

# The Bunce-Deddens Algebras as Crossed Products by Partial Automorphisms

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**Abstract.** We describe both the Bunce-Deddens  $C^*$ -algebras and their Toeplitz versions, as crossed products of commutative  $C^*$ -algebras by partial automorphisms. In the latter case, the commutative algebra has, as its spectrum, the union of the Cantor set and a copy of the set of natural numbers  $\mathbb{N}$ , fitted together in such a way that  $\mathbb{N}$  is an open dense subset. The partial automorphism is induced by a map that acts like the odometer map on the Cantor set while being the translation by one on  $\mathbb{N}$ . From this we deduce, by taking quotients, that the Bunce-Deddens  $C^*$ -algebras are isomorphic to the (classical) crossed product of the algebra of continuous functions on the Cantor set by the odometer map.

## 1. Introduction

Recall from [4] that a weighted shift operator is a bounded operator on  $l_2 = l_2(\mathbb{N})$  given, on the canonical basis  $\{e_n\}_{n=0}^\infty$ , by  $S_a(e_n) = a_{n+1}e_{n+1}$ , where the weight sequence  $a = \{a_n\}_{n=1}^\infty$ , is a bounded sequence of complex numbers. A weighted shift is said to be  $p$ -periodic if its weights satisfy  $a_n = a_{n+p}$  for all  $n$ .

Given a strictly increasing sequence  $\{n_k\}_{k=0}^\infty$  of positive integers, such that  $n_k$  divides  $n_{k+1}$  for all  $k$ , the Bunce-Deddens-Toeplitz  $C^*$ -algebra  $A = A(\{n_k\})$  is defined to be the  $C^*$ -algebra of operators on  $l_2$ , generated by the set of all  $n_k$ -periodic weighted shifts, for all  $k$ . These algebras were first studied by Bunce and Deddens in [5]. It was observed by them that the algebra  $\mathcal{K}$ , of compact operators on  $l_2$ , is contained in  $A$  and that the quotient  $A/\mathcal{K}$  is a simple  $C^*$ -algebra. The latter became known as the Bunce-Deddens  $C^*$ -algebra and has been extensively studied (see, for example, [1], [2, 10.11.4], [3], [6], [9, p.

248], [11], [12], [13]). We shall denote these algebras by  $B(\{n_k\})$  or simply by  $B$ , if the weight sequence is understood.

The goal of the present work is to describe both  $A(\{n_k\})$  and  $B(\{n_k\})$  as the crossed product of commutative  $C^*$ -algebras by partial automorphisms [7], in much the same way as we have described general AF-algebras [8] as partial crossed products.

In the case of  $A(\{n_k\})$ , we shall see that it is given by a curious (partial) dynamical system consisting of a topological space  $X$  which can be thought of as a compactification of the (discrete) space  $\mathbb{N}$  of natural numbers, the complement of  $\mathbb{N}$  in  $X$  being homeomorphic to the Cantor set  $\mathbb{K}$ . The transformation  $f$  of  $X$ , by which the partial crossed product is taken, leaves both  $\mathbb{N}$  and  $\mathbb{K}$  invariant. Its behavior on  $\mathbb{N}$  is that of the translation by one, while the action on  $\mathbb{K}$  is by means of the odometer map (see, for example, [12]) which is defined as follows. Given a sequence  $\{q_k\}_{k=0}^{\infty}$  of positive integers (below we shall use  $q_k = n_{k+1}/n_k$ ), consider the Cantor set, as given by the model

$$\mathbb{K} = \prod_{j=0}^{\infty} \{0, 1, \dots, q_j - 1\}.$$

The odometer map is the map  $\mathcal{O}: \mathbb{K} \rightarrow \mathbb{K}$ , given by formal addition of  $(1, 0, \dots)$  with carry over to the right. Note that

$$\mathcal{O}(q_0 - 1, q_1 - 1, \dots) = (0, 0, \dots)$$

since the carry over process, in this case, extends all the way to infinite. For further reference let us call by the name of "partial odometer" the restriction of  $\mathcal{O}$  to a map from

$$X - \{(q_0 - 1, q_1 - 1, \dots)\} \quad \text{to} \quad X - \{(0, 0, \dots)\}$$

so that, for this map, the carry over process always terminates in finite time.

As already mentioned,  $B(\{n_k\})$  is the quotient of  $A(\{n_k\})$  by  $\mathcal{K}$ . But  $\mathcal{K}$  can be seen to correspond to the restriction of the above dynamical system to  $\mathbb{N}$  (see [7]). So, we deduce that the Bunce-Deddens algebras  $B(\{n_k\})$  are isomorphic to the crossed product of the Cantor set by the odometer map. This result is already well known [2, 10.11.4] but it is

interesting to remark how little bookkeeping is necessary to deduce it from the machinery of partial automorphisms [7]. Moreover, this should be compared with [8], Theorem 3.2, according to which UHF-algebras sit as subalgebras of  $\mathcal{B}(\{n_k\})$  (see also [12]).

This work was done while I was visiting the Mathematics Department at the University of New Mexico.

## 2. Circle Actions

For each  $z$  in the unit circle

$$S^1 = \{w \in \mathbb{C} : |w| = 1\}$$

let  $U_z$  denote the diagonal unitary operator on  $l_2$ , given by  $U_z(e_n) = z^n e_n$ . If  $S$  is any weighted shift, it is easy to see that  $U_z S U_z^{-1} = zS$ . Thus, denoting by  $\alpha_z$  the inner automorphism of  $\mathcal{B}(l_2)$  given by conjugation by  $U_z$ , one finds that  $A$  is invariant under  $\alpha_z$ . Moreover, one can see that this gives a continuous action of  $S^1$  on  $A$ , in the sense of [10], 7.4.1 (even though the corresponding action is not continuous on  $\mathcal{B}(l_2)$ ).

Let us denote the fixed point subalgebra by  $A_0$ . It is easy to see that  $A_0$  consists precisely of the operators in  $A$  which are diagonal with respect to the basis  $\{e_n\}$ . Now, given that the  $C^*$ -algebra of (bounded) diagonal operators is isomorphic to  $l_\infty = l_\infty(\mathbb{N})$ , it is convenient to view  $A_0$  as a subalgebra of  $l_\infty$ . For the purpose of describing  $A_0$ , observe that, since  $\mathcal{K}$  is contained in  $A$ , it follows that  $c_0$  (the subalgebra of  $l_\infty$  formed by sequences tending to zero) is contained in  $A_0$ . Carrying this analysis a bit further one can prove that  $A_0 = c_0 \oplus D$  where  $D$  is the subalgebra of  $l_\infty$  generated by all  $n_k$ -periodic sequences, for all  $k$ .

This decomposition is useful in determining the spectrum of  $A_0$ . Note, initially, that the spectrum of  $c_0$  is homeomorphic to  $\mathbb{N}$  (with the discrete topology) while the spectrum of  $D$  is the Cantor set, here denoted  $\mathbb{K}$ . This said, one has that the spectrum of  $A_0$  can be described, at least in set theoretical terms, as the union  $X = \mathbb{N} \cup \mathbb{K}$ . Moreover, since  $c_0$  is an essential ideal in  $A_0$ , one sees that  $\mathbb{N}$  is an open dense subset of  $X$ . To better grasp the entire topology of  $X$  we need a more precise notation. Assume, without loss of generality, that  $n_0 = 1$  and

let  $q_k = n_{k+1}/n_k$  for  $k \geq 0$ . Any integer  $n$  with  $0 \leq n < n_k$  has a unique representation as

$$n = \sum_{j=0}^{k-1} \beta_j^{(n)} n_j$$

where  $0 \leq \beta_j^{(n)} < q_j$ . Here the  $\beta_j^{(n)}$  play the role of digits in a decimal-like representation, except that the base varies along with the position of each digit. Accordingly, we let  $\beta^{(n)} = (\beta_0^{(n)}, \dots, \beta_{k-1}^{(n)})$  be the corresponding notation for  $n$  (which we shall use interchangeably without further warning). When convenient, we shall also view  $\beta^{(n)}$  as an element of the set

$$\mathbb{K}_k = \prod_{j=0}^{k-1} \{0, 1, \dots, q_j - 1\}.$$

For each  $k$  and each  $\beta$  in  $\mathbb{K}_k$ , we denote by  $e_\beta$  the  $n_k$ -periodic sequence (thus an element of  $D$ ) given by

$$e_\beta(n) = \begin{cases} 1 & \text{if } n \equiv \beta \pmod{n_k} \\ 0 & \text{otherwise} \end{cases}$$

Note that the length of  $\beta$  determines which  $n_k$  should be used in the above definition. Clearly the set  $\{e_\beta \in \cup_{k=1}^{\infty} \mathbb{K}_k\}$  generates  $D$ . Making use of the notation introduced, we shall adopt for the Cantor set, the model

$$\mathbb{K} = \prod_{j=0}^{\infty} \{0, 1, \dots, q_j - 1\}$$

so that, once we view  $D$  as the algebra of continuous functions on  $\mathbb{K}$  via the Gelfand transform, the support of the  $e_\beta$  form a basis for the topology of  $\mathbb{K}$ . In fact, the support of  $e_\beta$  is precisely the set of elements  $\gamma = (\gamma_i)$  in  $\mathbb{K}$  such that  $\gamma_j = \beta_j$  for all  $j = 0, \dots, k-1$  (assuming that  $\beta$  is in  $\mathbb{K}_k$ ). That is,  $\gamma$  is in the support of  $\beta$  if and only if its initial segment coincides with  $\beta$ .

Considering now, the whole of  $A_0$ , note that it is generated by the set of all idempotents  $p$  which, viewed as elements of  $l_\infty$ , have one of the two following forms: either it has a finite number of non-zero coordinates (in which case  $p$  is in  $c_0$ ), or it coincide with some  $e_\beta$ , except for finitely many coordinates. The set of such idempotents is closed under

multiplication, which therefore implies that their support, in the spectrum  $X$  of  $A_0$ , form a basis for the topology of  $X$ . With this we have precisely described the topology of  $X$ :

**Theorem 2.1.** *The spectrum of  $A_0$  consists of the union of the Cantor set  $\mathbb{K}$  and a copy of the set of natural numbers  $\mathbb{N}$ . Each element of  $\mathbb{N}$  is an isolated point and a fundamental system of neighborhoods of a point  $\gamma = (\gamma_i)$  in  $\mathbb{K}$  consists of the sets  $V_k$  defined to be the union of the sets*

$$\{\zeta \in \mathbb{K}: \zeta_i = \gamma_i, i < k\}$$

and

$$\{n \in \mathbb{N}: n \geq k \quad \text{and} \quad \beta_i^{(n)} = \gamma_i, i < k\}.$$

Note the interesting interplay between the digital representation of the natural numbers on one hand, and of elements of the Cantor set, on the other.

### 3. The Main Result

Recall from [7], Theorem 4.21, that a regular semi-saturated action of  $S^1$  on a  $C^*$ -algebra, causes it to be isomorphic to the covariance algebra of a certain partial automorphism of the fixed point subalgebra. In the case of the action  $\alpha$  of  $S^1$  on  $A$ , described above, it is very easy to see that it is regular and semi-saturated. The semi-saturation follows immediately, since every weighted shift belongs to the first spectral subspace of  $\alpha$ , henceforth denoted  $A_1$ . The fact that  $\alpha$  is regular depends on the existence of an isomorphism  $\theta: A_1^*A_1 \rightarrow A_1A_1^*$  and a linear isometry  $\lambda: A_1^* \rightarrow A_1A_1^*$  such that, for  $x, y \in A_1, a \in A_1^*A_1$  and  $b \in A_1A_1^*$

- (i)  $\lambda(x^*b) = \lambda(x^*)b$
- (ii)  $\lambda(ax^*) = \theta(a)\lambda(x^*)$
- (iii)  $\lambda(x^*)^*\lambda(y^*) = xy^*$
- (iv)  $\lambda(x^*)\lambda(y^*)^* = \theta(x^*y)$ .

See [7], 4.3 and 4.4 for more information. It is easy to see that  $\theta$  and  $\lambda$ , given by  $\theta(a) = SaS^*$  and  $\lambda(x^*) = Sx^*$ , satisfy the desired properties, where  $S$  denotes the unilateral (unweighted) shift.

Note that, in the present case,  $A_1^*A_1 = A_0$  while  $A_1A_1^*$  is the ideal

of  $A_0$  formed by all sequences for which the first coordinate vanishes. That is, under the standard correspondence between ideals and open subsets of the spectrum,  $A_1 A_1^*$  corresponds to  $X - \{0\}$ . The isomorphism  $\theta: C(X) \rightarrow C(X - \{0\})$  therefore induces a homeomorphism  $f: X \rightarrow X - \{0\}$  which we would now like to describe.

If an integer  $n$  is thought of as an element of  $X$ , as seen above, then the element  $\delta_n$  of  $l_\infty$ , represented by the sequence having the  $n^{\text{th}}$  coordinate equal to one and zeros everywhere else, corresponds to the characteristic function of the singleton  $\{n\}$  and, given that  $\theta(\delta_n) = \delta_{n+1}$ , we see that  $f(n) = n + 1$ .

Now, if  $\gamma = (\gamma_i)$  is in  $\mathbb{K}$ , let  $\gamma|_k$  be the  $k^{\text{th}}$  truncation of  $\gamma$ , that is  $\gamma|_k = (\gamma_0, \dots, \gamma_{k-1})$  so that we can consider  $e_{\gamma|_k}$ , as defined above. Also let  $f_k$  be the element of  $c_0$  given by  $f_k = (1, \dots, 1, 0, \dots)$  where the last "1" occurs in the position  $k - 1$ , counting from zero. The support of the Gelfand transform of the idempotent element  $(1 - f_k)e_{\gamma|_k}$  is precisely the set  $V_k$  referred to in 2.1. So, as  $k$  varies, the intersection of these sets is precisely the singleton  $\{\gamma\}$ . Therefore, to find out what  $f(\gamma)$  should be, it is enough to look for the intersection of the supports of the Gelfand transforms of the elements

$$\theta((1 - f_k)e_{\gamma|_k}) = S(1 - f_k)e_{\gamma|_k} S^*.$$

The reader is now invited to verify, using this method, that the effect that  $f$  has on  $\gamma$  is precisely the effect of the odometer map. One should exercise special attention to check that the above method does indeed give  $f(q_0 - 1, q_1 - 1, \dots) = (0, 0, \dots)$ , in contrast with the partial automorphisms that produce UHF-algebras [8], since  $(q_0 - 1, q_1 - 1, \dots)$  is removed from the domain of the maps considered there. Summarizing our findings so far, we have:

**Theorem 3.1.** *The Bunce-Deddens-Toeplitz  $C^*$ -algebra  $A(\{n_k\})$  is isomorphic to the crossed product of  $C(X)$  by the partial automorphism  $\theta: C(X) \rightarrow C(X - \{0\})$  induced by the (inverse of the) map  $f: X \rightarrow X - \{0\}$  acting like the odometer on  $\mathbb{K}$  and like translation by one on  $\mathbb{N}$ .*

Recall that the Bunce-Deddens  $C^*$ -algebras were defined to be the quotient  $B = A/\mathcal{K}$ . If one identifies  $A$  with the crossed product above,

it is easy to see that the ideal  $\mathcal{K}$  corresponds to the crossed product of  $c_0$  by the corresponding restriction of  $\theta$ . Therefore, the quotient can be described as the (classical) crossed product of the Cantor set by the odometer map. That is:

**Theorem 3.2.** ([2], 10.11.4) *The Bunce-Deddens  $C^*$ -algebra  $B(\{n_k\})$  is isomorphic to the crossed product of  $C(\mathbb{K})$  by the automorphism induced by the odometer map.*

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