$, then the (local) maximal function MF of F is defined to be

$$MF(x) = \|L_x c_Q \cdot F\|_\infty = \sup_{y \in xQ} |F(y)| \quad (2.8)$$

$MF(x)$ controls F in a neighbourhood of $x \in \mathcal{G}$. For a function space Y on \mathcal{G} we define

$$\mathcal{M}(Y) = \{F \text{ such that } MF \in Y\} \quad (2.9a)$$

with norm

$$\|F\|_{\mathcal{M}(Y)} = \|MF\|_Y, \quad (2.9b)$$

and write $\mathcal{M}_c(Y)$ for the subspace of continuous functions in $\mathcal{M}(Y)$. Then $\mathcal{M}_c(Y) \rightarrow \mathcal{M}(Y) \rightarrow Y$ are continuous embeddings and $\mathcal{M}(Y)$, $\mathcal{M}_c(Y)$ are independent of Q . The general theory of such spaces has

been worked out by H. FEICHTINGER in [F1], under the name of Wiener-type spaces, and has many applications in harmonic analysis. The control of the local behaviour of functions in $\mathcal{M}_C(Y)$ yields immediately the following

2.5. Proposition ([FG2], Lemma 3.8). *If $(x_i)_{i \in I}$ is relatively separated and $F \in \mathcal{M}_C(Y)$, then $(F(x_i)_{i \in I})$ is in $Y_d(X)$ and*

$$\|F(x_i)_{i \in I}|Y_d\| \leq C \|F|Y\|, \quad (2.10)$$

where C is a constant independent of F .

Working with nonabelian groups we shall need the right versions of these spaces:

$$M^R F(x) = \|R_x c_Q \cdot F\|_\infty \quad \text{and} \quad \mathcal{M}^R(Y) = \{F \text{ such that } M^R F \in Y\} \quad (2.11)$$

with the obvious norm $\|F| \mathcal{M}^R(Y)\| = \|M^R F|Y\|$.

The following convolution relation was proved in [FG3], Thm. 7.1, and is vital for our approach:

2.6. Theorem. *Assume that Y satisfies 2.2(i)-(iv), then*

$$Y * \mathcal{M}(L_v^1) \subseteq \mathcal{M}(Y) \quad (2.12)$$

and

$$Y * \mathcal{M}_C(L_v^1) \subseteq \mathcal{M}_C(Y) \quad (2.13)$$

2.7. Square-integrable representations. An irreducible, unitary, continuous representation π of \mathcal{G} on a Hilbert space \mathcal{H} is called square-integrable if for at least one representation coefficient $\langle \pi(x)g, g \rangle$ ($g \in \mathcal{H}$)

$$\int_{\mathcal{G}} |\langle \pi(x)g, g \rangle|^2 dx < \infty \quad (2.14)$$

(see e.g. [DM], [GMP], or any standard text on representation theory).

In this case there exists a positive, densely defined self-adjoint operator A on \mathcal{H} such that the orthogonality relations hold true:

$$\int_{\mathcal{G}} \overline{\langle \pi(x)g_1, f_1 \rangle} \langle \pi(x)g_2, f_2 \rangle dx = \langle A g_2, A g_1 \rangle \langle f_1, f_2 \rangle \quad (2.15)$$

for all $f_1, f_2 \in \mathcal{H}$ and $g_1, g_2 \in \text{dom } A$. If we write $V_g(f)(x) = \langle \pi(x)g, f \rangle$

for the representation coefficients, then for a fixed $g \in \text{dom } A$ the mapping $f \rightarrow V_g(f)$ is from \mathcal{H} into $L^2(\mathcal{G})$ and intertwines $\pi(x)$ and L_x .

For the special choice $g_1 = g_2 = g \in \text{dom } A$ and $\|Ag\| = 1$, $f_1 = \pi(y)g$, $f_2 = f \in \mathcal{H}$ one obtains the reproducing formula for the representation coefficients $V_g(f)$ from (2.15):

$$V_g(f) * V_g(g) = V_g(f) \quad (2.16)$$

This formula and its extensions are the very reasons for the unification of all the different examples mentioned in Section 1.

2.8. As was pointed out in [FG2], it is necessary to require stronger integrability conditions on π , in order to handle other spaces than Hilbert space. Therefore, given a weight function w , i.e. $w(xy) \leq w(x)w(y)$ and $w(x) \geq 1$ for all $x, y \in \mathcal{G}$, chosen as in (2.5), we shall always assume that the representation π is w -integrable: in other words

$$\mathcal{A}_w = \{g \in \mathcal{H} : \int |\langle \pi(x)g, g \rangle| w(x) dx < \infty\} \text{ is non-trivial.} \quad (2.17)$$

With this assumption $\mathcal{A}_w \neq \{0\}$, formula (2.16) can be easily analyzed. Note that the functions $G := \langle \pi(x)g, g \rangle$ are convolution idempotents in L_w^1 whenever g is normalized by $\|Ag\| = 1$. Therefore the convolution operator $TF = F * G$ is a bounded projection from Y onto the subspace $Y * G$ in the situation of (2.6).

3. Coorbit Spaces and Examples

We recall the definition of coorbit spaces and review their basic properties. Finally we show how the classical function spaces fit into this construction. For a detailed exposition of the abstract coorbit spaces we refer to [FG2], §4, and [FG3]. There it was shown how to attach an abstract Banach space $\mathcal{C} \circ Y$ to a w -integrable representation π and a function space Y on the group \mathcal{G} . Starting only with a Hilbert space given, the first problem is to provide spaces of “test functions” and “distributions”.

Whereas function spaces on manifolds, domains etc. are always defined as subspaces of distributions, of all holomorphic functions etc, in our situation we first have to construct a suitably large object that

may serve as a reservoir for selection. Here is how this is done ([FG1] and [FG2]):

3.1. Fix an irreducible, unitary, continuous representation π on \mathcal{H} and a weight function w on \mathcal{G} (arising as in (2.5) from a function space Y on \mathcal{G}) such that $\mathcal{A}_w \neq \{0\}$ and fix once and forever some nontrivial vector $g \in \mathcal{A}_w$.

Definition: (“Test functions” and “distributions”)

$$\mathcal{H}_w^1 = \{f \in \mathcal{H} \text{ such that } V_g(f) = \langle \pi(\cdot)g, f \rangle \in L_w^1\} \quad (3.1a)$$

with norm

$$\|f\|_{\mathcal{H}_w^1} = \|V_g(f)\|_{L_w^1} \quad (3.1b)$$

Then the anti-dual $\mathcal{H}_w^{1\sim}$, the space of all continuous conjugate-linear functionals on \mathcal{H}_w^1 , will serve as the reservoir for selection.

\mathcal{H}_w^1 and $\mathcal{H}_w^{1\sim}$ are π -invariant Banach spaces with the continuous embeddings

$$\mathcal{H}_w^1 \rightarrow \mathcal{H} \rightarrow \mathcal{H}_w^{1\sim} \quad (3.2)$$

and their definition is independent of the choice of the analyzing vector $g \in \mathcal{A}_w$. (It follows from (2.15) and the symmetry of the weight w , $w = w^\vee \Delta^{-1}$, that $\mathcal{H}_w^1 = \mathcal{A}_w$ as sets, see [FG1], Lemma 4.2) The inner product on $\mathcal{H} \times \mathcal{H}$ extends to a sesquilinear form on $\mathcal{H}_w^1 \times \mathcal{H}_w^{1\sim}$, therefore for $g \in \mathcal{H}_w^1$ and $f \in \mathcal{H}_w^{1\sim}$ the *extended representation coefficients* $V_g(f)(x) = \langle \pi(x)g, f \rangle$ are well-defined.

3.2. Let Y be a Banach space of functions on \mathcal{G} with canonical weight function w satisfying conditions 2.2 (i)–(iv).

Definition (the *coorbit* of Y with respect to π):

$$\mathcal{C}o Y = \{f \in \mathcal{H}_w^{1\sim} : V_g(f) \in Y\} \quad (3.3a)$$

with norm

$$\|f\|_{\mathcal{C}o Y} = \|V_g(f)\|_Y \quad (3.3b)$$

Then $\mathcal{C}o Y$ with this norm is a π -invariant Banach space, the definition is independent of $g \in \mathcal{A}_w$, and one obtains the obvious embeddings

$$\mathcal{H}_w^1 \rightarrow \mathcal{C}o Y \rightarrow \mathcal{H}_w^{1\sim} \quad (3.4)$$

Moreover, the basic spaces \mathcal{H} , \mathcal{H}_w^1 and $\mathcal{H}_w^{1\sim}$ can be identified with coorbits under π :

$$\mathcal{H} = \mathcal{C}_o L^2(\mathcal{G}), \quad \mathcal{H}_w^1 = \mathcal{C}_o L_w^1 \quad \text{and} \quad \mathcal{H}_w^{1\sim} = \mathcal{C}_o L_{1/w}^\infty.$$

These abstract spaces have been investigated thoroughly in [FG1], [FG2], [FG3]. [FG3] is devoted entirely to the Banach space structure of the coorbit spaces. We refer to these papers for more details and the proofs of the assertions.

A square-integrable representation can usually be realized by operators acting on a Hilbert space of functions on a homogeneous space of \mathcal{G} . Therefore for a concrete realization of π the elements in $\mathcal{C}_o Y$ are indeed such handy things as functions or distributions or measures. The advantage of this general definition is that — for frames, atomic decompositions, structure of the spaces — it does not matter in the least what the objects in $\mathcal{C}_o Y$ “are”.

3.3. Examples. In order to demonstrate the scope of the notion of coorbit space we give a list of examples. It is a remarkable fact that almost all classical function spaces in real and complex variable theory occur naturally as coorbits of certain integrable representations. We shall content ourselves with a few hints and refer to the literature for the standard theory of the mentioned spaces. In all of the examples it can be checked that the assumptions of Section 2 on the representation π and the function space Y are satisfied.

(a) *The Besov–Triebel–Lizorkin–Spaces.* Let \mathcal{G} be the n -dimensional $ax + b$ -group $\mathbb{R}^n \times \mathbb{R}^+$ with multiplication $(\bar{x}, t)(\bar{y}, v) = (\bar{x} + t\bar{y}, tv)$, $\bar{x}, \bar{y} \in \mathbb{R}^n$, $t, v \in \mathbb{R}^+$ and $\pi(\bar{x}, t)$ be the unitary representation of \mathcal{G} which acts on $L^2(\mathbb{R}^n)$ by translations and dilations, i.e. for $f \in L^2(\mathbb{R}^n)$

$$\pi(\bar{x}, t) f(\bar{z}) = t^{-n/2} f\left(\frac{\bar{z} - \bar{x}}{t}\right) \quad (3.5)$$

(strictly speaking, π is not irreducible. Extending \mathcal{G} by the orthogonal group $O(n)$ and π by rotations, one is back with the assumptions in 2.5). For convenience the analyzing vector g is chosen to be a radial Schwartz function with all moments vanishing and the reservoir for selection shall be the space of tempered distributions modulo polynomials. Then a result of TRIEBEL ([T]) implies that

$$\dot{B}_{p,q}^s = \mathcal{C} \circ L_{s+n/2-n/q}^{p,q} \tag{3.6a}$$

and

$$\dot{F}_{p,q}^s = \mathcal{C} \circ T_{s+n/2}^{p,q} \tag{3.6b}$$

where $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, \dot{B} are the homogeneous Besov-spaces on \mathbb{R}^n , \dot{F} the homogeneous Triebel—Lizorkin spaces, $L_s^{p,q}$ a certain mixed norm space on \mathcal{G} and $T_s^{p,q}$ a tent space on \mathcal{G} (cf. [CMS]). Written out in detail, this means for example: $f \in \dot{F}_{p,q}^s$, $p < \infty$, iff

$$\int_{\mathbb{R}^n} d\bar{x} \left(\iint_{|\bar{y}-\bar{x}| \leq t} |\langle \pi(\bar{y}, t)g, f \rangle|^q t^{-q(s+n/2)} \frac{d\bar{y} dt}{t^{n+1}} \right)^{p/q} < \infty \tag{3.7}$$

For $p = \infty$, the definition of $T_s^{\infty,q}$ is different (see [CMS]), nevertheless $BMO = \dot{F}_{\infty,2}^0$ and the smoothness spaces of Sharpley-de Vore $C^{p,\alpha}$ (coinciding with some $\dot{F}_{\infty,p}^s$) are coorbits. [SV]

(b) *The modulation spaces on \mathbb{R}^n .* Let \mathbb{H}_n be the Heisenberg group $\mathbb{R}^n \times \mathbb{R}^n \times T$ with multiplication $(\bar{x}_1, \bar{y}_1, \tau_1)(\bar{x}_2, \bar{y}_2, \tau_2) = (\bar{x}_1 + \bar{x}_2, \bar{y}_1 + \bar{y}_2, \tau_1 \tau_2 e^{i\bar{x}_2 \bar{y}_1})$, $\bar{x}_i, \bar{y}_i \in \mathbb{R}^n$, $\tau_i \in T$ and consider the Schrödinger representation on \mathbb{H}_n of $L^2(\mathbb{R}^n)$:

$$\pi(\bar{x}, \bar{y}, \tau)f(\bar{z}) = \tau e^{i\bar{y} \cdot (\bar{z} - \bar{x})} f(\bar{z} - \bar{x}) \tag{3.8}$$

Then the modulation spaces $M_{p,q}^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, $1 \leq p, q \leq \infty$, were introduced by FEICHTINGER ([F2], [F3]) in formal analogy with the Besov spaces:

$$f \in M_{p,q}^s \text{ if and only if } \int \int |\langle \pi(\bar{x}, \bar{y}, \tau)g, f \rangle|^p d\bar{x}]^{q/p} (1 + |\bar{y}|)^{s q} d\bar{y} < \infty \tag{3.9}$$

(g being an analyzing vector in \mathcal{S} , the reservoir being \mathcal{S}') or

$$M_{p,q}^s = \mathcal{C} \circ L_s^{p,q} \tag{3.10}$$

Among the $M_{p,q}^s$ can be found $L^2(\mathbb{R}^n)$, certain Sobolev spaces, the Bessel potential spaces and the Segal algebra S_o of FEICHTINGER ([F3]).

(c) Similarly, modulation spaces can be studied on all locally compact abelian groups (cf. [F3]).

(d) For the *Besov—Triebel—Lizorkin—spaces on the Heisenberg groups \mathbb{H}_n* (cf. [FS]) neither a new theory nor a new notation is required. The only adjustments to be made are the following substitutions: in example (a) replace \mathbb{R}^n by \mathbb{H}_n , translations and dilations on \mathbb{R}^n by the

corresponding operations on \mathbb{H}_n , the group \mathcal{G} by the semidirect product $\mathbb{H}_n \rtimes \mathbb{R}^+$, \mathbb{R}^+ acting by dilations on \mathbb{H}_n . Then everything said in (a) carries over to the function spaces $B_{p,q}^s(\mathbb{H}_n)$ and $F_{p,q}^s(\mathbb{H}_n)$ on \mathbb{H}_n .

From the abstract point of view, the next example is nothing new and equal to example (b), but the actual appearance is quite different. We use the well-known fact that the unitary representations of the Heisenberg and the $ax + b$ -groups can also be realized on spaces of holomorphic functions:

(e) The *Bargmann–Fock spaces on \mathbb{C}^n* : It is convenient to write the Heisenberg group \mathbb{H}_n in a slightly different form: $\mathbb{H}_n \cong \mathbb{C}^n \times T$ with multiplication $(c_1, \tau_1) \cdot (c_2, \tau_2) = (c_1 + c_2, \tau_1 \tau_2 e^{i \operatorname{Im} \bar{c}_1 \cdot c_2 / 2})$ where $c_i \in \mathbb{C}^n$, $\tau_i \in T$ and $c_1 \cdot c_2$ denotes the standard bilinear form on \mathbb{C}^n . The identification with \mathbb{H}_n of example (b) is given by $(\bar{x}, \bar{y}, \tau) \rightarrow \left(\frac{\bar{x} + i\bar{y}}{\sqrt{2}}, \tau e^{-i\bar{x}\bar{y}/2} \right)$.

For any $\alpha > 0$ the Bargmann–Fock representation σ_α on

$$\mathcal{H}_\alpha = \left\{ F \text{ holomorphic on } \mathbb{C}^n : \int_{\mathbb{C}^n} |F(z)|^2 e^{-\alpha|z|^2} dz < \infty \right\}$$

is defined by

(3.11)

$$\sigma_\alpha(c, \tau) F(z) = \tau e^{\sqrt{\alpha} \bar{c} \cdot z - |c|^2 / 2} F(z - c / \sqrt{\alpha})$$

(cf. [B]). σ_α and the Schrödinger representation π (3.8) are equivalent by means of the intertwining operator $T_\alpha: L^2(\mathbb{R}^n) \rightarrow \mathcal{H}_\alpha$, sometimes called the Bargmann transform,

$$\begin{aligned} T_\alpha f(z) &= \int_{\mathbb{R}^n} \exp\left(-\frac{\alpha}{2} z^2 - \frac{1}{2} \xi^2 + \sqrt{2\alpha} z \xi\right) f(\xi) d\xi = \\ &= \exp\left(\frac{\alpha}{2}(x^2 + y^2)\right) \langle \pi(\sqrt{2\alpha} x, -\sqrt{2\alpha} y, e^{-i\alpha xy}) g_o, f \rangle \end{aligned}$$

where $x = \operatorname{Re} z \in \mathbb{R}^n$, $y = \operatorname{Im} z \in \mathbb{R}^n$ and $g_o(\xi) = e^{-\xi^2/2}$ the Gaussian.

By the general intertwining theorem of [FG2], Thm. 4.8, T_α establishes an isometric isomorphism between the coorbits with respect to π and those with respect to σ_α :

$$f \in \mathcal{C}_{o_\pi} Y \Leftrightarrow T_\alpha f \in \mathcal{C}_{o_{\sigma_\alpha}} Y$$

Since T_α is essentially a generalized representation coefficient with respect to π , the coorbits w.r.t. σ_α are easily described. After a trivial manipulation we obtain: a holomorphic function F on \mathbb{C}^n is in $\mathcal{C} \circ_{\sigma_\alpha} Y$ iff $\exp(-\alpha|z|^2/2)F(z) \in Y$. (Observe that we can assume Y to be a function space on \mathbb{C}^n instead of \mathbb{H}_n , because the action of the torus T can be neglected.)

The simplest examples are the coorbits of L^p :

$$\mathcal{F}_\alpha^p = \mathcal{C} \circ_{\sigma_\alpha} L^p = \left\{ F \text{ holomorphic on } \mathbb{C}^n : \int_{\mathbb{C}^n} |F(z)|^p e^{-p\alpha|z|^2/2} dz < \infty \right\} \quad (3.12)$$

These spaces have recently emerged in the study of Hankel operators on Bargmann–Fock space \mathcal{H}_α [JPR]. Although rather different in their appearance, they have the same properties as the modulation spaces.

(f) *The Bergman spaces on the upper half plane.* They are defined on $U = \{z \in \mathbb{C} : \text{Im } z > 0\}$ for $p \geq 1$, $\alpha > -1$, by

$$A^{p,\alpha} = \left\{ f \text{ holomorphic on } U : \iint_U |f(x+iy)|^p y^{+\alpha} dx dy < \infty \right\} \quad (3.13)$$

(cf. [R]) and are related to the discrete series π_m of $\text{SL}(2, \mathbb{R})$. For instance, in [FG1], 7.3, the following identification was obtained

$$A^{p,p^{m/2}} = \mathcal{C} \circ_{\pi_m} L^p \quad (3.14)$$

3.4. Before we can deal with atomic decompositions and frames of general coorbit spaces, we need some facts from [FG1] and [FG2].

Theorem (the Correspondence Principle, [FG2], Prop. 4.3). *Let π , Y and w satisfy the assumptions of Section 2 and set $G = V_g(g) \in L_w^1$ for a fixed analyzing vector $g \in \mathcal{A}_w$ with normalization $\|Ag\| = 1$. Then: A function $F \in Y$ is of the form $V_g(f)$ for some $f \in \mathcal{C} \circ Y$ if and only if*

$$F * G = F \quad (3.15)$$

In other words, $\mathcal{C} \circ Y$ is isometrically isomorphic to the closed subspace $Y * G$ of Y . In this case,

$$F(x) = \langle L_x G, F \rangle \quad (3.16)$$

exhibits the subspace $Y * G$ as a *Banach space with reproducing kernel*.

This theorem explains why the unification of all the mentioned examples works. As soon as a space is known to be a coorbit with respect to a representation, one may forget its original definition and deal with certain functions on the group. The theorem describes the mechanism of transference between $\mathcal{C}_o Y$ and functions on \mathcal{G} .

Definition: By a slight abuse of notation, call the inverse operator from $Y * G$ onto $\mathcal{C}_o Y V_g^{-1}$. The intertwining of $\pi(x)$ and L_x by V_g implies that

$$V_g^{-1} (L_x G) = \pi(x) g \quad (3.17)$$

The theorem is essentially a consequence of the orthogonality relations for π ((2.15) and (2.16)).

3.5. Dealing with coherent frames for $\mathcal{C}_o Y$ we need information about the sequences $(\langle \pi(x_i) g, f \rangle)_{i \in I}$, where (x_i) is some Q -dense, relatively separated set. In other words, we need a control of the local behaviour of the extended representation coefficients $V_g(f)$. To this end we have to restrict the set of analyzing vectors.

Definition ([FG1]). *The set of basic atoms* is defined to be

$$\mathcal{B}_w = \{g \in \mathcal{H} : \langle \pi(x) g, g \rangle \in \mathcal{M}(L_w^1)\}$$

It follows immediately, that

$$\mathcal{B}_w \subseteq \mathcal{H}_w^1 \quad (3.18)$$

and that

$$\langle \pi(x) g, g \rangle \in \mathcal{M}_c(L_w^1) \cap \mathcal{M}_c^R(L_w^1) \quad (3.19)$$

(because $G = V_g(g)$ is continuous and ∇ -invariant, $G = G^\nabla$, moreover $F \varepsilon \mathcal{M}^R(Y)$ iff $F^\nabla \varepsilon \mathcal{M}(Y)$ is true in general).

Remark: If \mathcal{G} has a compact invariant neighbourhood, e.g. the Heisenberg group, then $\mathcal{A}_w = \mathcal{B}_w$. In general, however, \mathcal{A}_w and \mathcal{B}_w are different. See also Section 6.

3.6. Applying Thm. 2.6 to the reproducing formula (3.15) yields the desired control of the local behaviour of $V_g(f)$.

Theorem ([FG3], Thm. 8.1). *Let π , Y , w be as in Section 2. If $g \in \mathcal{B}_w$, then $V_g(f) \in \mathcal{M}_c(Y)$ for all $f \in \mathcal{C}_o Y$. In particular, if $(x_i)_{i \in I}$ is a relatively separated family in \mathcal{G} , then*

$$\|(V_g(f)(x_i))_{i \in I} | Y_d(X)\| \leq C \|f| \mathcal{C}_o Y\| \quad (3.20)$$

with a constant C independent of f .

4. The Analysis of Discretization Operators

In this section different types of discretization operators are studied which finally lead to the general construction of atomic decompositions and frames. As a consequence of Thm. 3.4. this is equivalent to discretization of the reproducing formula (3.15)

$$F \approx \sum \lambda_i L_{x_i} G. \quad (4.1)$$

One way of doing this is to approximate $F = F * G = \int F(y) L_y G dy$ by a Riemann type sum, as was shown in [FG2]. In the following we give a systematic treatment of different approximations to the reproducing formula (3.15). The use of “maximal functions” instead of a Cotlar-type decomposition of the discretization operators simplifies many arguments and enables us to construct both atomic decompositions *and* Banach frames for the coorbit spaces.

4.1. For the discretization of formula (3.15) we make use of uniform *partitions of unity* $\Psi = (\psi_i, X, U)$ subordinate to (the compact neighbourhood of e) $U \subseteq \mathcal{G}$ with the following properties:

(i) $X = (x_i)_{i \in I}$ is a U -dense and relatively separated family in \mathcal{G} (2.3) such that

$$\begin{aligned} & \text{(ii) } \text{supp } \psi_i \subseteq x_i U \\ & \text{(iii) } 0 \leq \psi_i \leq 1 \quad \text{for all } i \in I \\ & \text{(iv) } \sum_{i \in I} \psi_i \equiv 1 \end{aligned} \quad (4.2)$$

It is convenient but not necessary to assume the functions ψ_i to be continuous. On the other hand, the characteristic functions of a partition of \mathcal{G} , which is deduced from the covering $(x_i U)_{i \in I}$ of \mathcal{G} , are also admitted. With the symbol Ψ is always given the information on ψ_i , x_i and U . We order partitions of unity by inclusion of their neighbourhood U and write $\Psi \rightarrow 0$ if $U \rightarrow \{e\}$. To avoid trivialities we furthermore assume that U is always contained in some fixed (large) compact neighbourhood $Q \subseteq \mathcal{G}$.

4.2. Discretization operators. We try to approximate the convolution operator

$$T: F \rightarrow F * G \quad (4.3)$$

by one of the following discretization operators

$$T_\Psi F = \sum_{i \in I} \langle \psi_i, F \rangle L_{x_i} G \quad (4.4)$$

$$S_\Psi F = \sum_{i \in I} F(x_i) \psi_i * G \quad (4.5)$$

$$U_\Psi F = \sum_{i \in I} F(x_i) c_i L_{x_i} G, \quad (4.6)$$

where Ψ is a partition of unity as in 4.1 and c_i is the mass of ψ_i : $c_i = \int \psi_i$. T_Ψ involves only the averages of F around the points $x_i \in \mathcal{G}$ and was therefore used in the approach of [FG2]. The pointwise evaluations at x_i make S_Ψ and U_Ψ more subtle to treat and it is to be expected that some kind of “maximal inequality” shows up in the proof. U_Ψ is a combination of T_Ψ and S_Ψ .

4.3. Remarks (i) It is not clear a priori whether the formal expressions (4.4)–(4.6) make sense at all and on which spaces they are bounded operators. Remember that in the reproducing formula (3.15) $F = V_g(f)$ and $G = V_g(g)$ for some $g \in \mathcal{A}_w$. In view of the identification of $\mathcal{C}_c Y$ with $Y * G$ we want to check the boundedness of the T , S , U on $Y * G$. Moreover, (4.5), (4.6) and Thm. 3.6 suggest that g should be taken from \mathcal{B}_w to get the desired estimates.

(ii) Some comment is required on the meaning of the sum over I : we order the finite subsets of I by inclusion, then $\sum_{i \in I} \dots$ will be understood as the limit of the net of partial sums over the finite subsets of I . If the bounded functions with compact support are dense in Y , then this convergence should take place in the norm of Y , otherwise in a weak sense.

Example: If $g \in \mathcal{B}_w$, i.e. $G = V_g(g) \in \mathcal{M}_c^R(L_w^1) \cap \mathcal{M}_c(L_w^1)$, then $(G(x_i^{-1}y))_{i \in I} \in l_w^1$ (by Prop. 2.5) and by the same argument $F(x_i) = V_g(f)(x_i) \in Y_d(X) \subseteq l_{1/w}^\infty$ (according to [FG2], Lemma 3.5). Therefore $\sum_{i \in I} F(x_i) L_{x_i} G(y)$ is defined for all $y \in \mathcal{G}$ by $l_{1/w}^\infty - l_w^1$ duality, hence the sum converges at least pointwise. In the sequel we show what else can be said about these sums.

Since the index set is countable we could enumerate it in some way and then interpret $\sum_{i \in I} \dots$ as the limit of the sequence of ordinary partial sums. Because Y and Y_d etc. are Banach lattices, this is equivalent to the first description.

With this interpretation in mind, it is never a problem to justify the use of the distribute law (summation against convolution) or the interchanging of sums and integrals. In the proofs these details will be omitted in most cases.

We need some preparation for the main results on the discretization operators:

4.4. Lemma. *For any $F \in \mathcal{M}_c(Y)$ and any uniform partition of unity Ψ subordinate to $Q \subseteq \mathcal{G}$ the following inequality holds true:*

$$\left\| \sum_{i \in I} |F(x_i)| \psi_i | \mathcal{M}(Y) \right\| \leq C \|F| \mathcal{M}_c(Y)\| \tag{4.7}$$

with an absolute constant C depending only on Q .

Proof: Note that in

$$M\left(\sum_{i \in I} |F(x_i)| \psi_i\right)(x) = \left\| L_x c_Q \cdot \left(\sum_{i \in I} |F(x_i)| \psi_i\right) \right\|_{\infty} = : A$$

the sum runs only over the finite index set

$$I_x = \{i \in I : x Q \cap x_i Q \neq \emptyset\} = \{i \in I : x_i \in x Q Q^{-1}\}.$$

Therefore, we continue with

$$A \leq \|L_x c_{Q Q^{-1}} \cdot F\|_{\infty} = M'(F)(x),$$

where M' is the maximal function taken with respect to $Q Q^{-1}$.

Therefore,

$$\begin{aligned} \left\| \sum |F(x_i)| \psi_i | \mathcal{M}(Y) \right\| &= \|M(\sum |F(x_i)| \psi_i) | Y\| \leq \|M'(F) | Y\| \leq \\ &\leq C \|M F | Y\| = C \|F| \mathcal{M}(Y)\| \end{aligned}$$

because different maximal functions yield equivalent norms on $\mathcal{M}(Y)$. (cf. 2.4. and [F1], Thm. 2) \square

4.5. Finally we need a related maximal function for the description of *local oscillations*. (For function spaces defined by local oscillations see also [S]).

Definition: If $U \subseteq \mathcal{G}$ is a compact neighbourhood of e , then

$$G_U^\#(x) = \sup_{u \in U} |G(ux) - G(x)| \tag{4.8}$$

is the U -oscillation of G .

4.6. Lemma. (i) A function G is in $\mathcal{M}^R(L_w^1)$ if and only if $G \in L_w^1$ and $G_U^\# \in L_w^1$ for one (and hence for all) compact neighbourhoods U of e .

(ii) If, in addition, G is continuous, then

$$\lim_{U \rightarrow \{e\}} \|G_U^\#|L_w^1\| = 0 \quad (4.9)$$

(iii) If $y \in xU$, then

$$|L_y G - L_x G| \leq L_y G_U^\# \quad (4.10)$$

holds pointwise, i.e.

$$|G(y^{-1}z) - G(x^{-1}z)| \leq G_U^\#(y^{-1}z) \quad \text{for all } z \in \mathcal{G}.$$

Proof: (i) Take the maximal function M^R of (2.11) with respect to U . Then

$$M^R G(x) = \|R_x c_U \cdot G\|_\infty \leq G_U^\#(x^{-1}) + |G(x^{-1})| \quad (4.11)$$

and consequently

$$\begin{aligned} \|G|\mathcal{M}^R(L_w^1)\| &= \|M^R G|L_w^1\| \leq \|G_U^\#|L_w^1\| + \|G|L_w^1\| = \\ &= \|G_U^\#|L_w^1\| + \|G|L_w^1\| \end{aligned} \quad (4.12)$$

Since $\mathcal{M}^R(L_w^1) \subseteq L_w^1$ and $G_U^\#(x) \leq M^R G(x^{-1}) + |G(x^{-1})|$ the converse is also obvious.

(ii) If, in addition, G is continuous, then by (i) $G \in \mathcal{M}_c^R(L_w^1)$. Note that $U \subseteq U_0$ implies $G_U^\# \leq G_{U_0}^\#$, therefore for $\varepsilon > 0$ a compact set $K \subseteq \mathcal{G}$ may be chosen such that

$$\int_{\mathcal{A}_K} G_U^\#(x) w(x) dx \leq \frac{\varepsilon}{2} \quad (4.13)$$

for all $U \subseteq U_0$. Since G is now uniformly continuous on K , we can find a neighbourhood $U_1 \subseteq U_0$ such that

$$G_{U_1}^\#(x) < \frac{\varepsilon}{2\lambda(K)w} \quad (4.14)$$

holds for all $x \in K$, where $w = \sup_{x \in K} w(x)$.

Consequently,
$$\int_K G_U^\#(x) w(x) dx < \frac{\varepsilon}{2} \quad (4.15)$$

whenever $U \subseteq U_1$, and all together yields

$$\|G_U^\#|L_w^1\| < \varepsilon \quad \text{for } U \subseteq U_1.$$

(iii) is a consequence of

$$|G(y^{-1}z) - G(x^{-1}z)| \leq \sup_{u \in U} |G(y^{-1}z) - G(uy^{-1}z)| = G_U^\#(y^{-1}z),$$

because $y \in xU$ implies $x^{-1} = uy^{-1}$. \square

4.7. If $G = V_g(g)$ then the condition of 4.6. (i) again implies that $g \in \mathcal{B}_w$. Now it is very easy to show the boundedness of the discretization operators T_Ψ , S_Ψ and U_Ψ .

4.8. Proposition. (i) *If $G \in L_w^1$, then $\{S_\Psi\}$, Ψ running through all uniform partitions of unity subordinate to Q , is a uniformly bounded family of operators from $\mathcal{M}_c(Y)$ into Y .*

(ii) *If $G \in \mathcal{M}^R(L_w^1)$, then $\{U_\Psi\}$ is a uniformly bounded operator family from $\mathcal{M}_c(Y)$ into Y .*

(iii) *If $G \in \mathcal{M}^R(L_w^1)$, then $\{T_\Psi\}$ is a uniformly bounded family of operators from Y into Y .*

Proof: (i) We use (2.6), the embedding $\mathcal{M}_c(Y) \rightarrow Y$ and Lemma 4.4. to obtain

$$\begin{aligned} \|S_\Psi F|Y\| &= \left\| \left(\sum F(x_i) \psi_i \right) * G|Y \right\| \leq \left\| \sum F(x_i) \psi_i|Y \right\| \|G|L_w^1\| \leq \\ &\leq \left\| \sum F(x_i) \psi_i|\mathcal{M}(Y) \right\| \|G|L_w^1\| \leq C \|F|\mathcal{M}_c(Y)\| \|G|L_w^1\| \end{aligned} \tag{4.16}$$

and the constant is independent of Ψ .

(ii) We use (4.10) and estimate first

$$|c_i L_{x_i} G - \psi_i * G| = \left| \int \psi_i(y) (L_{x_i} G - L_y G) dy \right| \leq \int \psi_i(y) L_y G_Q^\# dy = \psi_i * G_Q^\# \tag{4.17}$$

(because $\text{supp } \psi_i \subseteq x_i Q$). Therefore,

$$\begin{aligned} \|U_\Psi F - S_\Psi F|Y\| &= \left\| \sum_{i \in I} F(x_i) (c_i L_{x_i} G - \psi_i * G)|Y \right\| \leq \\ &\leq \left\| \left(\sum_{i \in I} |F(x_i)| \psi_i \right) * G_Q^\#|Y \right\| \end{aligned} \tag{4.18}$$

As in (4.16) we obtain

$$\|U_\Psi F - S_\Psi F|Y\| \leq C \|F| \mathcal{M}_c(Y)\| \|G_Q^\#|L_w^1\| \quad (4.19)$$

Finally, from (4.16) and (4.19)

$$\|U_\Psi F|Y\| \leq C (\|G|L_w^1\| + \|G_Q^\#|L_w^1\|) \|F| \mathcal{M}_c(Y)\| \quad (4.20)$$

holds with a bound independent of Ψ .

(iii) has already been shown in [FG2], Prop. 5.3. \square

4.9. Remarks: (i) Almost the same arguments show, that for any Q -dense and relatively separated set $X = (x_i)_{i \in I}$ in \mathcal{G} the corresponding “synthesis operators”

$$S_X(\lambda_i) = \left(\sum_{i \in I} \lambda_i \psi_i \right) * G \quad (4.21)$$

$$U_X(\lambda_i) = \sum \lambda_i c_i L_{x_i} G \quad (4.22)$$

and

$$T_X(\lambda_i) = \sum \lambda_i L_{x_i} G \quad (4.23)$$

are bounded from $Y_d(X)$ (cf. 2.3) into Y with a norm independent of X . This is needed in the “synthesis problem”. (cf. [FG2], and Thm. 5.2).

(ii) Note that in the original situation where $G = V_g(g)$ is an L_w^1 -convolution idempotent (because of (2.16)), the image of these operators is contained in the closed subspace $Y * G$ of Y . If $g \in \mathcal{B}_w$, i.e. $V_g(g) = G \in \mathcal{M}_c(L_w^1) \cap \mathcal{M}_c^R(L_w^1)$, then $Y * G$ is contained in $\mathcal{M}_c(Y)$ by Thm. 2.6. We conclude from Prop. 4.7., that the discretization operators S_Ψ , T_Ψ , U_Ψ are uniformly bounded from $Y * G$ into $Y * G$.

Now we can study how the convolution operator T , $TF = F * G$, is approximated by the discretizations S_Ψ , T_Ψ , U_Ψ . The use of the local oscillation $G_U^\#$ makes this almost trivial for T_Ψ (compared to [FG2], 5.4)

4.10. Theorem ([FG2], Prop. 5.4). *Assume that Y satisfies the standard assumptions of 2.2. (i)–(iv). If $G \in \mathcal{M}_c^R(L_w^1)$. Then*

$$\lim_{\Psi \rightarrow 0} \| \|T - T_\Psi|Y\| \| = 0, \quad (4.24)$$

i.e. for any $\varepsilon > 0$ there exists a neighbourhood U of e such that the operator norm on Y satisfies $\| \|T - T_\Psi|Y\| \| < \varepsilon$ for all uniform partitions of unity subordinate to U .

Proof: We use (4.10) and obtain pointwise

$$\begin{aligned}
 |TF - T_\Psi F| &= \left| \sum_{i \in I} \int F(y) \psi_i(y) (L_y G - L_{x_i} G) dy \right| \leq \\
 &\leq \sum_{i \in I} \int |F(y)| \psi_i(y) L_y G_U^\# dy = |F| * G_U^\# \quad (4.25)
 \end{aligned}$$

(Taking only partial sums over finite subsets and then taking the limit, shows that the interchange of summation and integration is perfectly justified). The norm in Y is therefore

$$\|TF - T_\Psi F|Y\| \leq \|F|Y\| \|G_U^\#|L_w^1\| \quad (4.26)$$

because of (2.6), and the operator norm in Y is

$$\|T - T_\Psi|Y\| \leq \|G_U^\#|L_w^1\| \quad (4.27)$$

Now apply Lemma 4.6(ii) \square

Whereas T_Ψ approximates T on the whole space Y and only $G \in \mathcal{M}_c(L_w^1)$ is required, for the treatment of S_Ψ and U_Ψ more structure is needed: the full power of the reproducing formula (3.15) and the interpretation of $L_x G$, $x \in \mathcal{G}$, as the reproducing kernel of $Y * G$.

4.11. Theorem. *Assume that Y satisfies 2.2 (i)—(iv) and that G is a self-adjoint convolution idempotent in $\mathcal{M}_c^R(L_w^1)$, i.e. $G = G^\nabla = G * G$. Then*

$$\lim_{\Psi \rightarrow 0} \|T - S_\Psi|Y * G\| = 0 \quad (4.28)$$

where the operator norm is taken of the restriction to the subspace $Y * G$.

Proof: With (4.5), (2.6), (3.15) and Lemma 4.4. one obtains

$$\|TF - S_\Psi F|Y\| \leq \|F - \sum_{i \in I} F(x_i) \psi_i|Y\| \|G|L_w^1\|. \quad (4.29)$$

Thus we are led to estimate the differences $F(y) - F(x_i)$ for $y \in x_i U$. For that purpose we use the reproducing kernel, i.e. the expression $F(x) = \langle L_x G, F \rangle$ (3.16) and the fundamental estimate (4.10) for $L_y G$ in Lemma 4.6. (iii).

For $y \in x_i U$, i.e. $x_i = y u^{-1}$, and $u \in U$ this yields

$$\begin{aligned}
 |F(y) - F(x_i)| &\leq \sup_{u \in U} |F(y) - F(y u^{-1})| = \sup_{u \in U} |\langle L_y G - L_{y u^{-1}} G, F \rangle| = \\
 &= \sup_{u \in U} \left| \iint_{\mathcal{G}} (\bar{G}(y^{-1} z) - \bar{G}(u y^{-1} z)) F(z) dz \right| \leq \\
 &\leq \int G_U^\#(y^{-1} z) |F(z)| dz = |F| * G_U^\#(y) \quad (4.30)
 \end{aligned}$$

Therefore,

$$\begin{aligned} \left| F(y) - \sum_{i \in I} F(x_i) \psi_i(y) \right| &\leq \sum_{i \in I} \sup_{u \in U} |F(y) - F(yu^{-1})| \psi_i(y) \leq \\ &\leq \sum_{i \in I} |F| * G_U^{\# \nabla}(y) \psi_i(y) = |F| * G_U^{\# \nabla}(y) \end{aligned} \quad (4.31)$$

where we have used (4.30) and finally

$$\begin{aligned} \|TF - S_\psi F|Y\| &\leq \|F - \sum_{i \in I} F(x_i) \psi_i|Y\| \|G|L_w^1\| \leq \\ &\leq \| |F| * G_U^{\# \nabla} |Y\| \|G|L_w^1\| \leq \|F|Y\| \|G_U^{\#}|L_w^1\| \|G|L_w^1\|. \end{aligned} \quad (4.32)$$

This is true under the assumption that $F = F * G$, consequently on $Y * G$ the operator norm is

$$\| \|T - S_\psi|Y * G\| \| \leq \|G_U^{\#}|L_w^1\| \|G|L_w^1\| \quad (4.33)$$

Again, Lemma 4.6(ii) proves the claim. \square

4.12. Remark: If one is inclined to introduce another function, the right U -oscillation

$$F_U^b(x) = \sup_{u \in U} |F(xu^{-1}) - F(x)| \quad (4.34)$$

then this proof can be rephrased in terms of the following pointwise inequalities:

$$(i) \quad F_U^b(x) = (F * G)_U^b(x) \leq |F| * (G_U^{\#})^\nabla(x) \quad (4.35a)$$

and

$$(ii) \quad |F - \sum F(x_i) \psi_i| \leq F_U^b \quad (4.35b)$$

It remains to prove the same for the discretization operator U_ψ . As in Prop. 4.8(ii) this case can be reduced to S_ψ .

4.13. Theorem. *Under the assumptions of Thm. 4.11*

$$\lim_{\psi \rightarrow 0} \| \|T - U_\psi|Y * G\| \| = 0 \quad (4.36)$$

holds true.

Proof: The inequality

$$\| \|T - S_\psi|Y * G\| \| \leq \|G_U^{\#}|L_w^1\| \|G|L_w^1\|$$

has just been proved (4.33). If we take into account that by (2.13)

$$\|F|\mathcal{M}_c(Y)\| \leq \|F|Y\| \|G|\mathcal{M}_c(L_w^1)\| \quad (4.37)$$

estimate (4.19) turns into

$$\|U_\Psi F - S_\Psi F|Y\| \leq c \|G|\mathcal{M}_c(L_w^1)\| \|G_U^\#|L_w^1\| \|F|Y\| \quad (4.38)$$

(Ψ is now subordinate to U , the constant c can be given more explicitly). Combining these estimates proves Thm. 4.13:

$$\| \|T - U_\Psi|Y * G\| \| \leq \|G_U^\#|L_w^1\| \|G|L_w^1\| + c \|G_U^\#|L_w^1\| \|G|\mathcal{M}_c(L_w^1)\| \quad (4.39)$$

4.14. Remark: The estimate of $\| \|T - T_\Psi|Y\| \|$ by $\|G_U^\#|L_w^1\|$ in (4.27) is slightly sharper than the one given in [FG2]. There the modulus of continuity $\Omega_U(G)$ was used, and it was shown in Section 6 that $G_U^\# \in \mathcal{M}^R(L_w^1)$ and that $\|G_U^\#|L_w^1\| \leq \|G_U^\#|\mathcal{M}^R(L_w^1)\| \leq \Omega_U(G)$. In numerical work this observation might lead to more efficient (i.e. less redundant) decompositions of functions.

5. Atomic Decompositions and Frames

In this section we have a rich harvest. Almost without effort the atomic decompositions and Banach frames for general coorbit spaces are obtained from the results on the discretization operators. The general theorem will be illustrated on some of the examples of Section 3, and a discussion and comparison with other results will conclude this section.

5.1. Recall once more that π is an irreducible, unitary representation of \mathcal{G} which is w -integrable (2.7 and 2.8) and that Y is a function space on \mathcal{G} satisfying conditions 2.2 (i)—(iv) with canonical weight w (2.5). We considered the coorbit of Y under π , $\mathcal{C}_o Y$, and found that it can be identified with a subspace of Y that has a reproducing kernel. For any fixed element $g \in \mathcal{A}_w$ with the normalization $\|Ag\| = 1$ this subspace consists exactly of all functions $F = V_g(f)$, $f \in \mathcal{C}_o Y$, and the reproducing formula $F * G = F$ holds true for exactly these functions by Thm. 3.4 ($G = V_g(g)$).

In Section 4 we tried to approximate this convolution T by the discretization operators T_Ψ , S_Ψ , U_Ψ . But because T is a projection onto

$Y * G$ (cf. 2.8), it is the identity operator on the relevant subspaces $Y * G \cong \mathcal{C} \circ Y$. Therefore the essence of Thms. 4.10, 4.11 and 4.13 is that T_Ψ , S_Ψ , U_Ψ are invertible on $Y * G$ as soon as the partition of unity Ψ is fine enough! This is sufficient to extract the atomic decomposition and a frame for $\mathcal{C} \circ Y$.

5.2. Theorem T. *Assume that Y satisfies 2.2 (i)—(iv), that w is the canonical weight, that the irreducible, unitary representation π is w -integrable and choose $g \in \mathcal{B}_w$ normalized by $\|Ag\| = 1$. Choose a neighbourhood U so small that*

$$\|G_U^\# |L_w^1\| < 1 \quad (5.1)$$

Then for any U -dense and relatively separated set $X = (x_i)_{i \in I}$, $\mathcal{C} \circ Y$ has the following atomic decomposition: If $f \in \mathcal{C} \circ Y$, then

$$f = \sum_{i \in I} \lambda_i(f) \pi(x_i) g \quad (5.2)$$

where the sequence of coefficients $\Lambda(f) = (\lambda_i(f))_{i \in I}$ depends linearly on f and satisfies

$$\|\Lambda(f)\|_{Y_d(X)} \leq C \|f\|_{\mathcal{C} \circ Y} \quad (5.3)$$

with a constant C depending only on g .

Conversely, if $\Lambda = (\lambda_i) \in Y_d(X)$, then $f = \sum_{i \in I} \lambda_i \pi(x_i) g$ is in $\mathcal{C} \circ Y$ and

$$\|f\|_{\mathcal{C} \circ Y} \leq C' \|\Lambda\|_{Y_d(X)} \quad (5.4)$$

The convergence of the sums (5.2) and (5.3) is in the norm of $\mathcal{C} \circ Y$, if the bounded functions of compact support are dense in Y , and is w^ in $\mathcal{H}_w^{1 \sim}$ otherwise.*

Proof: Note first that condition (5.1) can be satisfied as a consequence of Lemma 4.6(ii). By the preceding discussion and Thm. 4.10, in particular (4.27), (5.1) implies that T_Ψ is invertible on $Y * G$. Therefore for $F = V_g(f) \in Y * G$

$$F = T_\Psi T_\Psi^{-1} F = \sum_{i \in I} \langle \psi_i, T_\Psi^{-1} F \rangle L_{x_i} G \quad (5.5)$$

and by the Correspondence Principle (Thm. 3.4 and (3.17))

$$f = \sum_{i \in I} \langle \psi_i, T_\Psi^{-1} V_g(f) \rangle \pi(x_i) g \quad (5.6)$$

yields the desired decomposition.

The other assertions follow as in [FG2], Thm 6.1. \square

Theorem T is the main result of [FG2]; what is new here, is the explicit condition (5.1) on the size of U . As we remarked earlier, (5.1) enables us to choose U larger than in [FG2]. For more details on the “ T -method” and the remaining details of the proof we refer to [FG2].

We now turn to the existence of coherent frames for $\text{Co } Y$. From the atomic decomposition and the results of [FG3], the structure of $\mathcal{C}_o Y$ is known well enough to guess that the sequence space that appears with these frames is nothing else but Y_d (from 2.3).

5.3. Theorem S. *Impose the same assumptions on Y , w , π and $g \in \mathcal{B}_w$ as in Theorem 5.2. Choose the neighbourhood U of e such that*

$$\|G_U^\#|L_w^1\| < 1/\|G|L_w^1\| \quad (5.7)$$

Then for every U -dense and relatively separated family $X = (x_i)_{i \in I}$ in \mathcal{G} the set $\{\pi(x_i)g : i \in I\}$ is a Banach frame for $\mathcal{C}_o Y$. This means that

$$(i) \quad f \in \mathcal{C}_o Y \Leftrightarrow (\langle \pi(x_i)g, f \rangle)_{i \in I} \in Y_d(X) \quad (5.8)$$

(ii) There exist two constants $C_1, C_2 > 0$ depending only on $g \in \mathcal{B}_w$ such that

$$C_1 \|f|_{\mathcal{C}_o Y}\| \leq \| \langle \pi(x_i)g, f \rangle_{i \in I} \|_{Y_d(X)} \leq C_2 \|f|_{\mathcal{C}_o Y}\| \quad (5.9)$$

(iii) f can be unambiguously reconstructed from the coefficients $\langle \pi(x_i)g, f \rangle, i \in I$. If L_w^∞ , the space of bounded functions with compact support, is dense in Y , this reconstruction may be achieved as follows: there exists a system $e_i \in \mathcal{H}_w^1, i \in I$, such that

$$f = \sum_{i \in I} \langle \pi(x_i)g, f \rangle e_i \quad (5.10)$$

with convergence in $\mathcal{C}_o Y$.

Proof: By the same reasoning as above, condition (5.7) implies that S_ψ is invertible on $Y * G$. (see Thm. 4.11 and the estimate (4.33)). For $F = V_g(f) \in Y * G, f \in \mathcal{C}_o Y$, we obtain therefore the representation:

$$F = S_\psi^{-1} S_\psi F = S_\psi^{-1} \left(\sum_{i \in I} F(x_i) \psi_i * G \right) \quad (5.11)$$

This is a reconstruction of F starting from the coefficients $F(x_i)$. By the Correspondence Principle Thm. 3.4, we obtain

$$f = V_g^{-1} S_\psi^{-1} \left(\sum_{i \in I} \langle \pi(x_i)g, f \rangle \psi_i * G \right). \quad (5.12)$$

Next we observe that S_ψ is simultaneously invertible on all spaces $Y * G$ provided that the canonical weights w of Y are the same. Therefore the reconstruction (5.12) is valid simultaneously for all $\mathcal{C}_o Y$, $\mathcal{H}_w^1 \rightarrow \mathcal{C}_o Y \rightarrow \mathcal{H}_w^{1\sim}$. This follows from (4.33), which depends only on g and w , not on Y .

Thus any $f \in \mathcal{H}_w^{1\sim}$ is reconstructed this way. If for some $f \in \mathcal{H}_w^{1\sim}$ $V_g(f)(x_i) \in Y_d(X)$, then (5.11) yields a function F in $Y * G$ (use Remark 4.9 and $S_\psi^{-1}: Y * G \rightarrow Y * G$) and $f = V_g^{-1}(F) \in \mathcal{C}_o Y$. The converse, $f \in \mathcal{C}_o Y \Rightarrow \langle \pi(x_i)g, f \rangle_{i \in I} \in Y_d(X)$, is contained in Thm. 3.6.

The equivalence of norms (5.9) follows readily from (5.11)

$$\begin{aligned}
\|f|_{\mathcal{C}_o Y}\| &= \|F|_Y\| \leq \|S_\psi^{-1}|_{Y * G}\| \left\| \sum_{i \in I} F(x_i) \psi_i * G \right\|_Y \leq \\
&\leq \|S_\psi^{-1}\| \left\| \sum_{i \in I} F(x_i) \psi_i \right\|_Y \|G|_{L_w^1}\| \leq \\
&\leq \|S_\psi^{-1}\| \|G|_{L_w^1}\| \|(F(x_i))_{i \in I}\|_{Y_d(X)} \leq \\
&\leq C \|S_\psi^{-1}\| \|G|_{L_w^1}\| \|f|_{\mathcal{C}_o Y}\| \tag{5.13}
\end{aligned}$$

where we have used (2.6), the definition of Y_d , and Thm. 3.6.

It remains to prove the explicit representation (5.10) in the case that L_w^∞ is dense in Y . In this case the following lemma can be proved:

5.4. Lemma. *Set $E_i = S_\psi^{-1}(\psi_i * G)$. Then E_i is in $L_w^1 * G \subseteq \mathcal{M}_c(L_w^1)$ and the operator $\Lambda = (\lambda_i)_{i \in I} \rightarrow \sum_{i \in I} \lambda_i E_i$ is bounded from $Y_d(X)$ into $Y * G$.*

Proof: As argued above, S_ψ is also invertible on $L_w^1 * G$ under condition (5.7), and therefore $S_\psi^{-1}(\psi_i * G) := E_i$ is in $L_w^1 * G$ and in $\mathcal{M}_c(L_w^1)$ (by Thm. 3.6 once more). Now let $E \subseteq I$ be finite and $\Lambda = (\lambda_i) \in Y_d(X)$: then

$$\begin{aligned}
\left\| \sum_{i \in E} \lambda_i E_i \right\|_Y &= \left\| S_\psi^{-1} \left(\sum_{i \in E} \lambda_i \psi_i * G \right) \right\|_Y \leq \\
&\leq \|S_\psi^{-1}|_{Y * G}\| \|G|_{L_w^1}\| \|(\lambda_i)_{i \in E}\|_{Y_d(X)} \tag{5.13a}
\end{aligned}$$

is obtained as in (5.13).

Because the bounded functions with compact support are supposed to be dense in Y , the finite sequences are dense in $Y_d(X)$ ([FG2], Lemma 3.5). Therefore the mapping $\Lambda \rightarrow \sum \lambda_i E_i$ extends to a bounded mapping from $Y_d(X)$ into $Y * G$. \square

We have $g \in \mathcal{B}_w$, $f \in \mathcal{C}_o Y$, $F = V_g(f)$, and Thm. 3.6. implies $(F(x_i))_{i \in I} \in Y_d(X)$. By the Lemma the expression $\sum F(x_i) S_{\bar{\psi}}^{-1}(\psi_i * G)$ is a well-defined function in $Y * G$. Now the approximation by partial sums over finite index sets shows that we can interchange $S_{\bar{\psi}}^{-1}$ and the summation in (5.11) to obtain

$$F = S_{\bar{\psi}}^{-1}(\sum F(x_i) \psi_i * G) = \sum F(x_i) E_i. \quad (5.14)$$

Call $e_i \in \mathcal{H}_w^1$ the unique vector, such that $V_g(e_i) = E_i \in L_w^1 * G$, then the Correspondence Principle applied to (5.14) yields

$$f = \sum_{i \in I} \langle \pi(x_i)g, f \rangle e_i,$$

as desired. The convergence in $\mathcal{C}_o Y$ follows from (5.13). Thus Thm. S is proved completely. \square

Before we further exploit this construction of frames, let us formulate the analogous theorem for the discretization operators U_{ψ} .

5.5. Theorem U. *Assume that Y , w , π and $g \in \mathcal{B}_w$ are the same as in Theorem 5.2 and 5.3. Choose the neighbourhood U small enough, such that*

$$\|G_U^\# \|L_w^1\| (\|G \|L_w^1\| + c \|G \| \mathcal{M}_c(L_w^1)\|) < 1 \quad (5.15)$$

Then for any U -dense and relatively separated family $X = (x_i)_{i \in I}$ in \mathcal{G} , the set $\{\pi(x_i)g, i \in I\}$ is both a set of atoms and a Banach frame for $\mathcal{C}_o Y$. Moreover, there exists a ‘‘dual frame’’ $\{e_i, i \in I\}$ in \mathcal{H}_w^1 such that

(i) *The following norms are equivalent:*

$$\|f\|_{\mathcal{C}_o Y} \cong \|\langle e_i, f \rangle_{i \in I} Y_d(X)\| \cong \|\langle \pi(x_i)g, f \rangle_{i \in I} Y_d(X)\|. \quad (5.16)$$

(ii) *For $f \in \mathcal{C}_o Y$*

$$f = \sum_{i \in I} \langle e_i, f \rangle \pi(x_i)g, \quad (5.17)$$

with norm convergence in $\mathcal{C}_o Y$, if L_o^∞ is dense in Y , and w^ -convergence in $\mathcal{H}_w^{1 \sim}$ otherwise.*

(iii) *If L_o^∞ is dense in Y , then the decomposition*

$$f = \sum_{i \in I} \langle \pi(x_i)g, f \rangle e_i \quad (5.18)$$

is also valid for $f \in \mathcal{C}_o Y$.

Proof: As in the proofs of Thm. T and Thm. S, the stated condition (5.14) implies that the discretization operator U_Ψ is invertible on $Y * G$. Therefore for $f \in \mathcal{C}_0 Y$, $F = V_g(f) \in Y * G$

$$F = U_\Psi U_\Psi^{-1} F = \sum_{i \in I} (U_\Psi^{-1} F)(x_i) c_i L_{x_i} G \quad (5.19)$$

and

$$F = U_\Psi^{-1} U_\Psi F = U_\Psi^{-1} \left(\sum_{i \in I} F(x_i) c_i L_{x_i} G \right). \quad (5.20)$$

The rest of the proof now repeats the arguments given earlier, i.e. (5.19) leads to the atomic decomposition (5.17) proceeding as in Thm. T, and (5.20) ends with the construction of a frame for $\mathcal{C}_0 Y$ and (5.18).

Set $E_i = c_i U_\Psi^{-1}(L_{x_i} G)$, then $E_i \in L_w^1 * G$ and $E_i = V_g(e_i)$ for a unique $e_i \in \mathcal{H}_w^1$. Then

$$f = \sum_{i \in I} \langle \pi(x_i) g, f \rangle e_i$$

provided that L_σ^∞ is dense in Y .

$$\text{Claim: } U_\Psi^{-1} F(x_i) = \langle e_i, f \rangle \quad (5.21)$$

By construction, $U_\Psi^{-1} F$ is in the space $Y * G$ with reproducing kernel $L_x G$, therefore by (3.16) $U_\Psi^{-1} F(x_i) = \langle L_{x_i} G, U_\Psi^{-1} F \rangle$. Now a simple computation shows that U_Ψ is “self-adjoint” with respect to $\langle \cdot, \cdot \rangle$, i.e. $\langle L_x G, U_\Psi F \rangle = \langle U_\Psi L_x G, F \rangle$ for all $F \in L_{1/w}^\infty * G (\cong Y * G)$ and all $x \in \mathcal{X}$. Therefore the same applies to $U_\Psi^{-1} = \sum_{n=0}^{\infty} (Id - U_\Psi)^n$:

$$\langle L_x G, U_\Psi^{-1} F \rangle = \langle U_\Psi^{-1} L_x G, F \rangle.$$

Now (5.21) follows:

$$\begin{aligned} c_i (U_\Psi^{-1} F)(x_i) &= c_i \langle L_{x_i} G, U_\Psi^{-1} F \rangle = c_i \langle U_\Psi^{-1} L_{x_i} G, F \rangle = \\ &= \langle V_g(e_i), V_g(f) \rangle = \langle e_i, f \rangle \end{aligned} \quad (5.22)$$

With this final piece of information and the Correspondence Principle, (5.19) becomes

$$f = \sum_{i \in I} \langle e_i, f \rangle \pi(x_i) g,$$

and the equivalence of the norms is shown by the following estimate

$$\begin{aligned} \|f|_{\mathcal{C}_0} Y\| &\leq C \|\langle e_i, f \rangle_{i \in I} Y_d(X)\| = C \|(c_i U_{\bar{\psi}}^{-1} F(x_i))_{i \in I} Y_d(X)\| \leq \\ &\leq C' \|U_{\bar{\psi}}^{-1} F|Y * G\| \leq C'' \|F|Y\| = C'' \|f|_{\mathcal{C}_0} Y\|, \end{aligned} \quad (5.23)$$

where the first inequality follows from $U_{\bar{\psi}}^{-1} F \varepsilon Y * G$ and Thm. 3.6. as usual. \square

In view of the examples of Section 3 Thms. T, S, U contain all the atomic decompositions with respect to coherent states so far known and provide the first construction of frames for these spaces. Let us write down explicitly what this means in some of the examples. All necessary calculations are straightforward and left to the reader.

5.6. *Examples.* (a) Wavelet Theory: An atomic decomposition for the spaces of Besov – Triebel – Lizorkin type $\dot{B}_{p,q}^s$ or $\dot{F}_{p,q}^s$ on \mathbb{R}^n is of the form

$$f(x) = \sum_{\substack{j \in \mathbb{Z} \\ k \in \mathbb{Z}^n}} \lambda_{j,k} \beta^{-nj/2} g(\beta^{-j}x - \alpha k) \quad (5.24)$$

Here we use the lattice $x_i \sim (\alpha k \beta^j, \beta^j)$ in the $ax + b$ -group as the simplest example of a U -dense and relatively separated set in \mathcal{G} . $\alpha > 0$ and $\beta > 1$ depend only on g (and possibly on the smoothness parameter s) in the way specified in Thms. T, S, U.

A very general condition for g to be in \mathcal{B}_w , $w(x, t) = t^{-s}$, is

$$\iint |\langle \pi(\bar{x}, t)g, g \rangle| (1 + t + |\bar{x}|)(t^{-s} + t^s) \frac{d\bar{x} dt}{t^{n+1}} < \infty \quad (5.25)$$

in other words, g is in a certain *weighted* Besov space. Finally, the conditions on the coefficients $\Lambda = (\lambda_{j,k})$ are (cf. also [LM] and [G2] for the explicit calculation of the sequence spaces)

$$f \in \dot{B}_{p,q}^s \Leftrightarrow \|f|_{\dot{B}_{p,q}^s}\| \cong \|\Lambda|_{s,p,q}\| = \left(\sum_j \left(\sum_k |\lambda_{j,k}|^p \right)^{q/p} \beta^{-jq(s + n/2 - np)} \right)^{1/q} < \infty \quad (5.26)$$

and

$$\begin{aligned} f \in \dot{F}_{p,q}^s &\Leftrightarrow \|f|_{\dot{F}_{p,q}^s}\| \cong \|\Lambda|_{p,q}^s\| = \\ &= \left\| \left(\sum_{j,k} |\lambda_{j,k}|^q \beta^{-jq(s + n/2)} c_{[\beta^j \alpha k, \beta^j \alpha(k+1)]}(x) \right)^{1/q} \Big| L^p(dx) \right\| < \infty \end{aligned} \quad (5.27)$$

(with nontrivial modifications if $q = \infty$). On the other hand, the same functions $g_{j,k}$, given as $g_{j,k} = \pi(\alpha k \beta^j, \beta^j) g = \beta^{-j n/2} g(\beta^{-j}x - \alpha k)$ are

frames for all these spaces. Thus, $f \in \dot{B}_{p,q}^s$ is completely determined by the coefficients $\langle g_{j,k}, f \rangle$, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$, which have a nice interpretation in signal analysis. The algorithm described in the proofs of the theorems allows to construct f provided that only the values $\langle g_{j,k}, f \rangle$ are known. Moreover

$$\|f|_{\dot{B}_{p,q}^s}\| \cong \|\langle g_{j,k}, f \rangle_{j,k}|_{l_s^{p,q}}\|.$$

Similarly, $f \in \dot{F}_{p,q}^s$ if and only if

$$\|f|_{\dot{F}_{p,q}^s}\| \cong \|\langle g_{j,k}, f \rangle_{j,k}|_{t_s^{p,q}}\|.$$

Since $L^p(\mathbb{R}^n)$, all Sobolev spaces, the real Hardy space \mathcal{H}^1 , BMO and many others are $\dot{F}_{p,q}^s$, cf. [T2] this description applies to most function spaces in classical analysis. Since much of classical analysis is focussed around this example, it is not surprising that other treatments of wavelet theory exist. In [FJ3] similar non-orthogonal expansions are obtained by discretizing Calderon's reproducing formula. More powerful tools are the orthogonal bases for these spaces, [LM], [M], [D2], [MA] etc. Both types of description are useful in applications: The orthogonal bases, when a concise characterization of a function without redundancy is important, but the form of the basic wavelet g is not essential, the non-orthogonal expansions and frames, when the basic function g is given by the problem and flexibility is required.

Let us mention that a discussion of Besov–Triebel–Lizorkin spaces on the Heisenberg groups \mathbb{H}_n — a little explored topic — is analogous and requires no additional efforts.

(b) The modulation spaces of (3.9) have a decomposition of the form

$$f(x) = \sum_{m,k \in \mathbb{Z}^n} \lambda_{m,k} e^{im\alpha x} g(x - \beta k) \quad (5.28)$$

where $\alpha, \beta > 0$ depend only on g . There are easy necessary and sufficient conditions for g to be a basic atom. The Bessel potential space $M_{2,2}^s = \{f \in \mathcal{S}' : \int |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s d\xi < \infty\}$ is characterized by the condition

$$\|f|_{M_{2,2}^s}\| \cong \|(\lambda_{m,k})|_{l_s^2}\| = \left(\sum_{m,k} |\lambda_{m,k}|^2 (1 + |m|^2)^s \right)^{1/2} < \infty \quad (5.29)$$

(In (5.28) we have put the trivial action of the third coordinate of \mathbb{H}_n by scalars into the coefficients. This simple change does not affect any of our considerations).

Dually to (5.28) the set $g_{m,k}(x) = e^{im\alpha x} g(x - \beta k)$, $m, k \in \mathbb{Z}^n$, is a frame for the $M_{p,q}^s$ e.g.

$$\|\langle g_{m,k}, f \rangle_{m,k}\|_{l_s^2} \cong \|f\|_{M_{2,2}^s} \quad (5.30)$$

with the possibility of the complete reconstruction of f from the values $\langle g_{m,k}, f \rangle$ by one of the algorithms of Thm. S and U .

Thus in this approach the construction of frames for other spaces than $L^2(\mathbb{R}^n)$ poses no additional problem. Moreover, (5.30) has an important consequence for signal analysis: if the basic atom g has compact support, then the representation coefficients $\langle \pi(\bar{x}, \bar{y}, 1)g, f \rangle$ are known as the short time Fourier transform of f . By (5.30) it suffices to know the STFT on a discrete lattice only in order to store the full information on f , see also [D2], [BA1]. The general theory shows that it need not even be a lattice, but it may be a rather irregular sampling of the STFT. Furthermore, the smoothness of f is reflected in the decay of the coefficients $\langle g_{n,k}, f \rangle$.

(c) Mapping example (b) to the Bargmann–Fock representation, the existence of frames for the general Bargmann–Fock spaces F_α^p follows easily and does not require any additional efforts (compare [DG]). If $G(z)$ is in \mathcal{F}_α^1 , i.e. G is holomorphic on \mathbb{C}^n and

$$\int_{\mathbb{C}^n} |G(z)| e^{-\alpha|z|^2/2} dz < \infty \quad (5.31)$$

then for $\beta > 0$, $\gamma > 0$ small enough

$$G_{m,k}(z) = \exp(\sqrt{\alpha}(\beta m - i\gamma k)z - (\beta^2 m^2 + \gamma^2 k^2)/2) \cdot G(z - (\beta m + i\gamma k)/\sqrt{\alpha}) \quad (5.32)$$

is a frame for \mathcal{F}_α^p (see (3.12)) and

$$F \in \mathcal{F}_\alpha^p \Leftrightarrow \langle G_{m,k}, F \rangle \in l^p \quad (5.33)$$

If G is taken to be the reproducing kernel of \mathcal{F}_α^p , one recovers easily the sampling theorem Theorem 8.4 of [JPR]. It is left to the reader to write down frames and atomic decompositions for the other examples of Section 3.

Discussion of the Theorems

5.7. The frame operator D : Assume the same situation as in Thm. U and choose the neighbourhood U small enough to make the discretization operator U_ψ invertible on $Y * G$. (condition 5.15). Set

$$D: \mathcal{C}_0 Y \rightarrow \mathcal{C}_0 Y \tag{5.34}$$

$$Df = V_g^{-1} U_\psi V_g(f) = \sum_{i \in I} \langle \pi(x_i)g, f \rangle c_i \pi(x_i)g.$$

Since U_ψ is invertible on $Y * G$ and $U_\psi^{-1} = \sum_{n=0}^{\infty} (\text{Id} - U_\psi)^n$ on $Y * G$, we find that D is invertible on $\mathcal{C}_0 Y$ and

$$D^{-1} = \sum_{n=0}^{\infty} (\text{Id} - D)^n. \tag{5.35}$$

The reconstruction of f from the frame coefficients $\langle \pi(x_i)g, f \rangle$ now involves only operations in $\mathcal{C}_0 Y$ and (5.35) is a “simple” iteration algorithm to do that. The first approximation of f is of course Df , higher approximations require the repeated application of $\text{Id} - D$. Numerically, this means just repeated multiplication with the matrix

$$g_{ij} = \langle \pi(x_i)g, \pi(x_j)g \rangle \tag{5.36}$$

and all information on the numeric procedure is completely contained in this matrix!

Thus for a numerical implementation one may completely disregard the group theoretic background of the problem.

In the special case of the original Hilbert space \mathcal{H} one recovers the technique of I. DAUBECHIES [D1]. In \mathcal{H} the situation is very simple: the mere existence of frame bounds, i.e. the equivalence $\|f\|_{\mathcal{H}} \cong \|\langle \pi(x_i)g, f \rangle_{i \in I}\|_{l^2}$, already guarantees the invertibility of D . This argument seems to break down completely outside Hilbert space. Moreover, the frame bounds seem to play only a minor role in the general case; as we have seen, the invertibility and the quality of approximation of f by Df are determined by $\|G_\psi^\#|L_w^1\|$.

5.8. (a) By construction, the family $\{\pi(x_i)g, i \in I\}$ is a frame simultaneously for all spaces $\mathcal{C}_0 Y$, where $\mathcal{H}_w^1 \rightarrow \mathcal{C}_0 Y \rightarrow \mathcal{H}_w^{1\sim}$ (provided that $g \in \mathcal{B}_w$)

(b) The dual frame $\{e_i\}$, i.e. the vectors for the synthesis of f from the coefficients $\langle \pi(x_i)g, f \rangle$ (5.18), belongs automatically to \mathcal{H}_w^1 . Therefore the pathologies that are described in [D1] do not occur here.

(c) The sequence $\langle \pi(x_i)g, f \rangle$ characterizes to which space f belongs. Even if the interest is only directed to the Hilbert space case, this remark enables us to detect smoothness of f and may increase the quality and speed of reconstruction. Such questions obviously do not make sense in an approach with only one space under consideration.

(d) In the examples related to the Heisenberg group and the $ax + b$ -group it is easy to give explicit sufficient conditions for $g \in \mathcal{B}_w$. This makes the choice of the basic atom g highly flexible and allows to take g according to one's purposes, e.g. in examples (a) and (b) of 3.3. g may be chosen to have compact support, or vanish on certain intervals or be band-limited etc. This flexibility is important in many applications.

(e) All existing theories of frames work only with lattices (x_i) in the group, e.g. those presented in 2.3. This is certainly highly desirable and is the most economic way for computations, but it does not always occur in practice where sampling errors have to be taken into account. We consider it an important message of the general theory that a complete reconstruction of functions is guaranteed even for highly irregular samplings.

5.9. Formulas (5.13) and (5.19) have an interesting interpretation on the level of representation coefficients. A general representation coefficient $\langle \pi(x)g, f \rangle$ is completely determined by its values on a sufficiently dense, discrete subset (x_i) in \mathcal{G} . This reminds strikingly of the Shannon–Whittaker sampling theorem for band-limited functions and shows that representation coefficients of the discrete series exhibit a behaviour similar to holomorphic functions.

5.10. Duality of frames and atomic decompositions: If Y has an absolutely continuous norm, then the dual $(\mathcal{C}_o Y)' = \mathcal{C}_o Y^a$ is again a coorbit space embedded into $\mathcal{H}_w^{1\sim}$. ([FG2], Thm. 4.9). Under this hypothesis on the norm of Y , the atomic decomposition

$$f = \sum_{i \in I} \langle e_i, f \rangle \pi(x_i)g$$

is norm convergent for all $f \in \mathcal{C}_o Y$ and this implies at least w^* -convergence of

$$h = \sum_{i \in I} \langle \pi(x_i)g, h \rangle e_i \quad (5.37)$$

for $h \in (\mathcal{C} \circ Y)' = \mathcal{C} \circ Y^a$. Moreover, with (5.37) and Thm. 3.6 it is easy to see that $\|h\|_{\mathcal{C} \circ Y^a}$ and $\|\langle \pi(x_i)g, h \rangle_{i \in I}\|_{Y_d^a}$ are equivalent. Thus under this weak hypothesis the set of atoms $\{\pi(x_i)g\}$ for $\mathcal{C} \circ Y$ is a frame for the dual space $(\mathcal{C} \circ Y)'$, and vice versa by the same argument.

5.11. Corollary. ([Y], p. 189). *In a Hilbert space sets of atoms and frames coincide.*

This duality of atomic decompositions and frames is also built in the discretization operators: S_Ψ and T_Ψ are adjoint to each other at least in a formal sense.

6. Complementary Results

In this final section, a few converse results are given, which should help to clarify the meaning of some of the assumptions in the main theorems.

Recall that in the construction of a frame or an atomic decomposition for $\mathcal{C} \circ Y$ one starts with a basic atom $g \in \mathcal{B}_w$. Then a neighbourhood U of e can be found such that $\{\pi(x_i)g, i \in I\}$ is a frame and a set of atoms, respectively, for any U -dense and relatively separated set $(x_i)_{i \in I}$. In the theorems the size of U is determined by the explicit formulas (5.1), (5.7) and (5.15).

6.1. The necessary conditions can be understood roughly as follows: the “atoms” $\pi(x_i)g$ must of course be elements of $\mathcal{C} \circ Y$; but since the construction should work simultaneously for all coorbit spaces $\mathcal{C} \circ Y \rightarrow \mathcal{H}_w^1$, this means that the $\{\pi(x_i)g\}$ must be contained in the minimal space which is embedded into all these $\text{Co } Y$. This minimal space is precisely \mathcal{H}_w^1 ([FG1], Cor. 4.7), therefore at least $\pi(x_i)g \in \mathcal{H}_w^1$. In order to sum up infinite series, additional properties are required, i.e. $g \in \mathcal{B}_w$, which on the level of the examples can be expressed by decay conditions, smoothness and cancellation properties.

In order to get a frame, the coefficient mapping $f \rightarrow \langle \pi(x_i)g, f \rangle_{i \in I}$ must be one-to-one at least, which indicates that a sufficient density of the (x_i) is required. Next we want to sum up $\sum_{i \in I} \lambda_i \pi(x_i)g$ for general coefficients $(\lambda_i) \in Y_d(X) (\subseteq l_{1/w}^\infty(X)$ by [FG2], Lemma 3.5). This requires that the (x_i) be separated, otherwise such a series might not be convergent. This qualitative discussion is made more precise by

6.2. Proposition. *Suppose that the “operator of synthesis” $T_X: (\lambda_i)_{i \in I} \rightarrow \sum_{i \in I} \lambda_i \pi(x_i) g$ is bounded from $l_{1/w}^\infty$ into $\mathcal{H}_w^{1\sim}$ for all relatively separated sets $X = (x_i)_{i \in I}$ in \mathcal{G} . Then $g \in \mathcal{B}_w$.*

Proof: We choose a fixed vector $h \in \mathcal{A}_w$ and take all representation coefficients with respect to h . Then for $\Lambda = (\lambda_i)_{i \in I} \in l_{1/w}^\infty$ we obtain

$$V_h(T_X \Lambda)(y) = V_h\left(\sum_{i \in I} \lambda_i \pi(x_i) g\right)(y) = \sum_{i \in I} \lambda_i V_h(g)(x_i^{-1} y) \quad (6.1)$$

By assumption, for any relatively separated $X = (x_i)_{i \in I}$

$$\|V_h(T_X \Lambda)|_{l_{1/w}^\infty}\| \cong \|T_X \Lambda|_{\mathcal{H}_w^{1\sim}}\| \leq C_X \|\Lambda|_{l_{1/w}^\infty}\| \quad (6.2)$$

holds true for all $\Lambda \in l_{1/w}^\infty$. (6.1) and (6.2) imply that

$$V_h(g)(x_i^{-1} y) = \overline{V_g(h)(y^{-1} x_i)} \in l_w^1 \quad (6.3)$$

Since this is true for all relatively separated sets $X = (x_i)_{i \in I}$, this implies that $V_g(h) \in \mathcal{M}_c(L_w^1)$ (by Lemma 3.8 in [FG2]). Using the orthogonality relations (2.15) (with the special choice $g_1 = g_2 = h, f_1 = \pi(y)g, f_2 = g$) enables us to isolate $V_g(g)$:

$$\|Ah\|^2 V_g(g) = V_h(g) * V_g(h) \in L_w^1 * \mathcal{M}_c(L_w^1) \subseteq \mathcal{M}_c(L_w^1) \quad (6.4)$$

by means of (2.13). Therefore

$$V_g(g) = V_g(g)^\vee \in \mathcal{M}_c^R(L_w^1) \cap \mathcal{M}_c(L_w^1)$$

or $g \in \mathcal{B}_w$ by Def. 3.5. \square

6.3. Thus the condition $g \in \mathcal{B}_w$ is necessary. In groups with a compact invariant neighbourhood, e.g. the Heisenberg group, the basic spaces \mathcal{B}_w and \mathcal{H}_w^1 coincide ([FG3], Lemma 7.2) and there is no problem to describe the basic atoms. In general, however \mathcal{A}_w and \mathcal{B}_w are distinct. This may be seen as follows:

If $g \in \mathcal{B}_w$, choose $h \in \mathcal{A}_w$ such that $\langle Ag, Ah \rangle \neq 0$. By (2.15) we obtain that

$$V_g(g) * V_h(h) = \langle Ag, Ah \rangle V_h(g) \in \mathcal{M}^R(L_w^1) * L_w^1 \subseteq \mathcal{M}^R(L_w^1). \quad (6.5)$$

On the other hand, if for some $h \in \mathcal{A}_w$ the function $V_h(g)$ is in $\mathcal{M}^R(L_w^1)$, then $V_g(g)$ is in $\mathcal{M}^R(L_w^1)$ as in (6.4). Thus we have proved that

$$g \in \mathcal{B}_w \Leftrightarrow V_h(g) \in \mathcal{M}^R(L_w^1) \text{ for some } h \in \mathcal{A}_w. \quad (6.6)$$

This means (almost) that

$$\mathcal{B}_w = \mathcal{C} \circ \mathcal{M}^R(L_w^1). \tag{6.7}$$

In the next step it can be shown that the sequence space associated to $\mathcal{M}^R(L_w^1)$ is different from l_w^1 unless \mathcal{G} has a compact invariant neighbourhood. Since the sequence spaces are distinct, the coorbit spaces must be distinct by [FG3], Thm. 8.14.

Example: Let \mathcal{G} be the one-dimensional $ax + b$ -group, π the representation on $L^2(\mathbb{R})$ by translations and dilations (3.5) and consider the unweighted case $w \equiv 1$. Then (6.7) is actually true and

$$\begin{aligned} \mathcal{C} \circ L^1 &= \dot{B}_{1,1}^{1/2} \\ \mathcal{A} &= \dot{B}_{1,1}^{1/2} \cap \dot{B}_{1,1}^{-1/2} = \mathcal{C} \circ L_{1+\Delta}^1 \end{aligned}$$

and the following embeddings are valid

$$\dot{B}_{1,1}^w \subseteq \mathcal{B} = \mathcal{C} \circ \mathcal{M}^R(L^1) \subsetneq \dot{B}_{1,1}^{1/2} \tag{6.8}$$

(where $w(x, t) = (1 + |x| + |t|)$ is the left translation norm on $\mathcal{M}^R(L^1)$ and $B_{1,1}^w$ the weighted Besov space defined in (5.25) with $s = 0$)

6.4. Proposition. *Suppose that for some $g \in \mathcal{H}_w^{1\sim}$, $X = (x_i)_{i \in I}$ in \mathcal{G} and $1 < p \leq \infty$, the “synthesis operator” $T_X(\lambda_i) = \sum_{i \in I} \lambda_i \pi(x_i) g$ is bounded from l_w^p into $\mathcal{C} \circ L_w^p$. Then X is relatively separated. The conclusion also holds true if T_X is bounded from $l_{1/w}^\infty$ into $\mathcal{H}_w^{1\sim}$.*

Proof: We fix a vector $h \in \mathcal{A}_w$ such that $\text{Re} \langle h, g \rangle > 0$ and set $H = V_h(g)$. By assumption

$$\left\| \sum_{i \in I} \lambda_i L_{x_i} H \right\|_{L_w^p} \leq C \left(\sum |\lambda_i|^p w(x_i)^p \right)^{1/p} \tag{6.9}$$

holds true for all $\Lambda = (\lambda_i)_{i \in I} \in l_w^p$ with a constant C independent of Λ . Choose a compact set Q with nonvoid interior such that

$$Q^{-1} Q \subseteq \left\{ x \in \mathcal{G} : \text{Re} H(x) > \frac{1}{4} \text{Re} \langle h, g \rangle \right\}$$

and set $w = \sup \{w(x) : x \in Q^{-1} Q\}$. If $(x_i)_{i \in I}$ is not relatively separated, there exists a sequence $\{z_n\}$ in \mathcal{G} such that

$$I_n := \# \{i \in I : x_i \in z_n Q\} \geq n \tag{6.10}$$

holds for all $n \in \mathbb{N}$ (compare [FG2], Lemma 3.3). Set

$$\lambda_i^{(n)} = \begin{cases} w(x_i)^{-1} & \text{if } x_i \in z_n Q \\ 0 & \text{otherwise} \end{cases} \quad (6.11)$$

then each sequence $\Lambda_n = (\lambda_i^{(n)})_{i \in I}$ is in l_w^p and $\|\Lambda_n\|_{l_w^p} = I_n^{1/p}$.

Next we evaluate $T_X \Lambda_n$ on $z_n Q$ and observe that for $x_i \in z_n Q$ and $u \in Q$ both

$$\lambda_i = w(x_i)^{-1} \geq w^{-1} w(z_n u)^{-1} \quad \text{and} \quad \operatorname{Re} H(x_i^{-1} z_n u) \geq \frac{1}{4} \operatorname{Re} \langle h, g \rangle$$

hold true. Therefore

$$\begin{aligned} \operatorname{Re} T_X \Lambda_n(z_n u) &= \sum_{x_i \in z_n Q} w(x_i)^{-1} \operatorname{Re} H(x_i^{-1} z_n u) \\ &\geq w^{-1} w(z_n u)^{-1} \frac{1}{4} \operatorname{Re} \langle h, g \rangle I_n \end{aligned}$$

and consequently (with the obvious modification for $p = \infty$)

$$\begin{aligned} \|T_X \Lambda_n\|_{L_w^p} &\geq \left(\int_Q |\operatorname{Re} T_X \Lambda_n(z_n u)|^p w(z_n u)^p du \right)^{1/p} \\ &\geq \frac{1}{4w} \operatorname{Re} \langle h, g \rangle I_n \lambda(Q)^{1/p} \end{aligned} \quad (6.12)$$

If $p > 1$ and n large enough, (6.12) will certainly exceed $\|\Lambda_n\|_{l_w^p} = I_n^{1/p}$. Therefore (6.10) contradicts the boundedness of T_X and the claim is proved. \square

6.5. The final result concerns small perturbations and the stability of frames.

Proposition. *Assume that $\{\pi(x_i)g, i \in I\}$ is a frame for $\mathcal{C}_o Y$, as constructed in Theorem S or U.*

(a) *Then there exists a neighbourhood V of e such that $\{\pi(y_i)g, i \in I\}$ is a frame for $\mathcal{C}_o Y$ whenever $x_i^{-1} y_i \in V$ for all $i \in I$.*

(b) *There exists a $\delta > 0$ such that $\{\pi(x_i)g', i \in I\}$ is a frame for $\mathcal{C}_o Y$ whenever $g' \in \mathcal{B}_w$ and*

$$\|V_g(g - g')\|_{\mathcal{M}_c^R(L_w^1)} < \delta. \quad (6.13)$$

Proof. We consider only the discretization by S_Ψ , the other method is treated in an analogous manner. The given sampling values $\{y_i\}_{i \in I}$ suggest to compare S_Ψ , which is invertible by assumption, to

$$S'_\Psi F = \sum_{i \in I} F(y_i) \psi_i * G \quad (6.14)$$

on $Y * G$ with $F = V_g(f)$ as usual. Then the same arguments used in (4.31) and (4.32) lead to

$$\| \|S_\Psi - S'_\Psi | Y * G \| \| \leq \|G\|_{L_w^1} \|G^\# \|_{L_w^1} \quad (6.15)$$

If V is small enough, S'_Ψ must be also invertible on $Y * G$ (using Lemma 4.6 ii). In that case, S'^{-1}_Ψ provides a method of reconstruction from the coefficients $\langle \pi(y_i)g, f \rangle$ and $e'_i = V_g^{-1} S'^{-1}_\Psi (\psi_i * G)$ is the “dual frame”.

(b) In this case we consider the discretization

$$S''_\Psi V_g(f) = \sum_{i \in I} V_{g'}(f)(x_i) \psi_i * G \quad (6.16)$$

Because of $g' \in \mathcal{B}_w$ and the one-to-one correspondence $V_g(f) \leftrightarrow f \leftrightarrow \leftrightarrow V_{g'}(f)$, S''_Ψ is well-defined and bounded from $Y * G$ into $Y * G$. Now recall that

$$V_{g'}(f) = V_g(f) * V_g(g')^\vee \quad (6.17)$$

holds true on $\mathcal{H}_w^{1\sim} \cong \mathcal{C} \circ Y$ (this is a consequence of the orthogonality relations (2.16) for $f \in \mathcal{H}$ and can be extended to $\mathcal{H}_w^{1\sim}$ whenever $g, g' \in \mathcal{A}_w$). Now:

$$\begin{aligned} \| (S_\Psi - S''_\Psi) V_g(f) | Y \| &= \\ &= \left\| \sum_{i \in I} V_{g-g'}(f)(x_i) \psi_i * G | Y \right\| \leq && \text{(by (2.6))} \\ &\leq \left\| \sum_{i \in I} V_{g-g'}(f)(x_i) \psi_i | Y \right\| \|G\|_{L_w^1} \leq && \text{(Lemma 4.4)} \\ &\leq c \|V_g(f) - V_{g'}(f)\|_{\mathcal{M}_c(Y)} \|G\|_{L_w^1} \leq && \text{(by 6.17)} \\ &\leq c \|V_g(f) * V_g(g - g')^\vee\|_{\mathcal{M}_c(Y)} \|G\|_{L_w^1} \leq && \text{(by 2.13)} \\ &\leq c \|V_g(f) | Y\| \|V_g(g - g')\|_{\mathcal{M}_c^R(L_w^1)} \|G\|_{L_w^1} \end{aligned}$$

Thus

$$\| \|S_\Psi - S''_\Psi | Y * G \| \| \leq C' \|V_g(g - g')\|_{\mathcal{M}_c^R(L_w^1)} \quad (6.18)$$

implies that S''_ψ is invertible on $Y * G$ as soon as $\|V_g(g - g')\|_{\mathcal{M}_c^R(L_w^1)}$ is small enough, and the proof is complete. \square

6.6. Concluding remarks. (a) The discretization methods used for the construction of Banach frames and atomic decompositions work in the presence of a reproducing formula $F * G = F$ with a sufficiently “smooth” reproducing kernel G . Therefore this program should work in more general situations than that of an irreducible, unitary representation. The irreducibility of π is only needed to guarantee the validity of a reproducing formula. Examples suggest that this should be true whenever π is a cyclic representation which is supported on a compact-open set in the dual \mathcal{G} of \mathcal{G} . In this case the techniques of Section 4 and 5 would provide immediately frames for Besov–Triebel–Lizorkin-spaces on homogeneous groups. But to achieve this is now no longer a problem of function spaces, but one of representation theory.

(b) *Square-integrable representations:* If (π, H) is no longer integrable, but only square-integrable and irreducible, then the definition of a coorbit does not make sense, but the Hilbert space \mathcal{H} has still a reproducing kernel and the question arises whether it is possible to construct a coherent frame $\{\pi(x_i)g, i \in I\}$ for \mathcal{H} . The analysis of Section 4 breaks down because a convolution $F * G = F$ on $L^2 * L^2 \rightarrow L^2$ depends highly on the phase, not only on the absolute value $|F|$ and none of the arguments in Section 4 seems applicable. Nevertheless, it is possible to modify some of the techniques and construct coherent frames for \mathcal{H} also in this case.

(c) *Tight frames:* A Hilbert frame $\{\pi(x_i)g : i \in I\} \subseteq \mathcal{H}$ is called tight if coincides with its “dual frame” $\{e_i\} \subseteq \mathcal{H}$, i.e.

$$f = \sum c_i \langle \pi(x_i)g, f \rangle \pi(x_i)g \quad \text{for all } f \in \mathcal{H} \quad (6.19)$$

It is known that tight Hilbert frames exist for all realizations of the representations of the Heisenberg and the $ax + b$ -group ([DGM]) with $c_i = 1$ and for Lorentz groups [BO]. For theoretical considerations it would be very interesting to know whether tight frames $\{\pi(x_i)g, i \in I\}$ with $g \in \mathcal{B}_w$ do exist in general. If $g \in \mathcal{B}_w$, then the decomposition (6.19) holds automatically for all $f \in \mathcal{C}_o Y \rightarrow \mathcal{H}_w^{1\sim}$ and

$$\|f|_{\mathcal{C}_o Y}\| \leq c_1 \|\langle \pi(x_i)g, f \rangle_{i \in I}\|_{Y_d(X)} \leq c_2 \|f|_{\mathcal{C}_o Y}\| \quad (6.20)$$

is true (use (4.23) and Lemma 4.4). This means: if $\{\pi(x_i)g\}$ is a tight frame for only the Hilbert space \mathcal{H} and if $g \in \mathcal{B}_w$, then $\{\pi(x_i)g, i \in I\}$ is automatically a Banach frame for all coorbit spaces $\mathcal{C}_o Y$ embedded into $\mathcal{H}_w^{1\sim}$ and the reconstruction method is particularly simple. This implies that the smooth tight frames of [DGM] are actually frames for all modulation spaces and all Besov–Triebel–Lizorkin spaces!

It is not known whether smooth tight frames exist in the general situation.

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