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Convergence in distribution of sums of bivariate Appell polynomials with long-range dependence*

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Summary. Normalized quadratic forms of moving averages converge to double Wiener-Itô integrals if the summands are sufficiently dependent. This result extends to sums of bivariate Appell polynomials of arbitrary degree.

1 Introduction and main results

Quadratic forms of stationary sequences, appropriately normalized, may have Gaussian or non-Gaussian limits. Under what circumstances will the limits be of one type or the other? This question, partially investigated in Rosenblatt [17], Fox and Taqqu [9], Avram [1], and Terrin and Taqqu [19], is further clarified here. Suppose the sequence is Gaussian with a regularly varying covariance function. If the matrix in the quadratic form is Toeplitz with entries that are also regularly varying, then the quadratic form is characterized by two parameters, denoted here as α and β. Specifically, suppose the sequence has spectral density $f(x)=|x|^{-\alpha}L_1(x)$, $\alpha<1$, and the entries of the matrix are the Fourier coefficients of a function $g(x)=|x|^{-\beta}L_2(x)$, $\beta<1$, where L_1 and L_2 are slowly varying. Then if $\alpha+\beta<1/2$, the limit of the normalized quadratic form is Gaussian (Fox and Taqqu [9]), and if $\alpha+\beta>1/2$, the limit is a Wiener-Itô integral on $\mathbf{R}^2$ (Terrin and Taqqu [19]).

When $\alpha>0$, the spectral density f has a singularity at the origin, and is associated with the phenomenon known as long-range dependence (long memory). Statistical models with this type of spectral density include fractional Gaussian noise (Mandelbrot and Van Ness [16] and Mandelbrot and Taqqu [15]) and fractional ARMA (Granger and Joyeux [12] and Hosking [13]), and the estimation of their parameters frequently involves quadratic forms (see for example Dahlhaus [6], Yajima [21] and Fox and Taqqu [8]).

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Non-linear functions of long memory processes may also exhibit long-range dependence (Taqqu [18]). Therefore, it is of interest to explore the limit behavior of quadratic forms for such sequences, or more generally forms of the type

$$\sum_{j=1}^{N} \sum_{k=1}^{N} a_{j-k}\, G(X_j, X_k) \tag{1.1}$$

where G is a general bivariate function and X_k is a stationary, possibly non-Gaussian, long memory process. A first step toward understanding (1.1) is the study of forms of the type

$$\sum_{j=1}^{N} \sum_{k=1}^{N} a_{j-k}\, P_{m,n}(X_j, X_k) \tag{1.2}$$

where $P_{m,n}$ is a bivariate Appell polynomial defined in (1.7) and X_j is a moving average of independent and identically distributed random variables. In the event that $\{X_j\}$ is a Gaussian sequence, the bivariate Appell polynominals are called bivariate Hermite polynomials. Although in general the Appell polynomials do not form an orthogonal basis, there are classes of analytic functions that can be expanded in Appell polynomials (Giraitis and Surgailis [10]).

If one keeps the assumptions on the covariances and coefficients described above, then as Avram [1] recently showed, the limit of (1.2) adequately normalized is *Gaussian* if X_j is Gaussian and if $\alpha \geqq 0$, $0 \leqq \beta < 1/2$, $(m+n)\alpha + 2\beta < m+n-1$, and either $m\alpha + 2\beta < m$ or $n\alpha + 2\beta < n$. In this paper we show that when $(m+n)\alpha + 2\beta > m+n-1$ and both $n\alpha > n-1$ and $m\alpha > m-1$, the limit is *non-Gaussian* and is representable as a Wiener-Itô integral of order $m+n$. This result holds even if X_j is non-Gaussian. Our parameter space is depicted in Fig. 1 below in the case $m > n$.

When (α, β) is in region 1, the limit is Gaussian, and when (α, β) is in region 2,

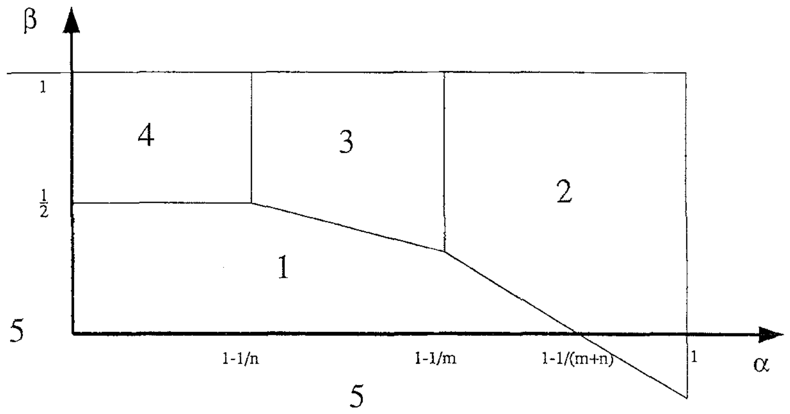

Fig. 1. The parameter space

the limit is an $m+n$ fold Wiener-Itô integral. Regions 3 and 4, where the limit takes a different non-Gaussian form, will be handled in a subsequent paper. Since the terms $a_{j-k}\, P_{m,n}(X_j, X_k)$ are less dependent when α or β are negative, it is likely that much of region 5 will produce Gaussian limits.

Convergence to the Wiener-Itô integral is established in two theorems, because the case $m=n=1$ does not require the additional conditions $n\alpha>n-1$ and $m\alpha>m-1$. Since the moving averages $\{X_j\}$ are not necessarily Gaussian, it is not possible to represent the N^{th} partial sum as a Wiener-Itô integral, as was done in Terrin and Taqqu [19]. Instead, techniques used for sums of univariate Appell polynomials in Giraitis and Surgailis [11] are extended here to obtain our results. First, some notation is needed. Let

$$X_j=\sum_{s=-\infty}^{\infty} b(j-s)\,\xi_s, \quad j=0,1,\ldots \tag{1.3}$$

where

$$b(k)=\int_{-\pi}^{\pi} e^{ikx}|x|^{-\alpha/2} L_1^{1/2}(|x|^{-1})\,dx, \tag{1.4}$$

L_1 is slowly varying at ∞ and bounded on bounded intervals, and $\ldots, \xi_{-1}, \xi_0, \xi_1 \ldots$ is a sequence of independent and identically distributed random variables, with $E\xi_0=0$, and $E\xi_0^2=\sigma^2$. Since

$$EX_j X_k=2\pi\sigma^2 \int_{-\pi}^{\pi} e^{i(j-k)x}|x|^{-\alpha} L_1(|x|^{-1})\,dx,$$

the sequence $X_0, X_1 \ldots$ has long-range dependence when $\alpha>0$.
Define the *Wick power*

$$:Y_1 \ldots Y_k:$$

of random variables $Y_1, \ldots, Y_k$ inductively, as in Avram and Taqqu [2] (see also Giraitis and Surgailis [10]). When $k=0$, set the Wick power equal to 1. When $k>0$, define $:Y_1 \ldots Y_k:$ recursively by

$$E:Y_1 \ldots Y_k:=0 \tag{1.5}$$

and

$$\partial :Y_1 \ldots Y_k:/\partial Y_i = :Y_1 \ldots Y_{i-1}\, Y_{i+1} \ldots Y_k: . \tag{1.6}$$

The *multivariate Appell polynomials* $P_{n_1,\ldots,n_k}(Y_1, \ldots, Y_k)$ are indexed by non-negative integers $n_1, n_2, \ldots$. They are defined by

$$P_{n_1,\ldots,n_k}(Y_1, \ldots, Y_k) = : Y_1 \ldots Y_1\, Y_2 \ldots Y_2 \ldots Y_k \ldots Y_k: \tag{1.7}$$

where Y_i is repeated n_i times, $i=1, \ldots, k$. The polynomials have mean zero and behave like powers in the sense that

$$\partial P_{n_1,\ldots,n_k}(Y_1, \ldots, Y_k)/\partial Y_i = n_i\, P_{n_1,\ldots,n_i-1,\ldots,n_k}(Y_1, \ldots, Y_k).$$

Let G_0 be the spectral measure on $\mathbf{R}$ with density $|x|^{-\alpha}$, and let Z_{G_0} be the complex-valued Gaussian random measure satisfying

$$EZ_{G_0}(A)=0, \quad E|Z_{G_0}(A)|^2=G_0(A), \quad \overline{Z_{G_0}(-A)}=Z_{G_0}(A) \tag{1.8}$$

for any Borel set A of $\mathbf{R}$ with finite G_0 measure.
Define the constants a_k, $k=\ldots-1, 0, 1, \ldots$, in (1.2) by

$$a_k=\int_{-\pi}^{\pi} e^{ikx}|x|^{-\beta} L_2(|x|^{-1})\,dx, \tag{1.9}$$

where L_2 is slowly varying at ∞ and bounded on bounded intervals.

Theorem 1.1 *Suppose* $E|\xi_0|^{2(m+n)}<\infty$,

$$(m+n)\,\alpha+2\beta>m+n-1, \tag{1.10}$$

$$m\alpha>m-1, \quad n\alpha>n-1, \tag{1.11}$$

$$\alpha<1, \quad \beta<1, \tag{1.12}$$

and

$$d_N=N^H L_1^{m/2}(N)\,L_2(N),$$

where

$$H=(m+n)(\alpha-1)/2+\beta+1. \tag{1.13}$$

Then for $m, n=1, 2\ldots$

$$\frac{1}{d_N}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1} a_{j-k}\,P_{m,n}(X_j, X_k) \tag{1.14}$$

converges in distribution to the $m+n$ *fold Wiener-Itô integral*

$$(2\pi\sigma^2)^{(m+n)/2} \int''_{\mathbf{R}^{m+n}} K_0(x_1, \ldots, x_{m+n})\,dZ_{G_0}(x_1)\ldots dZ_{G_0}(x_{m+n}), \tag{1.15}$$

where

$$K_0(x_1, \ldots, x_{m+n})=\int_{-\infty}^{\infty} \Delta\left(\sum_1^m x_p+u\right)\Delta\left(\sum_{m+1}^{m+n} x_p-u\right)|u|^{-\beta}\,du, \tag{1.16}$$

and

$$\Delta(y)=\frac{e^{iy}-1}{iy}. \tag{1.17}$$

Remarks. Conditions (1.10) and (1.12) imply $1/2<H<2$ and $\beta>-1/2$. Condition (1.11) implies $\alpha>0$, which is not necessary when $m=n=1$, as will be seen in the following theorem. The integral $\int''_{\mathbf{R}^{m+n}}$ is defined in Major [14]. The double prime indicates that the integration excludes the hyperdiagonals $x_i=\pm x_j$, $i\neq j$.

The next theorem focuses on the case $m=n=1$.

Theorem 1.2 *If $H=\alpha+\beta>1/2$, $\alpha<1$, $\beta<1$, and $d_N=N^H L_1^{1/2}(N)\,L_2(N)$, then*

$$\frac{1}{d_N}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1} a_{j-k}\,P_{1,1}(X_j, X_k)\equiv\frac{1}{d_N}\left[\sum_{j=0}^{N-1}\sum_{k=0}^{N-1} a_{j-k}\,X_j X_k - E\sum_{j=0}^{N-1}\sum_{k=0}^{N-1} a_{j-k}\,X_j X_k\right]$$

converges in distribution to

$$2\pi\sigma^2 \int_{\mathbf{R}^2}'' K_0(x_1, x_2)\,\mathrm{d}Z_{G_0}(x_1)\,\mathrm{d}Z_{G_0}(x_2),$$

where

$$K_0(x_1, x_2)=\int_{-\infty}^{\infty}\frac{\mathrm{e}^{i(x_1+u)}-1}{i(x_1+u)}\,\frac{\mathrm{e}^{i(x_2-u)}-1}{i(x_2-u)}\,|u|^{-\beta}\,\mathrm{d}u.$$

The theorems are proved in Sect. 4 by applying a proposition given in Sect. 3. To satisfy the hypotheses of the proposition, one must check the convergence of certain integrals. This is done in Sect. 2 using power counting techniques.

2 Applications of power counting

The proofs of Theorems 1.1 and 1.2 depend on the convergence of certain integrals. In this section, we apply power counting methods to establish convergence. We start with some notation and terminology, and then state a power counting theorem established in Terrin and Taqqu [20] as Corollary 1.1.

For $i=1, \ldots, m$, let $M_i(\mathbf{x})$ be a linear functional on $\mathbf{R}^n$. Let θ_i be real constants and set

$$T=\{M_i : i=1, \ldots, m\}$$

and

$$T'=\{M_1+\theta_1, \ldots, M_m+\theta_m\}.$$

T is a set of linear functionals and T' is a set of affine-linear functionals. For $0<a_i<b_i<\infty$, $c_i>0$ and real constants α_i and β_i define

$$P(\mathbf{x})=f_1(M_1(\mathbf{x})+\theta_1)\ldots f_m(M_m(\mathbf{x})+\theta_m) \tag{2.1}$$

where $|f_i|$ is bounded above on (a_i, b_i) and

$$|f_i(y)|\le\begin{cases} c_i|y|^{\alpha_i} & \text{if } |y|<a_i\\ c_i|y|^{\beta_i} & \text{if } |y|>b_i.\end{cases} \tag{2.2}$$

Define

(i) $s_T(W)=\operatorname{span}(W)\cap T$ for $W\subset T$,

(ii) $s_{T'}(W)=\operatorname{span}(W)\cap T'$ for $W\subset T'$,

(iii) $d_0(W)=r(W)+\sum_{s_{T'}(W)}\alpha_i$ for $W\subset T'$,

(iv) $d_\infty(W)=r(T)-r(W)+\sum_{T\setminus s_T(W)}\beta_i$ for $W\subset T$, where span(W) denotes the linear span of W, $r(W)$ is the number of linearly independent elements of W, and summation over a subset of T is the summation over all indices of functionals in the subset.

A set $W \subset T$ is said to be *padded* if for every linear functional M in W, M is also in $s_T(W \setminus \{M\})$. That is, M can be obtained as a linear combination of other elements in W. There is a corresponding definition for affine-linear functionals. Let $L_i = M_i + \theta_i$, $i = 1, \ldots, m$. A set $W \subset T'$ is *padded* if for every L in W, L is also in $s_{T'}(W \setminus \{L\})$.

Power counting theorem (Terrin and Taqqu [20], Corollary 1.1). *Suppose $r(T) = n$ and $\alpha_i > -1$, $\beta_i \geqq -1$, $i = 1, \ldots, m$. If*
(a) $d_0(W) > 0$ *for every padded, nonempty subset W of T' with $W = s_{T'}(W)$, and*
(b) $d_\infty(W) < 0$ *for every padded, proper subset W of T, with $W = s_T(W)$, including the empty set, then*

$$\int_{\mathbf{R}^n} |f_1(M_1(\mathbf{x}) + \theta_1) \ldots f_m(M_m(\mathbf{x}) + \theta_m)| \, \mathrm{d}^m x < \infty.$$

Remark 2.1 To show that an integral on $[-A, A]^n$, $A > 0$, is finite, it is sufficient to show that condition (a) is satisfied.

Let G be the measure on $[-\pi, \pi]$ with density $|x|^{-\alpha}$, and let G_N be the measure defined by

$$G_N(A) = N^{1-\alpha} G(N^{-1} A). \tag{2.3}$$

Then $dG(x) = |x|^{-\alpha} dx$, $|x| < \pi$, and $dG_N(x) = |x|^{-\alpha} dx$, $|x| < N\pi$. For each $N = 1, 2, \ldots$, let

$$h_N(x_1, \ldots, x_{m+n}) = \int_{-\pi}^{\pi} S_N\left(\sum_1^m x_p + u\right) S_N\left(\sum_{m+1}^{m+n} x_p - u\right) |u|^{-\beta} \, \mathrm{d}u, \tag{2.4}$$

where

$$S_N(y) = \sum_{j=0}^{N-1} \mathrm{e}^{ijy}, \tag{2.5}$$

and let

$$\begin{aligned} K_N(x_1, \ldots, x_{m+n}) &\equiv N^{-(1+\beta)} h_N(x_1/N, \ldots, x_{m+n}/N) \\ &= \int_{-N\pi}^{N\pi} \Delta_N\left(\sum_1^m x_p + u\right) \Delta_N\left(\sum_{m+1}^{m+n} x_p - u\right) |u|^{-\beta} \, \mathrm{d}u, \end{aligned} \tag{2.6}$$

where

$$\Delta_N(y) = (\mathrm{e}^{iy} - 1)/N(\mathrm{e}^{iy/N} - 1). \tag{2.7}$$

The goal of this section is to establish

Proposition 2.1 *Suppose conditions* (1.10), (1.11) *and* (1.12) *hold. If the parameter H is as defined in* (1.13), *and G_0 is the measure on* **R** *with density $|x|^{-\alpha}$, then*

$$(2.8)\quad \lim_{N\to\infty} \frac{1}{N^{2H}} \int_{[-\pi,\pi]^{m+n}} |h_N(\mathbf{x})|^2\, \mathrm{d}_{m+n} G \equiv \lim_{N\to\infty} \int_{[-N\pi,N\pi]^{m+n}} |K_N(\mathbf{x})|^2\, \mathrm{d}^{m+n} G_N = \int_{\mathbf{R}^{m+n}} |K_0(\mathbf{x})|^2\, \mathrm{d}^{m+n} G_0 < \infty,$$

where K_0 is defined in (1.16).

The proof of the proposition uses several lemmas.

Lemma 2.1 *If*

$$(2.9)\quad T' = T = \left\{\sum_1^m x_p + u, \sum_{m+1}^{m+n} x_p - u, \sum_1^m x_p + v, \sum_{m+1}^{m+n} x_p - v, u, v, x_1, \ldots, x_{m+n}\right\},$$

then condition (*a*) *of the power counting theorem is met if*
(a′) $d_0(W) > 0$ *for stets W satisfying $W = \bigcup_I W_i$, $I \subset \{2, \ldots, 8\}$, and $W = s_T(W)$, and condition* (*b*) *is met if*
(b′) $d_\infty(W) < 0$ *for sets W satisfying $W = \bigcup_I W_i$, $I \subset \{1, \ldots, 8\}$, $W = s_T(W)$, and $W \neq T$, where*

$$W_1 = \emptyset$$

$$W_2 = \left\{\sum_1^m x_p + u, u, x_1, \ldots, x_m\right\}$$

$$W_3 = \left\{\sum_{m+1}^{m+n} x_p - u, u, x_{m+1}, \ldots, x_{m+n}\right\}$$

$$W_4 = \left\{\sum_1^m x_p + v, v, x_1, \ldots, x_m\right\}$$

$$W_5 = \left\{\sum_{m+1}^{m+n} x_p - v, v, x_{m+1}, \ldots, x_{m+n}\right\}$$

$$W_6 = \left\{\sum_1^m x_p + u, \sum_1^m x_p + v, u, v\right\}$$

$$W_7 = \left\{\sum_{m+1}^{m+n} x_p - u, \sum_{m+1}^{m+n} x_p - v, u, v\right\}$$

$$W_8 = \left\{\sum_1^m x_p + u, \sum_{m+1}^{m+n} x_p - u, \sum_1^m x_p + v, \sum_{m+1}^{m+n} x_p - v\right\}.$$

Proof. Suppose W is a padded subset of T satisfying $W = s_T(W)$. It suffices to show that if $L \in W$, then $L \in W_i \subset W$ for some $i = 1, \ldots, 8$. Observe that there are three types of functionals in T:

(i) $L = x_i$, $i = 1, \ldots, m+n$,
(ii) $L \in \{u, v\}$,

(iii) $L \in W_8$.
By the symmetries of T, it suffices to examine the functionals (i) x_1, (ii) u, and (iii) $\sum_1^m x_p + u$.

(i) Suppose $x_1 \in W$. Then x_1 must also be part of the expression of another functional in W, since W is padded. The only possibilities are $\sum_1^m x_p + u$ and $\sum_1^m x_p + v$. Hence either $\sum_1^m x_p + u$, $x_1, \ldots, x_m$, and u all belong to W, or $\sum_1^m x_p + v$, $x_1, \ldots, x_m$, and v all belong to W. Thus either $x_1 \in W_2 \subset W$, or $x_1 \in W_4 \subset W$.

(ii) Suppose $u \in W$. Then u must be a part of the expression of another functional of W. The only possibilities are $\sum_1^m x_p + u$ and $\sum_{m+1}^{m+n} x_p - u$. Assume without loss of generality that $\left(\sum_1^m x_p + u\right) \in W$ and u is a linear combination of $\sum_1^m x_p + u$ and other elements of $W \setminus \{u\}$. There are two cases:
Case 1. $\{x_1, \ldots, x_m\} \subset W$. Then $u \in W_2 \subset W$.
Case 2. $\{x_1. \ldots, x_m\} \not\subset W$. Then $\sum_1^m x_p$ is part of the expression of another functional in W. The only possibility is $\sum_1^m x_p + v$. Since $W = s_T(W)$, $v \in W$, and hence $u \in W_6 \subset W$.

(iii) Suppose $\left(\sum_1^m x_p + u\right) \in W$. There are two cases:
Case 1. $\{x_1, \ldots, x_m\} \subset W$. Then, since $W = s_T(W)$, $\left(\sum_1^m x_p + u\right) \in W_2 \subset W$.
Case 2. $\{x_1, \ldots, x_m\} \not\subset W$. Then $\sum_1^m x_p$ is part of the expression of another functional in W. The only possibility is $\sum_1^m x_p + v$. If $v \in W$, then $u \in W$, since $W = s_T(W)$. Hence $\left(\sum_1^m x_p + u\right) \in W_6 \subset W$. If $v \notin W$, then $u \notin W$, and hence $x_i \notin W$, $i = 1, \ldots, m+n$, implying $W \subset W_8$. Thus, since W is padded, W must be equal to W_8. □

Remark 2.2 1. The sets of the form $W = \bigcup_I W_i$, $I \subset \{1, \ldots, 8\}$ satisfying $W = s_T(W)$ are as follows:

(2.10) $$\emptyset, W_2, W_6, W_8, W_2 \cup W_3, W_2 \cup W_4, W_6 \cup W_7, W_2 \cup W_4 \cup W_8, T,$$

and

(2.11) $$W_3, W_4, W_5, W_7, W_4 \cup W_5, W_3 \cup W_5, W_3 \cup W_5 \cup W_8.$$

To verify that there are not more, note that among the unions of two nonempty sets,

$$W_2 \cup W_6 = W_4 \cup W_6 = W_2 \cup W_4,$$
$$W_3 \cup W_7 = W_5 \cup W_7 = W_3 \cup W_5,$$
$$W_6 \cup W_8 = W_7 \cup W_8 = W_6 \cup W_7.$$

Furthermore, the sets W corresponding to the following index sets I do not satisfy $W = s_T(W)$:

$$\{2,5\}, \{2,7\}, \{2,8\}, \{3,4\}, \{3,6\}, \{3,8\}, \{4,7\}, \{4,8\}, \{5,6\}, \{5,8\}.$$

Among the unions W of three or more nonempty sets, all fall into one of the following categories:

(i) $W = W_2 \cup W_4 \cup W_8$,
(ii) $W = W_3 \cup W_5 \cup W_8$,
(iii) $W = T$,
(iv) W is equal to a union of two of the W_i's,
(v) $W \neq s_T(W)$.

2. Symmetries among the exponents may further reduce the number of subsets W which must be investigated. Only the sets in (2.10) will be relevant in the proofs that follow.

Lemma 2.2 *Assume conditions* (1.10), (1.11), *and* (1.12). *If*

$$f(x) = \begin{cases} 1 & |x| < 1 \\ |x|^{-1} & |x| > 1, \end{cases}$$

and

$$(2.12) \quad F(u, v, x_1, \ldots, x_{m+n}) = f\left(\sum_1^m x_p + u\right) f\left(\sum_{m+1}^{m+n} x_p - u\right) f\left(\sum_1^m x_p + v\right) f\left(\sum_{m+1}^{m+n} x_p - v\right),$$

then

$$(2.13) \quad \int_{\mathbf{R}^{m+n+2}} F(u, v, x_1, \ldots, x_{m+n}) |u|^{-\beta} |v|^{-\beta} |x_1|^{-\alpha} \ldots |x_{m+n}|^{-\alpha} \, du \, dv \, dx_1 \ldots dx_{m+n} < \infty.$$

Proof. We apply the power counting theorem. The set $T = T'$ of linear functionals associated with the integral (2.13) is given in (2.9). To verify condition (a), it suffices to show $d_0(W) > 0$ for any subset W of T. The elements of W associated with exponents $\alpha_i \neq 0$ are contained in $\{u, v, x_1, \ldots, x_{m+n}\}$. Thus here, $d_0(W) = r(W) + \sum_W \alpha_i$. The assumption $\alpha_i > -1$ for all i yields $d_0(W) > 0$.

To verify condition (b) we apply Lemma 2.1. Observe that conditions (1.10) and (1.11) are symmetric in m and n, and note that x_i, $i = 1, \ldots, m+n$ have the same exponent, $-\alpha$; u and v have the same exponent, $-\beta$; and the remaining elements of T have the same exponent, -1. In view of these symmetries, and

Remark 2.2, it suffices to show $d_\infty(W)<0$ for the eight proper subsets of T listed in (2.10). We show it for one set here, as an example. By condition (1.11),

$$d_\infty(W_2 \cup W_4 \cup W_8)=(m+n+2)-(m+3)-n\alpha=n-1-n\alpha<0$$

by (1.11). □

Lemma 2.3 *Assume conditions* (1.10), (1.11), *and* (1.12), *suppose* $0<\delta<\pi/4$, *and set*

$$(2.14)\quad E_N(\delta)=\left\{(u,v,x_1,\dots,x_{m+n})\in[-N\pi,N\pi]^{m+n+2}:\left|\sum_1^m x_p+u\right|<N(2\pi-\delta),\right.$$
$$\left.\left|\sum_{m+1}^{m+n} x_p-u\right|<N(2\pi-\delta),\left|\sum_1^m x_p+v\right|<N(2\pi-\delta),\left|\sum_{m+1}^{m+n} x_p-v\right|<N(2\pi-\delta)\right\}.$$

Then

$$(2.15)\quad \int_{E_N} \Delta_N\left(\sum_1^m x_p+u\right)\Delta_N\left(\sum_{m+1}^{m+n} x_p-u\right)\overline{\Delta_N}\left(\sum_1^m x_p+v\right)\overline{\Delta_N}\left(\sum_{m+1}^{m+n} x_p-v\right)$$
$$\cdot|u|^{-\beta}|v|^{-\beta}|x_1|^{-\alpha}\dots|x_{m+n}|^{-\alpha}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x_1\dots\mathrm{d}x_{m+n}$$

converges to

$$\int_{\mathbf{R}^{m+n}} |K_0(x)|^2\,\mathrm{d}^{m+n}G_0$$

as $N\to\infty$, *where* Δ_N *and* K_0 *are as defined in* (2.7) *and* (1.16) *respectively.*

Proof When $|y|<N(2\pi-\delta)$, $|\Delta_N(y)|<\min(1, C_\delta|y|^{-1})$. Here the integrand in (2.15) is bounded in absolute value by $C_\delta^4 F(u,v,x_1,\dots,x_{m+n})$ $|u|^{-\beta}|v|^{-\beta}|x_1|^{-\alpha}\dots|x_{m+n}|^{-\alpha}$, where F is as defined in (2.12). The lemma now follows from Lemma 2.2, the dominated convergence theorem, and the fact that $\Delta_N\to\Delta$ and $E_N\to\mathbf{R}^{m+n+2}$. □

Lemma 2.4 *Assume conditions* (1.10) *through* (1.12), *let* $H=(m+n)(\alpha-1)/2+\beta+1$, *and let* $E_N(\delta)$ *be as in* (2.14). *Then*

$$(2.16)\quad \int_{[-N\pi,N\pi]^{m+n+2}\setminus E_N} \Delta_N\left(\sum_1^m x_p+u\right)\Delta_N\left(\sum_{m+1}^{m+n} x_p-u\right)\overline{\Delta_N}\left(\sum_1^m x_p+v\right)\overline{\Delta_N}\left(\sum_{m+1}^{m+n} x_p-v\right)$$
$$\cdot|u|^{-\beta}|v|^{-\beta}|x_1|^{-\alpha}\dots|x_{m+n}|^{-\alpha}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x_1\dots\mathrm{d}x_{m+n}$$
$$\equiv\frac{1}{N^{2H}}\int_{[-\pi,\pi]^{m+n+2}\setminus(E_N/N)} S_N\left(\sum_1^m x_p+u\right)S_N\left(\sum_{m+1}^{m+n} x_p-u\right)$$
$$\cdot\overline{S_N}\left(\sum_1^m x_p+v\right)\overline{S_N}\left(\sum_{m+1}^{m+n} x_p-v\right)|u|^{-\beta}|v|^{-\beta}|x_1|^{-\alpha}\dots|x_{m+n}|^{-\alpha}\,\mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x_1\dots\mathrm{d}x_{m+n}$$

converges to 0 *as* $N\to\infty$.

Proof. The identity results from the change of variables $Nx_i \to x_i$, $i=1, \ldots, m+n$, $Nu \to u$, and $Nv \to v$. We now focus on the convergence. We have

$$|S_N(y)| \leqq 4N^\eta |y+\theta|^{\eta-1}, \qquad 0<\eta<1, \tag{2.17}$$

where $|\theta|=2k\pi$ when $(2k-1)\pi \leqq |y| < (2k+1)\pi$ (see for example Fox and Taqqu [9] pages 226–227 and 237).

First consider the case $\beta>0$. Let

$$\begin{aligned} \eta_1 &= \tfrac{1}{2}(\alpha m+\beta-m+1)-\varepsilon, \\ \eta_2 &= \tfrac{1}{2}(\alpha n+\beta-n+1)-\varepsilon, \end{aligned} \tag{2.18}$$

where $\varepsilon>0$ is chosen so that $0<\eta_i<1$, $i=1,2$. Observe that conditions (1.11) and (1.12) ensure that such an ε exists. Hence, by (2.17), the right side of (2.16) is bounded by a constant times a finite sum of integrals of the form

$$\begin{aligned} N^{2(\eta_1+\eta_2)-2H} & \int_{[-\pi,\pi]^{m+n+2}} \left|\sum_1^m x_p+u+\theta_1\right|^{\eta_1-1} \\ & \cdot \left|\sum_{m+1}^{m+n} x_p-u+\theta_2\right|^{\eta_2-1} \left|\sum_1^m x_p+v+\theta_3\right|^{\eta_1-1} \\ & \cdot \left|\sum_{m+1}^{m+n} x_p-v+\theta_4\right|^{\eta_2-1} |u|^{-\beta}|v|^{-\beta}|x_1|^{-\alpha}\ldots|x_{m+n}|^{-\alpha} \\ & \cdot \mathrm{d}u\,\mathrm{d}v\,\mathrm{d}x_1\ldots\mathrm{d}x_{m+n}, \end{aligned} \tag{2.19}$$

where $\theta_i=2k_i\pi$ for some integer k_i, $i=1, 2, 3, 4$. Note that $\theta_i \neq 0$ for some $i=1, 2, 3, 4$, because the set (E_N/N) is excluded from the domain of integration. Since the exponent $2(\eta_1+\eta_2)-2H=-4\varepsilon$ of N is negative, it suffices to show that the integral in (2.19) is finite.

It is sufficient to show that condition (a) of the power counting theorem is satisfied where

$$\begin{aligned} T' = \Bigg\{ & \sum_1^m x_p+u+\theta_1, \sum_{m+1}^{m+n} x_p-u+\theta_2, \sum_1^m x_p+v+\theta_3, \\ & \sum_{m=1}^{m+n} x_p-v+\theta_4, u, v, x_1. \ldots, x_{m+n} \Bigg\}. \end{aligned}$$

Let W' be a padded subset of T' satisfying $W'=s_{T'}(W')$, let T be as in (2.9), and let W be the subset of T which is W' without the θ_i's. Then $r(W') \geqq r(W)$. Since all exponents are negative, and the set of exponents associated with $s_{T'}(W')$ is contained in the set of exponents associated with $s_T(W)$, one has

$$d_0(W')=r(W')+\sum_{s_{T'}(W')} \alpha_i \geqq r(W)+\sum_{s_T(W)} \alpha_i = d_0(W)=d_0(s_T(W)).$$

Observe that W and hence $s_T(W)$ is padded. We will show that $d_0(W)>0$ for all nonempty, padded subsets W of T satisfying $W=s_T(W)$, except $W=T$. This will imply condition (a) for all relevant sets, except T'. Since the presence of the θ_i's in T' is crucial, as

$$\begin{aligned}(2.20)\quad d_0(T)&=(m+n+2)+2(\eta_1-1)+2(\eta_2-1)-(m+n)\alpha-2\beta\\ &=(m+n+2)+(m\alpha+\beta-m+1-2\varepsilon)-2+(n\alpha+\beta-n+1-2\varepsilon)\\ &\quad-2-(m+n)\alpha-2\beta=-4\varepsilon<0,\end{aligned}$$

the set T' will be handled later.

Note that u and v have the same exponent $-\beta$; $x_1, \ldots, x_{m+n}$ have the same exponent $-\alpha$; and conditions (1.10) and (1.11) are symmetric in m and n. In view of these symmetries, (2.18), Lemma 2.1 and Remark 2.2, condition (a) will be satisfied, except for T', if $d_0(W)>0$ for the seven nonempty, proper subsets W listed in (2.10). As a consequence of conditions (1.10), (1.11), (1.12), it is a straightforward exercise to show $d_0(W)>0$ for those seven sets. We do it for W_8, as an example. By (1.10), we may choose ε even smaller to that

$$\begin{aligned}(2.21)\quad d_0(W_8)&=3+2(\eta_1-1)+2(\eta_2-1)\\ &=3+(m\alpha+\beta-m+1-2\varepsilon)-2+(n\alpha+\beta-n+1-2\varepsilon)-2\\ &=(2\beta+(m+n)\alpha)-(m+n-1)-4\varepsilon>0.\end{aligned}$$

Focusing on T' now, assume without loss of generality that $\theta_1\neq 0$. Then

$$r(T')\geqq r\left(\left\{x_1, \ldots, x_{m+n}, u, v, \sum_1^m x_p+u+\theta_1\right\}\right)=m+n+3.$$

Hence

$$d_0(T')\geqq m+n+3+2(\eta_1-1)+2(\eta_2-1)-(m+n)\alpha-2\beta=1+d_0(T)=1-4\varepsilon>0$$

by (2.20).

This completes the proof in the case $\beta>0$.

If $\beta\leqq 0$, let

$$\eta_1=\eta_2=\tfrac{1}{4}((m+n)\alpha+2\beta-(m+n)+2)-\varepsilon.$$

By conditions (1.10) and (1.12), $\varepsilon>0$ can be chosen so that $1/4<\eta_i<1$. The right side of (2.16) is then bounded by a constant multiple of (2.19). Since $N^{2(\eta_1+\eta_2)-2H}=N^{-4\varepsilon}$, it suffices to show the integral in (2.19) is finite. Assume without loss of generality that $\theta_1\neq 0$. Thus $\left|\sum_1^m x_p+u\right|>2\pi-\delta$, and hence, since $|u|<\pi$, there is some $i\in\{1, \ldots, m\}$ such that $|x_i|>(\pi-\delta)/m$. Therefore $|x_i|^{-\alpha}$ is bounded. Since $|u|^{-\beta}$ and $|v|^{-\beta}$ are also bounded, the factors $|x_i|^{-\alpha}$, $|u|^{-\beta}$, and $|v|^{-\beta}$ can be removed from the integrand, and hence the only potentially padded set is W_8'. To see this, observe that x_p, $p\in\{1, \ldots, i-1, i+1, \ldots, m+n\}$, cannot be expressed as a linear combination of elements of $T'\backslash\{u, v, x_i, x_p\}$. Since $d_0(W_8')$ is the same as in the case $\beta>0$, the proof is complete. □

We are now in a position to prove Proposition 2.1.

Proof of Proposition 2.1 The identity in (2.8) results from a change of variables. Moreover,

$$\text{(2.22)} \qquad \int\limits_{[-N\pi, N\pi]^{m+n}} |K_N(\mathbf{x})|^2 \, \mathrm{d}^{m+n} G_N$$

$$= \int\limits_{[-N\pi, N\pi]^{m+n+2}} \Delta_N\Big(\sum_1^m x_p + u\Big) \Delta_N\Big(\sum_{m+1}^{m+n} x_p - u\Big) \overline{\Delta_n}\Big(\sum_1^m x_p + v\Big) \overline{\Delta_N}\Big(\sum_{m+1}^{m+n} x_p - v\Big)$$

$$\cdot |u|^{-\beta} |v|^{-\beta} |x_1|^{-\alpha} \ldots |x_{m+n}|^{-\alpha} \, \mathrm{d}u \, \mathrm{d}v \, \mathrm{d}x_1 \ldots \mathrm{d}x_{m+n}$$

converges to $\int\limits_{\mathbf{R}^{m+n}} |K_0(\mathbf{x})|^2 \, \mathrm{d}^{m+n} G_0$ as $N \to \infty$ by Lemmas 2.3 and 2.4. □

We state the following lemma, a consequence of Lemmas 2.2 and 2.4, which will be useful in Sect. 4. Since the proof is parallel to that of Lemma 10 of Terrin and Taqqu [19], it is omitted here.

Lemma 2.5

$$\int\limits_{\mathbf{R}^{m+n} \setminus [-A, A]^{m+n}} |K_N(\mathbf{x})|^2 \, \mathrm{d}^{m+n} G_N$$

converges to zero as $A \to \infty$, *uniformly for* $N = 0, 1, 2, \ldots$.

3 Limits of sums of Wick powers

This section extends a result of Giraitis and Surgailis [11]. We prove a proposition describing the limit behavior as $N \to \infty$ of sums of Wick powers of the form

$$\text{(3.1)} \qquad \sum_{s_1=-\infty}^{\infty} \cdots \sum_{s_m=-\infty}^{\infty} c_N(s_1, \ldots, s_m) :\xi_{s_1} \ldots \xi_{s_m}:$$

where

$$c_N(s_1, \ldots, s_m) = \int\limits_{[-\pi, \pi]^m} \mathrm{e}^{-i \sum_{p=1}^m s_p x_p} h_N(x_1, \ldots, x_m) \prod_{p=1}^m |x_p|^{-\alpha/2} L_1^{1/2}(|x_p|^{-1}) \, \mathrm{d}x_1 \ldots \mathrm{d}x_m,$$

and where $\alpha < 1$, L_1 is slowly varying, and h_N is a complex valued function in $L^2([-\pi, \pi]^m)$ satisfying $h_N(x_1, \ldots, x_m) = \overline{h_N(-x_1, \ldots, -x_m)}$. Observe that the c_N's are real and that replacing the exponential $\exp\Big\{-i \sum_{p=1}^m s_p x_p\Big\}$ by $\exp\Big\{i \sum_{p=1}^m s_p x_p\Big\}$ does not change (3.1).

Let G be the measure on $[-\pi, \pi]$ defined by

$$\text{(3.2)} \qquad G(dx) = |x|^{-\alpha} L_1(|x|^{-1}) \, dx,$$

where $\alpha < 1$ and L_1 is slowly varying at ∞ and bounded on bounded intervals. If G_N is the renormalized measure given by

$$(3.3) \qquad G_N(dx) \equiv N^{1-\alpha} G(dx/N)/L_1(N) = |x|^{-\alpha}[L_1(N|x|^{-1})/L_1(N)]\, dx$$

and G_0 is the measure on $\mathbf{R}$ with

$$(3.4) \qquad G_0(dx) = |x|^{-\alpha}\, dx$$

then

$$(3.5) \qquad G_N(A) \to G_0(A)$$

for all bounded Lebesgue measurable sets. Indeed, convergence of the measures follows from the integrability of $|x|^{-\alpha}$ on A and the fact that when $\delta > 0$,

$$(3.6) \qquad \frac{|x|^{\delta} L_1(N|x|^{-1})}{L_1(N)} = \frac{|N|x|^{-1}|^{-\delta} L_1(N|x|^{-1})}{N^{-\delta} L_1(N)} \to |x|^{\delta}$$

as $N \to \infty$ uniformly for $x \in A$ (see for example Bingham, Goldie and Teugels [4], Theorem 1.5.2).

Proposition 3.1 *Let G, G_N, and G_0 be as in* (3.2), (3.3), *and* (3.4) *respectively, and let Z_{G_0} be the Gaussian random measure defined as in* (1.8). *Assume* $\ldots \xi_{-1}$, ξ_0, ξ_1, $\ldots$ *are independent and identically distributed with $E\xi_0 = 0$, $E\xi_0^2 = \sigma^2$ and $E|\xi_0|^{2m} < \infty$ for some integer $m \geqq 2$. Suppose h_N, $N = 1, 2, \ldots$, is a complex-valued function on $[-\pi, \pi]^m$ satisfying $h_N(-x_1, \ldots, -x_m) = h_N(x_1, \ldots, x_m)$, and L_2 is slowly varying at ∞ and bounded on bounded intervals. Suppose the following three conditions hold:*

$$(3.7) \qquad \text{(i) } K_N(x_1, \ldots, x_m) \equiv N^{-H+m(\alpha-1)/2} L_2^{-1}(N)\, h_N(x_1/N, \ldots, x_m/N) \to K_0(x_1, \ldots, x_m)$$

uniformly on $[-A, A]^m$ for any $A > 0$, for some constant H, and some function K_0 that is continuous except on a set of G_0^m measure zero,

$$(3.8) \qquad \text{(ii) } \lim_{N \to \infty} \frac{1}{d_N^2} \int_{[-\pi,\pi]^m} |h_N|^2\, \mathrm{d}^m G = \int_{\mathbf{R}^m} |K_0|^2\, \mathrm{d}^m G_0 < \infty,$$

where $d_N = N^H L_1^{m/2}(N)\, L_2(N)$, and

$$\text{(iii) } \lim_{A \to \infty} \int_{\mathbf{R}^m \setminus [-A,A]^m} |K_N|^2\, \mathrm{d}^m G_N = 0$$

uniformly for $N = 0, 1, \ldots$.

Then the sequence

$$(3.9) \qquad U_N \equiv \frac{1}{d_N} \sum_{(s)_m} \left\{ \int_{[-\pi,\pi]^m} \mathrm{e}^{-i\sum_{p=1}^m s_p x_p} h_N(\mathbf{x}) \prod_{p=1}^m |x_p|^{-\alpha/2} L_1^{1/2}(|x_p|^{-1})\, \mathrm{d}^m \mathbf{x} \right\} :\xi_{s_1} \ldots \xi_{s_m}:$$

converges in distribution as $N \to \infty$ *to*

(3.10) $$(2\pi\sigma^2)^{m/2} \int_{\mathbf{R}^m}'' K_0(x_1, \ldots, x_m)\, Z_{G_0}(\mathrm{d}x_1) \ldots Z_{G_0}(\mathrm{d}x_m),$$

where $\sum_{(s)_m} = \sum_{s_1} \ldots \sum_{s_m}$.

Remark 3.1 1. The only explicit condition on α is $\alpha < 1$. Further conditions on α and H may be needed to obtain (i), (ii), or (iii).
2. The proposition still holds with each $|x_p|^{-\alpha/2}$ replaced by $|x_p|^{-\alpha_p/2}$ and L_1 replaced by $L_{(1,p)}$. That is, one may have m possibly different measures G_p, $p = 1, \ldots, m$, in place of m copies of G.

Let $\sum'_{(s)_m}$ be the summation of those terms in $\sum_{(s)_m}$ with $s_i \neq s_j$, $i \neq j$. The proposition is proved by letting $0 < A < N\pi$ and decomposing U_N as

$$\sum' \int_{[-A/N, A/N]^m} + \sum' \int_{[-\pi,\pi]^m \setminus [-A/N, A/N]^m} + \Big(\sum \int_{[-\pi,\pi]^m} - \sum' \int_{[-\pi,\pi]^m}\Big) = Y_N^A + R_N + R_N',$$

and showing that only Y_N^A contributes to the limit. Since the independence of the ξ's implies

$$:\xi_{s_1} \ldots \xi_{s_m}: = :\xi_{s_1}: \ldots :\xi_{s_m}: = \xi_{s_1} \ldots \xi_{s_m}$$

if the s_i's are all distinct (Factorization Lemma in Avram and Taqqu [2]), we let

(3.11) $$Y_N = \frac{1}{d_N} \sum'_{(s)_m} \Big\{ \int_{[-\pi,\pi]^m} e^{-i\sum_1^m s_p x_p} h_N(\mathbf{x}) \prod_1^m |x_p|^{-\alpha/2} L_1^{1/2}(|x_p|^{-1})\, \mathrm{d}^m x \Big\} \xi_{s_1} \ldots \xi_{s_m},$$

(3.12) $$Y_N^A = \frac{1}{d_N} \sum'_{(s)_m} \Big\{ \int_{[-A/N, A/N]^m} e^{-i\sum_1^m s_p x_p} h_N(\mathbf{x}) \prod_1^m |x_p|^{-\alpha/2} L_1^{1/2}(|x_p|^{-1})\, \mathrm{d}^m x \Big\} \xi_{s_1} \ldots \xi_{s_m},$$

$$R_N = Y_N - Y_N^A,$$
$$R_N' = U_N - Y_N.$$

We approximate h_N with step functions, using the following definition of a step function which excludes summation over diagonals. Let $A > 0$, let M be a positive integer, and let $\Delta_{-M}, \ldots, \Delta_M$ be a partition of $[-A, A]$ into $2M$ intervals of equal length. Denote by $\sum''$ the summation over $i(k) = -M, \ldots, M$, $k = 1, \ldots, m$, where $i(k) \neq \pm i(l)$ if $k \neq l$. We say that g_A is a *step function* if

(3.13) $$g_A(x_1, \ldots, x_m) = \sum'' g_{\Delta_{i(1)}, \ldots, \Delta_{i(m)}} 1_{\Delta_{i(1)} \times \ldots \times \Delta_{i(m)}}(x_1, \ldots, x_m).$$

Such step functions are dense in $L^2(\mu)$ for any atomless measure μ, as established by Major [14], pp. 28–29.

First we approximate the kernels K_N and K_0, as in the proof of Lemma 3 of Dobrushin and Major [7].

Lemma 3.1 *Assume the conditions of Proposition* 3.1. *Then there exist step functions* $\{g_A\}_{A>0}$ *such that for any* $\varepsilon>0$

$$\int_{[-A,A]^m} |K_N - g_A|^2 \, \mathrm{d}^m G_N < \varepsilon \tag{3.14}$$

when $A > A(\varepsilon)$ *and* $N=0$ *or* $N \geqq N(\varepsilon)$.

Let now g_A be as in Lemma 3.1 and set

$$I_N^A = \frac{1}{d_N} \sum_{(s)_m}{}' \left\{ \int_{[-A/N,A/N]^m} \mathrm{e}^{-i\sum_1^m s_p x_p} N^{H-m(\alpha-1)/2} L_2(N)\, g_A(N x_1, \ldots, N x_m) \cdot \prod_1^m |x_p|^{-\alpha/2} L_1^{1/2}(|x_p|^{-1}) \, \mathrm{d}x_1 \ldots \mathrm{d}x_m \right\} \xi_{s_1} \ldots \xi_{s_m}. \tag{3.15}$$

Lemma 3.2 *Assume the conditions of Proposition* 3.1. *Then*

$$\lim_A \limsup_N \operatorname{Var}(Y_N^A - I_N^A) = 0$$

Proof.

$$\operatorname{Var}(Y_N^A - I_N^A) = E\left| \frac{1}{d_N} \sum_{(s)_m}{}' B_N(s_1, \ldots, s_m)\, \xi_{s_1} \ldots \xi_{s_m} \right|^2 \tag{3.16}$$

where

$$B_N = \int_{[-A/N,A/N]^m} \mathrm{e}^{-i\sum_1^m s_p x_p} [h_N(x_1, \ldots, x_m) - N^{H-m(\alpha-1)/2} L_2(N) \cdot g_A(N x_1, \ldots, N x_m)] \prod_1^m |x_p|^{-\alpha/2} L_1^{1/2}(|x_p|^{-1}) \, \mathrm{d}x_1 \ldots \mathrm{d}x_m. \tag{3.17}$$

Observe that B_N may not be symmetric in $(s_1, \ldots, s_m)$, since h_N is not necessarily symmetric in $x_1, \ldots, x_m$. However

$$\begin{aligned} E\Big|\sum_{(s)_m}{}' B_N(s_1, \ldots, s_m)\, \xi_{s_1} \ldots \xi_{s_m}\Big|^2 &= E\Big|\sum_{(s)_m}{}' \operatorname{sym}(B_N(s_1, \ldots, s_m))\, \xi_{s_1} \ldots \xi_{s_m}\Big|^2 \\ &= m!\, \sigma^{2m} \sum_{(s)_m}{}' |\operatorname{sym}(B_N(s_1, \ldots, s_m))|^2 \\ &\leqq m!\, \sigma^{2m} \sum_{(s)_m}{}' |B_N(s_1, \ldots, s_m)|^2 \end{aligned}$$

where $\operatorname{sym} f(x_1, \ldots, x_m) = \frac{1}{m!} \sum_{\sigma} f(x_{\sigma(1)}, \ldots, x_{\sigma(m)})$ is the symmetrization of f. Hence (3.16) is bounded by

$$(3.18) \quad \frac{m!\,\sigma^{2m}}{d_N^2} \sum_{(s)_m}' |B_N(s_1, \ldots, s_m)|^2 \leqq \frac{m!\,\sigma^{2m}}{d_N^2} \sum_{(s)_m} |B_N(s_1, \ldots, s_m)|^2$$

$$= \frac{(2\pi)^m\, m!\,\sigma^{2m}}{d_N^2} \int_{[-A/N, A/N]^m} \cdot |h_N(x_1, \ldots, x_m) - N^{H - m(\alpha-1)/2} L_2(N)\, g_A(N x_1, \ldots, N x_m)|^2 \cdot \prod_1^m |x_p|^{-\alpha} L_1(|x_p|^{-1})\, dx_1 \ldots dx_m,$$

by Parseval's equality. With a change of variables, the right side of (3.18) becomes

$$(2\pi\sigma^2)^m\, m! \int_{[-A, A]^m} |K_N(\mathbf{x}) - g_A(\mathbf{x})|^2 \prod_1^m |x_p|^{-\alpha} [L_1(N|x_p|^{-1})/L_1(N)]\, \mathrm{d}^m x.$$

Thus

$$\limsup_N \operatorname{Var}(Y_N^A - I_N^A) \leqq \limsup_N (2\pi\sigma^2)^m\, m! \int_{[-A, A]^m} |K_N - g_A|^2\, \mathrm{d}^m G_N.$$

Hence, by Lemma 3.1,

$$\lim_A \limsup_N \operatorname{Var}(Y_N^A - I_N^A) = 0. \quad \square$$

The following lemma demonstrates that the remainder $R_N = Y_N - Y_N^A$ does not contribute to the limit.

Lemma 3.3 *Assume the conditions of Proposition* 3.1. *Then*

$$\lim_A \limsup_N \operatorname{Var}(R_N) = 0.$$

Proof. As in the proof of Lemma 3.2, we have

$$\operatorname{Var}(R_N) = E\left|\frac{1}{d_N} \sum_{(s)_m}' \tilde{B}_N(s_1, \ldots, s_m)\, \xi_{s_1} \ldots \xi_{s_m}\right|^2$$

$$\leqq \frac{m!}{d_N^2} \sum_{(s)_m} |\tilde{B}_N(s_1, \ldots, s_m)\, \xi_{s_1} \ldots \xi_{s_m}|^2$$

where

$$\tilde{B}_N = \int_{[-\pi, \pi]^m \setminus [-A/N, A/N]^m} \mathrm{e}^{-i \sum_1^m s_p x_p} h_N(\mathbf{x}) \prod_1^m |x_p|^{-\alpha/2} L_1^{1/2}(|x_p|^{-1})\, \mathrm{d}^m x.$$

Thus

$$\operatorname{Var}(R_N) \leq \frac{(2\pi\sigma^2)^m m!}{d_N^2} \int_{[-\pi,\pi]^m \setminus [-A/N, A/N]^m} |h_N(\mathbf{x})|^2 \prod_1^m |x_p|^{-\alpha} L_1(|x_p|^{-1})\, \mathrm{d}^m x$$

$$= (2\pi\sigma^2)^m m! \int_{[-N\pi, N\pi]^m \setminus [-A, A]^m} |K_N(\mathbf{x})|^2 \prod_1^m |x_p|^{-\alpha} [L_1(N|x_p|^{-1})/L_1(N)]\, \mathrm{d}^m x.$$

Hence

$$\lim_A \limsup_N \operatorname{Var}(R_N) \leq \lim_A \limsup_N (2\pi\sigma^2)^m m! \int_{\mathbf{R}^m \setminus [-A, A]^m} |K_N|^2\, \mathrm{d}^m G_N = 0$$

by (iii) of Proposition 3.1. □

As a consequence of Lemmas 3.2 and 3.3, one has the following

Lemma 3.4 *Assume the conditions of Proposition* 3.1. *Then*

$$\lim_A \limsup_N \operatorname{Var}(Y_N - I_N^A) = 0.$$

Next, we prove two lemmas which will be useful in finding the limit of I_N^A. Let Δ be a bounded set in $\mathbf{R}$,

(3.19) $$a_\Delta(s) = \int_{\Delta/N} \mathrm{e}^{-isx} |x|^{-\alpha/2} L_1^{1/2}(|x|^{-1})\, \mathrm{d}x,$$

and

(3.20) $$Z_N(\Delta) = N^{(1-\alpha)/2} L_1^{-1/2}(N) \sum_s a_\Delta(s) \xi_s.$$

Since $Z_N(\Delta)$ has representation

(3.21) $$Z_N(\Delta) = N^{(1-\alpha)/2} L_1^{-1/2}(N)\, 2\pi \int_{\Delta/N} |x|^{-\alpha/2} L_1^{1/2}(|x|^{-1})\, \mathrm{d}Z_\xi(x),$$

where $Z_\xi(x)$ is an orthogonal-increment process with $E|Z_\xi(dx)|^2 = \frac{\sigma^2}{2\pi} dx$ (see for example Theorem 4.10.1 of Brockwell and Davis [5]), $Z_N(\Delta)$ has the following properties:

(i) $Z_N(\Delta) = \bar{Z}_N(-\Delta)$
(ii) $\operatorname{Re} Z_N(\Delta)$ and $\operatorname{Im} Z_N(\Delta)$ have mean zero and are uncorrelated.
(iii) $\operatorname{Re} Z_N(\Delta)$ and $\operatorname{Im} Z_N(\Delta)$ each have variance equal to $\frac{1}{2} 2\pi\sigma^2 G_N(\Delta)$, if $\Delta \cap -\Delta = \emptyset$.
(iv) $\operatorname{Re} Z_N(\Delta_1), \operatorname{Im} Z_N(\Delta_1), \ldots, \operatorname{Re} Z_N(\Delta_m), \operatorname{Im} Z_N(\Delta_m)$ are uncorrelated, if $\pm\Delta_1, \ldots, \pm\Delta_m$ are disjoint.

(Re denotes the real part and Im denotes the imaginary part.)

Lemma 3.5 *Let $\Delta_1, \ldots, \Delta_m$ be bounded sets in* $\mathbf{R}$ *with $\pm\Delta_1, \ldots, \pm\Delta_m$ disjoint. Then*

$$(Z_N(\Delta_1), \ldots, Z_N(\Delta_m)) \xrightarrow{d} (2\pi\sigma^2)^{1/2} (Z_{G_0}(\Delta_1), \ldots, Z_{G_0}(\Delta_m))$$

where Z_N *and* Z_{G_0} *are as defined in* (3.20) *and* (1.8) *respectively.* (*The result holds for any* i.i.d. *sequence* ξ_s *satisfying* $E\xi_s=0$ *and* $E\xi_s^2=\sigma^2$.)

Proof. By Property (iv), it suffices to prove that $Z_N(\Delta)\xrightarrow{d} Z_{G_0}(\Delta)$ for any bounded set Δ in $\mathbf{R}$ with $\Delta\cap-\Delta=\emptyset$. We first prove that

$\operatorname{Re} Z_N(\Delta)\xrightarrow{d}(2\pi\sigma^2)^{1/2}\operatorname{Re} Z_{G_0}(\Delta)$, that is, $\operatorname{Re} Z_N(\Delta)\xrightarrow{d} N(0, \frac{1}{2}2\pi\sigma^2 G_0(\Delta))$. To make the dependence of $a_\Delta(s)$ on N explicit, write $a_\Delta(s)=a_\Delta(s,N)$. Suppose throughout that N is large enough for Δ to be contained in $(-N\pi, N\pi)$.
Observe that, as $N\to\infty$,

$$(3.22)\quad N^{1-\alpha}L_1^{-1}(N)\,\sigma^2\sum_s(\operatorname{Re} a_\Delta(s,N))^2=E|\operatorname{Re} Z_N(\Delta)|^2=\pi\sigma^2 G_N(\Delta)\to\pi\sigma^2 G_0(\Delta)$$

by Property (iii) of $Z_N(\Delta)$ and (3.5). Thus, for $\varepsilon_N\to 0$, there exists a sequence $V_0(N)=V_0(N,\varepsilon_N)$ such that

$$N^{1-\alpha}L_1^{-1}(N)\sum_{|s|>V_0(N)}(\operatorname{Re} a_\Delta(s,N))^2<\varepsilon_N,$$

and hence

$$(3.23)\qquad \lim_{N\to\infty} N^{1-\alpha}L_1^{-1}(N)\sum_{|s|>V_0(N)}(\operatorname{Re} a_\Delta(s,N))^2=0.$$

Since we can write

$$\begin{aligned}\operatorname{Re} Z_N(\Delta)&=N^{(1-\alpha)/2}L_1^{-1/2}(N)\sum_{|s|\leqq V_0(N)}\operatorname{Re} a_\Delta(s,N)\xi_s\\&\quad+N^{(1-\alpha)/2}L_1^{-1/2}(N)\sum_{|s|>V_0(N)}\operatorname{Re} a_\Delta(s,N)\xi_s,\end{aligned}$$

and since

$$\begin{aligned}&E\left|N^{(1-\alpha)/2}L_1^{-1/2}(N)\sum_{|s|>V_0(N)}\operatorname{Re} a_\Delta(s,N)\xi_s\right|^2\\&\quad=N^{1-\alpha}L_1^{-1}(N)\sigma^2\sum_{|s|>V_0(N)}(\operatorname{Re} a_\Delta(s,N))^2\to 0,\end{aligned}$$

it suffices to show

$$(3.24)\qquad N^{(1-\alpha)/2}L_1^{-1/2}(N)\sum_{|s|\leqq V_0(N)}\operatorname{Re} a_\Delta(s,N)\,\xi_s\xrightarrow{d} N(0,\pi\sigma^2 G_0(\Delta)).$$

Applying Schwartz's inequality, one has

$$\begin{aligned}&\max_s N^{(1-\alpha)/2}L_1^{-1/2}(N)|\operatorname{Re} a_\Delta(s,N)|\leqq\max_s N^{(1-\alpha)/2}L_1^{-1/2}(N)|a_\Delta(s,N)|\\&\quad=\max_s N^{(1-\alpha)/2}L_1^{-1/2}(N)\Big|\int_{\Delta/N} e^{-isx}|x|^{-\alpha/2}L_1^{1/2}(|x|^{-1})\,dx\Big|\\&\quad\leqq N^{(1-\alpha)/2}L_1^{-1/2}(N)\Big[\int_{\Delta/N}|x|^{-\alpha}L_1(|x|^{-1})\,dx\Big]^{1/2}\Big[\int_{\Delta/N}dx\Big]^{1/2}\\&\quad=[G_N(\Delta)(|\Delta|/N)]^{1/2}=o(1)\end{aligned}$$

since $G_N(\Delta) \to G_0(\Delta)$. Furthermore,

$$\lim_{N \to \infty} N^{1-\alpha} L_1^{-1}(N) \sigma^2 \sum_{|s| \leq V_0(N)} (\mathrm{Re}\, a_\Delta(s, N))^2 = \pi \sigma^2 G_0(\Delta),$$

by (3.22) and (3.23). Hence, Lindeberg's condition is satisfied (see Billingsley [3], Problem 27.6), and thus (3.24) holds. Similarly, using Properties (ii) and (iii), one gets $A(\mathrm{Re}\, Z_N(\Delta)) + B(\mathrm{Im}\, Z_N(\Delta)) \xrightarrow{d} A(\mathrm{Re}\, Z_{G_0}(\Delta)) + B(\mathrm{Im}\, Z_{G_0}(\Delta))$ for any real numbers A and B. □

Let $V = \{V_1, \ldots, V_l\}$ be a partition of $\{1, \ldots, m\}$, and let

(3.25) $$q_{V_j}(s) = \prod_{p \in V_j} N^{(1-\alpha)/2} L_1^{-1/2}(N)\, a_{\Delta_{i(p)}}(s)$$

where $\Delta_{i(1)}, \ldots, \Delta_{i(m)}$ are as in (3.13).

Lemma 3.6 *If $|V| = l < m$, $\ldots \xi_{-1}, \xi_0, \xi_1 \ldots$ are independent and identically distributed with $E|\xi_0|^{2m} < \infty$, and $A_{n_i}^{(i)}(x)$ is a polynomial of degree n_i, then*

$$\mathrm{Var} \sum_s{}' q_{V_1}(s_1) \ldots q_{V_l}(s_l)\, A_{|V_1|}^{(1)}(\xi_{s_1}) \ldots A_{|V_l|}^{(l)}(\xi_{s_l}) = o(1)$$

as $N \to \infty$.

Proof. By letting $\alpha' = 1 - \alpha$ and $\hat{a}(x) = |x|^{(\alpha'-1)/2} L_1^{1/2}(|x|^{-1})$, one may observe that the lemma is shown in the proof of Proposition 4.6 of Giraitis and Surgailis [11]. The only assumption on α' used in that proof is $\alpha' > 0$ (i.e. $\alpha < 1$). □

Let

(3.26) $$I_0^A = (2\pi\sigma^2)^{m/2} \sum{}'' g_{\Delta_{i(1)} \ldots \Delta_{i(m)}} Z_{G_0}(\Delta_{i(1)}) \ldots Z_{G_0}(\Delta_{i(m)})$$

where $\sum''$ is as in (3.13), and let I_N^A be as defined in (3.15).

Lemma 3.7 *Assume the conditions of Proposition* 3.1. *Then*

$$I_N^A \xrightarrow{d} I_0^A.$$

Proof. By (3.15), (3.13) and (3.19),

$$I_N^A = N^{m(1-\alpha)/2} L_1^{-m/2}(N) \sum_{(s)_m}{}' \sum{}'' g_{\Delta_{i(1)}, \ldots, \Delta_{i(m)}}\, a_{\Delta_{i(1)}}(s_1) \ldots a_{\Delta_{i(m)}}(s_m)\, \xi_{s_1} \ldots \xi_{s_m}.$$

The sum over $(s)_m$ may be written as

$$\sum_{(s)_m}{}' = \sum_{(s)_m} - \Big(\sum_{(s)_m} - \sum_{(s)_m}{}'\Big).$$

Focusing on $\sum_{(s)_m}$, one has

$$N^{m(1-\alpha)/2} L_1^{-m/2}(N) \sum_{(s)m} \sum'' g_{\Delta_{i(1)},\dots,\Delta_{i(m)}} a_{\Delta_{i(1)}}(s_1)\dots a_{\Delta_{i(m)}}(s_m)\, \xi_{s_1}\dots\xi_{s_m}$$
$$=\sum'' g_{\Delta_{i(1)},\dots,\Delta_{i(m)}} Z_N(\Delta_{i(1)})\dots Z_N(\Delta_{i(m)})$$

$$\xrightarrow{d} \sum'' g_{\Delta_{i(1)},\dots,\Delta_{i(m)}} (2\pi\sigma^2)^{m/2} Z_{G_0}(\Delta_{i(1)})\dots Z_{G_0}(\Delta_{i(m)}) = I_0^A$$

by Lemma 3.5. The remainder $\sum_{(s)_m} - \sum'_{(s)_m}$ may be written

$$\sum''\sum_V g_\Delta\, \gamma(V,\Delta)$$

where $\sum_V$ is over all partitions $V=\{V_1,\dots,V_l\}$ of $\{1,\dots,m\}$ such that $|V|=l<m$,

$$\gamma(V,\Delta)=\sum_{(s)_l}' q_{V_1}(s_1)\dots q_{V_l}(s_l)\, \xi_{s_1}^{|V_1|}\dots\xi_{s_l}^{|V_l|},$$

and q_{V_i} is as in (3.25). Observe that $|V_i|\geqq 2$ for at least one $i=1,\dots,l$. To complete the proof it suffices to show $E|\gamma|^2=o(1)$ as $N\to\infty$, but this follows from Lemma 3.6. □

Proof of Proposition 3.1 We want to show that U_N, defined in (3.9), converges to

$$Y=(2\pi\sigma^2)^{m/2} \int_{\mathbf{R}^m}'' K_0\, \mathrm{d}^m Z_{G_0}.$$

Recall that $U_N=Y_N+R'_N$ where Y_N is as in (3.11). We show first that $Y_N\xrightarrow{d} Y$ and then that Var $(R'_N)\to 0$. To show that $Y_N\xrightarrow{d} Y$, let I_N^A and I_0^A be as in (3.15) and (3.26) respectively. Then

$$E|I_0^A-Y|^2=(2\pi\sigma^2)^m \int_{\mathbf{R}^m} |g_A-K_0|^2\, \mathrm{d}^m G_0\to 0$$

as $A\to\infty$, and hence $I_N^A\xrightarrow{d} Y$. Moreover, $I_N^A\xrightarrow{d} I_0^A$, by Lemma 3.7. Therefore, since

$$\lim_A \limsup_N P[|I_N^A-Y_N|\geqq\varepsilon]\leqq \lim_A \limsup_N \frac{1}{\varepsilon^2}\operatorname{Var}(I_N^A-Y_N)=0,$$

by Lemma 3.4, one has $Y_N\xrightarrow{d} Y$, by Theorem 25.5 of Billingsley [3].

Now we show $\operatorname{Var}(R'_N)\to 0$. R'_N consists of those terms of U_N where the s_i are not all distinct. Let $V=\{V_1,\dots,V_l\}$ be a parition of $\{1,\dots,m\}$, and suppose $s_i=s_j$ if i and j belong to the same element of the partition. Then, by the Factorization Lemma in Avram and Taqqu [2],

$$:\xi_{s_1}\dots\xi_{s_m}:=Q_{|V_1|}(\xi_{s_1})\dots Q_{|V_l|}(\xi_{s_l}),$$

where Q is the univariate Appell polynomial associated with ξ. Hence,

$$U_N = \sum_V Y_N(V)$$

where

$$Y_N(V) = \frac{1}{d_N} \sum_{(s)_l}{}' \left\{ \int_{[-\pi,\pi]^m} e^{-i(s_1 \sum_{V_1} x_p + \ldots + s_l \sum_{V_l} x_p)} h_N(x_1, \ldots, x_m) \right.$$
$$\left. \cdot \prod_1^m |x_p|^{-\alpha/2} L_1^{1/2}(|x_p|^{-1}) \, dx_1 \ldots dx_m \right\} Q_{|V_1|}(\xi_{s_1}) \ldots Q_{|V_l|}(\xi_{s_l}).$$

To complete the proof, it suffices to show $\mathrm{Var}(Y_N(V)) \to 0$ if $|V| < m$. Recall that Y_N of (3.11) was written

$$Y_N = Y_N^A + R_N,$$

and Y_N^A of (3.12) was approximated by I_N^A of (3.15) in Lemma 3.2. Analogously we may write

$$Y_N(V) = Y_N^A(V) + R_N(V)$$

and approximate $Y_N^A(V)$ by $I_N^A(V)$, where

$$Y_N^A(V) = \frac{1}{d_N} \sum_{(s)_l}{}' \left\{ \int_{[-A/N,A/N]^m} e^{-i(s_1 \sum_{V_1} x_p + \ldots + s_l \sum_{V_l} x_p)} h_N(x_1, \ldots, x_m) \right.$$
$$\left. \cdot \prod_1^m |x_p|^{-\alpha/2} L_1^{1/2}(|x_p|^{-1}) \, dx_1 \ldots dx_m \right\} Q_{|V_1|}(\xi_{s_1}) \ldots Q_{|V_l|}(\xi_{s_l}),$$

and

$$I_N^A(V) = \frac{1}{d_N} \sum_{(s)_l}{}' \left\{ \int_{[-A/N,A/N]^m} e^{-i(s_1 \sum_{V_1} x_p + \ldots + s_l \sum_{V_l} x_p)} N^{H-m(\alpha-1)/2} L_2(N) \right.$$
$$\left. \cdot g_A(Nx_1, \ldots, Nx_m) \prod_1^m |x_p|^{-\alpha/2} L_1^{1/2}(|x_p|^{-1}) \, dx_1 \ldots dx_m \right\} Q_{|V_1|}(\xi_{s_1}) \ldots Q_{|V_l|}(\xi_{s_l}).$$

Indeed, setting

$$B_N(V) = \int_{[-A/N,A/N]^m} e^{-i(s_1 \sum_{V_1} x_p + \ldots + s_l \sum_{V_l} x_p)} [h_N(x_1, \ldots, x_m)$$
$$- N^{H-m(\alpha-1)/2} L_2(N) \, g_A(Nx_1, \ldots, Nx_m)]$$
$$\cdot \prod_1^m |x_p|^{-\alpha/2} L_1^{1/2}(|x_p|^{-1}) \, dx_1 \ldots dx_m,$$

one has

$$\text{(3.27)} \qquad \mathrm{Var}(Y_N^A(V) - I_N^A(V)) = E \left| \frac{1}{d_N} \sum_{(s)_l}{}' B_N(V) \, Q_{|V_1|}(\xi_{s_1}) \ldots Q_{|V_l|}(\xi_{s_l}) \right|^2$$
$$\leq \frac{l!}{d_N^2} C(V) \sum_{(s)_l}{}' |B_N(V)|^2 \leq \frac{l!}{d_N^2} C(V) \sum_{(s)_m} |B_N|^2,$$

where

$$C(V)=\max_{i,j}[EQ_{|V_i|}(\xi_0)\,Q_{|V_j|}(\xi_0)]^l,$$

and B_N is as in (3.17). The last inequality in (3.27) holds because every term in the sum on the left is included on the right. Hence

$$\lim_A \limsup_N \operatorname{Var}(Y_N^A(V)-I_N^A(V))\leq \lim_A \limsup_N \frac{l!}{d_N^2}\,C(V)\sum_{(s)_m}|B_N|^2$$
$$=\lim_A \limsup_N l!(2\pi)^m\,C(V)\int_{[-A,A]^m}|K_N-g_A|^2\,\mathrm{d}^m G_N=0$$

as in the proof of Lemma 3.2. One also has $\lim_A \limsup_N \operatorname{Var}(R_N(V))=0$, by an argument that parallels the proof of Lemma 3.3. To complete the proof of the proposition, it suffices to show

$$\lim_{N\to\infty}\operatorname{Var}(I_N^A(V))=0. \tag{3.28}$$

Since

$$I_N^A(V)=\sum{}''\,g_\Delta\sum_{(s)_l}{}'\,q_{V_1}(s_1)\dots q_{V_l}(s_l)\,Q_{|V_1|}(\xi_{s_1})\dots Q_{|V_l|}(\xi_{s_l}),$$

where q_{V_i} is as in (3.25), Lemma 3.6 implies (3.28). □

4 Proof of the main theorems

The same techniques used to expand univariate Appell polynomials in Theorem 1 of Avram and Taqqu [2] apply in the bivariate case, and thus, using the notation of Sect. 1, we have

$$P_{m,n}(X_j,X_k)=\sum_{(s)_{m+n}} b_{j-s_1}\dots b_{j-s_m}\,b_{k-s_{m+1}}\dots b_{k-s_{m+n}}:\xi_{s_1}\dots\xi_{s_{m+n}}:$$

where $\sum_{(s)_{m+n}}=\sum_{s_1=-\infty}^{\infty}\dots\sum_{s_{m+n}=-\infty}^{\infty}$ and $:\xi_{s_1}\dots\xi_{s_{m+n}}:$ is the Wick power defined in (1.5) and (1.6). Hence

$$\frac{1}{d_N}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}a_{j-k}\,P_{m,n}(X_j,X_k)=\frac{1}{d_N}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\int_{-\pi}^{\pi}\mathrm{e}^{i(j-k)u}|u|^{-\beta}L_2(|u|^{-1})\,\mathrm{d}u \tag{4.1}$$
$$\cdot\sum_{(s)_{m+n}}\int_{[-\pi,\pi]^{m+n}}\prod_1^m \mathrm{e}^{i(j-s_p)x_p}\prod_{m+1}^{m+n}\mathrm{e}^{i(k-s_p)x_p}$$
$$\cdot\prod_1^{m+n}|x_p|^{-\alpha/2}L_1^{1/2}(|x_p|^{-1})\,\mathrm{d}^{m+n}x:\xi_{s_1}\dots\xi_{s_{m+n}}:$$
$$=\frac{1}{d_N}\sum_{(s)_{m+n}}\int_{[-\pi,\pi]^{m+n}}\mathrm{e}^{-i\left(\sum_1^{m+n}s_px_p\right)}h_N(\mathbf{x})$$
$$\cdot\prod_1^{m+n}|x_p|^{-\alpha/2}L_1^{1/2}(|x_p|^{-1})\,\mathrm{d}^{m+n}x:\xi_{s_1}\dots\xi_{s_{m+n}}:$$

where

$$h_N(x_1, \dots, x_{m+n}) = \int_{-\pi}^{\pi} S_N\left(\sum_1^m x_p + u\right) S_N\left(\sum_{m+1}^{m+n} x_p - u\right) |u|^{-\beta} L_2(|u|^{-1})\, du, \tag{4.2}$$

S_N is as in (2.5),

$$d_N = N^H L_1^{(m+n)/2}(N)\, L_2(N), \tag{4.3}$$

and

$$H = (m+n)(\alpha-1)/2 + \beta + 1. \tag{4.4}$$

Proof of Theorem 1.1 By (4.1) it suffices to verify conditions (i), (ii), and (iii) of Proposition 3.1 for h_N given by (4.2) and H as in (4.4). Relations (1.10) and (1.12) imply $-\frac{1}{2} < \beta < 1$. One has

$$\begin{aligned} K_N(x_1, \dots, x_{m+n}) &= N^{-H+(m+n)(\alpha-1)/2} L_2^{-1}(N)\, h_N(x_1/N, \dots, x_{m+n}/N) \\ &= N^{-(\beta+1)} L_2^{-1}(N) \int_{-\pi}^{\pi} S_N\left(\sum_1^m x_p/N + u\right) S_N\left(\sum_{m+1}^{m+n} x_p/N - u\right) |u|^{-\beta} L_2(|u|^{-1})\, du \\ &= \int_{-N\pi}^{N\pi} \Delta_N\left(\sum_1^m x_p + u\right) \Delta_N\left(\sum_{m+1}^{m+n} x_p - u\right) |u|^{-\beta} [L_2(N|u|^{-1})/L_2(N)]\, du. \end{aligned}$$

Suppose first that L_1 and L_2 are asymptotically constant. If K_0 is as in (1.16), then $K_N \to K_0$ uniformly for $x \in [-A, A]^{m+n}$ by Lemma 9 in Terrin and Taqqu [19], because K_N is a function only of $\sum_1^m x_p$ and $\sum_{m+1}^{m+n} x_p$. Proposition 2.1 and Lemma 2.5 directly imply conditions (ii) and (iii).

Suppose now that L_1 or L_2 is not asymptotically constant. Since all conditions on α and β involve only strict inequalities, we still get $K_N \to K_0$ uniformly as well as conditions (ii) and (iii) by slightly modifying α and β and using properties of slowly varying functions (see for example Relation (3.6).) □

Proof of Theorem 1.2. As in the proof of Theorem 1.1 it suffices to verify conditions (i), (ii), and (iii) for

$$h_N = \int_{-\pi}^{\pi} S_N(x_1 + u)\, S_N(x_2 - u)\, |u|^{-\beta} L_2(|u|^{-1})\, du$$

and $H = \alpha + \beta$. The following results in Terrin and Taqqu [19], Lemma 9, Proposition 1, and Lemma 10, directly imply conditions (i), (ii), and (iii), if L_1 and L_2 are asymptotically constant. Since the conditions on α and β involve only strict inequalities, (i), (ii) and (iii) also hold when L_1 and L_2 are slowly varying. □

References

1. Avram, F.: Asymptotics of sums with dependent indices and convergence to the Gaussian distribution. (Preprint, 1989)
2. Avram, F., Taqqu, M.S.: Noncentral limit theorems and Appell polynomials. Ann. Probab. **15**, 767–775 (1987)
3. Billingsley, P.: Probability and measure. New York: Wiley 1979
4. Bingham, N.H., Goldie, C.M., Teugels, J.L.: Regular variation. Cambridge: Cambridge University Press 1987
5. Brockwell, P.J., Davis, R.A.: Time Series: theory and methods. Berlin Heidelberg New York: Springer 1987
6. Dahlhaus, R.: Efficient parameter estimation for self similar processes. Ann. Stat. **17**, 1749–1766 (1989)
7. Dobrushin, R.L., Major, P.: Non-central limit theorems for non-linear functions of Gaussian fields. Z. Wahrscheinlichkeitstheor. Verw. Geb. **50**, 27–52 (1979)
8. Fox, R., Taqqu, M.S.: Large-sample properties of parameter estimates for strongly dependent stationary Gaussian time series. Ann. Stat. **14**, 517–532 (1986)
9. Fox, R., Taqqu, M.S.: Central limit theorems for quadratic forms in random variables having long-range dependence. Probab. Th. Rel. Fields **24**, 213–240 (1987)
10. Giraitis, L., Surgailis, D.: Multivariate Appell polynomials and the central limit theorem. In: Eberlein, E., Taqqu, M.S. (eds.) Dependence in probability and statistics. Boston: Birkhäuser 1986
11. Giraitis, L., Surgailis, D.: A limit theorem for polynomials of linear processes with long range dependence. Lietuvos Matematikos Rinkinys **29**, 290–311 (1989)
12. Granger, C.W.J., Joyeux, R. An introduction to long-memory time series and fractional differencing. J. Time Ser. Anal. 1:15–30, 1980
13. Hosking, J.R.M.: Fractional differencing. Biometrika **68**, 165–176 (1981)
14. Major, P.: Multiple-Wiener-Itô integrals. (Lect. Notes Math, vol. 849) Berlin Heidelberg New York: Springer 1981
15. Mandelbrot, B.B., Taqqu, M.S.: Robust R/S analysis of long-run serial correlation. In: Proceedings of the 42nd Session of the International Statistical Institute. Manila, 1979. Bulletin of the I.S.I. Vol. 48, Book 2, pp. 69–104, 1979
16. Mandelbrot, B.B., Van Ness, J.W.: Fractional Brownian motions, fractional noises and applications. SIAM Rev. **10**, 422–437 (1968)
17. Rosenblatt, M.: Independence and dependence. In: Math. Stat. Probab. **2**, 411–443 (1961) Proc. 4th Berkeley Symp. Berkeley
18. Taqqu, M.S.: Weak convergence to fractional Brownian motion and to the Rosenblatt process. Z. Wahrscheinlichkeitstheor. Verw. Geb. **31**, 287–302 (1975)
19. Terrin, N., Taqqu, M.S.: A noncentral limit theorem for quadratic forms of Gaussian stationary sequences. J. Theor. Probab. **3**, 449–475 (1990)
20. Terrin, N., Taqqu, M.S.: Power counting theorem on R^n. In: Durrett, R., Kesten, H. (eds.) Spitzer Festschrift. Boston: Birkhäuser 1991
21. Yajima, Y.: On estimation of long-memory time series models. Aust. J. Stat. **27**, 303–320 (1985)