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On Convergence of the Averages $\frac{1}{N}\sum_{n=1}^{N} f_1(R^n x) f_2(S^n x) f_3(T^n x)$

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Abstract. In this note, we will prove that for commuting ergodic measure preserving transformations R, S and T, if RT^{-1}, ST^{-1} and TR^{-1} are also ergodic, then the limit

$$\lim \frac{1}{N}\sum_{n=1}^{N} f_1(R^n x) f_2(S^n x) f_3(T^n x)$$

exists in L^1-norm. The method used in this note was developed by CONZE, FURSTENBERG, LESIGNE and WEISS.

1. Introduction

For several commuting measure preserving transformations $R_1, \ldots, R_k$ on a probability space $(X, \mathscr{B}, \mu)$, limit theorems of the type

$$\frac{1}{N}\sum_{n=1}^{N} f_1(R_1^n x) \cdots f_k(R_k^n x),$$

where $f_1, \ldots, f_k$ are bounded measurable functions, have been studied by J. BOURGAIN, J. CONZE, H. FURSTENBERG, E. LESIGNE, and B. WEISS (see [2, 3, 4, 7, 8, 9, 10]). Although most people believe the existence of the limit in a certain sense (e.g. pointwise or L^1-norm), the cases in which one knows the answer are scarce. This paper follows these works in this direction and focuses on the case with mean convergence for three commuting measure preserving transformations R, S and T.

In Section 2, we bring some facts about Kronecker factors and Conze–Lesigne type modules. Readers should be able to find more details in [3, 4, 5, 9, 10, 12, 13]. In Section 3, some properties of

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one-dimensional (E)-cocycles are discussed and in Section 4, we used those properties to prove the convergence of (E)-cocycles. In Section 5, lemmas and propositions are prepared for proofs of our main theorems. Finally, in Section 6, we give some conditions for convergence of

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(R^n x) f_2(S^n x) f_3(T^n x).$$

Also a proof of a known theorem (Theorem 6.2), due to Conze and Lesigne, Furstenberg and Weiss, is given.

We will use $\mathbf{R}$, $\mathbf{C}$ and $U(d)$ denote the set of real numbers, the set of complex numbers and the set of all d-dimensional unitary matrices. In particular, $U(1)$ is the multiplicative group of all complex numbers with norm one. We sometimes also use $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ to denote $U(1)$. By a probability space, we mean a regular space (see [6, pages 103–104]) with the measure of the whole space being equal to one. For any set A, we will use A^k to denote the set

$$\underbrace{A \times \cdots \times A.}_{k}$$

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2. Preliminaries

A measure preserving system $(X, \mathscr{B}, \mu, R)$ is a measure preserving transformation R on a probability space $(X, \mathscr{B}, \mu)$. Sometimes we also use (X, R) as short abbreviation. We will say a measure preserving transformation T has *f.m.e.c.* (*finitely many ergodic components*) if its ergodic decomposition has finitely many components. Let $E(R)$ denote the closed subspace spanned by all eigenfunctions of R. It is clear that if R has f.m.e.c., then the eigenspace for each eigenvalue has finite dimension. Actually, the dimension of the eigenspace is equal to the number of ergodic components.

Let $R_1, \ldots, R_k$ be commuting measure preserving transformations on a probability space $(X, \mathscr{B}, \mu)$ such that $R_i, R_i R_j^{-1}$ have f.m.e.c. for all

$i \neq j$. Then it is clear that for $i \neq j$,

$$E(R_i) = E(R_i) = E(R_i R_j^{-1}). \tag{1}$$

Let E denote this subspace and let $\mathscr{D}$ denote the sub-σ-algebra of $\mathscr{B}$ such that $L^2(X, \mathscr{D}, \mu) = E$. Let Γ be the abelian group generated by $R_1, \ldots, R_k$. If Γ is ergodic on X, then there exist a compact abelian group G and $\alpha_1, \ldots, \alpha_k \in G$ such that $(G, \{\rho_{\alpha_1}, \ldots, \rho_{\alpha_k}\})$ is the maximal group rotation factor (Kronecker factor) of $(X, \{R_1, \ldots, R_k\})$. We will call G a *Kronecker group*. Let $p: X \to G$ be the factor map and let $\mathscr{B}_G$ be the σ-algebra on G generated by open sets. It is clear that $p(\mathscr{D}) = \mathscr{B}_G$. Abusing terminologies, we will use $\mathscr{D}$ to denote $\mathscr{B}_G$ and call a $\mathscr{D}$-measurable function on X a function on G.

Assume that Γ, being a finitely generated abelian group of measure preserving transformations on X, acts ergodicly. Let G be a Kronecker group of $(X, \mathscr{B}, \mu, \Gamma)$ and let $\pi: X \to G$ be the projection map. Then there exists a family of conditional probability measures $\{\mu_g : g \in G\}$ on $(X, \mathscr{B})$ with following properties:

i) $\mu_g(\pi^{-1}(g)) = 1$ for almost all $g \in G$.

ii) For every $f \in L^1(X, \mathscr{B}, \mu)$, the function $g \to \int f \, d\mu_g$ is measurable and $\int f \, d\mu = \int \{ \int f \, d\mu_g \} d\mu_G$. Here μ_G is the Haar measure on G.

The Symbols $\langle \cdot, \cdot \rangle_g$, $\| \cdot \|_g$ will be used to denote the inner product and the norm with respect to the measure μ_g.

A closed subspace $\mathscr{M} \subset L^2(X, \mathscr{B}, \mu)$ is called a *G-module* if $f\mathscr{M} \subset \mathscr{M}$ for any bounded function f on G. A finite set $\{\varphi_1, \ldots, \varphi_k\} \subset L^2(X)$ spans a G-module $\mathscr{M}$ if the set

$$\{\psi_1 \varphi_1 + \cdots + \psi_k \varphi_k; \text{ for any } \psi_1, \ldots, \psi_k \text{ on } G\}$$

is dense in $\mathscr{M}$. In this case, we also say that $\mathscr{M}$ is a *finite dimensional* G-module since it can be spanned by finitely many functions. For any finite dimensional G-module $\mathscr{M}$, one always can find a finite set $\varphi_1, \ldots, \varphi_k$ spanning $\mathscr{M}$ such that $\langle \varphi_i, \varphi_j \rangle_y = 0$ for $i \neq j$, and $\| \varphi \|_y = 1$ or 0. We will call this set a *global orthonormal basis* or just a *basis* of $\mathscr{M}$.

Let $\varphi_1, \ldots, \varphi_k$ be a basis of $\mathscr{M}$. For $R \in \Gamma$, if $\mathscr{M}$ is R-invariant (i.e., $R\mathscr{M} \subset \mathscr{M}$), there exists a matrix-valued function H on G such that

$$\begin{pmatrix} \varphi_1(Rx) \\ \vdots \\ \varphi_k(Rx) \end{pmatrix} = H(x) \begin{pmatrix} \varphi_1(x) \\ \vdots \\ \varphi_k(x) \end{pmatrix}.$$

We will say that H is induced by R with respect to a global orthonormal basis $\varphi_1, \ldots, \varphi_k$. In particular, if R is ergodic, then H is unitary and $\|\varphi_i\| = 1$ for all i. For any positive integer n, $H^{(n)}$ will be used to denote the product $H(R^{n-1}x)H(R^{(n-1)}x)\cdots H(x)$.

Let ρ_α be the rotation on G with respect to R on X. A R-invariant G-module is *irreducible* if it does not contain any non-trivial R-invariant sub-G-module. A matrix-valued function H on G is *irreducible* with respect to a rotation ρ_α if for any matrix-valued function A satisfying

$$A(g+\alpha)H(g) = H(g)A(g). \tag{2}$$

A must be a product of a constant and the identity matrix, i.e. $A(g) = cI$. One can show that a finite dimensional R-invariant G-module is irreducible if and only if the matrix-valued function H induced by R is irreducible with respect to the rotation ρ_α.

As before, we assume that Γ is a finitely generated abelian group of measure preserving transformations on a probability space $(X, \mathscr{B}, \mu)$. Let G be the Kronecker group. For any $R \in \Gamma$, we will use $K(G, R)$ to denote the space spanned by all finite dimensional R-invariant G-modules. One can show that there is an R-invariant sub-σ-algebra $\mathscr{B}_R$ such that $L^2(X, \mathscr{B}_R, \mu) = K(G, R)$. More details about finite dimensional modules can be found in [5, 6, 7, 13].

A $k \times k$ unitary matrix-valued function H on a compact abelian group G is called an *(E)-cocycle* with respect to a rotation ρ_α if there exist measurable functions $\lambda_t: G \to U(1)$ and $F: G \times G \to U(k)$ such that

$$F(t, g+\alpha)H(g) = \lambda_t H(g+t)F(t, g). \tag{3}$$

The pair $\{F(\cdot,\cdot), \lambda(\cdot)\}$ will be called an *(E)-pair*.

Now let R be ergodic. For a finite dimensional R-invariant G-module $\mathscr{M}$, if the unitary matrix-valued function H induced by R with respect to certain basis is an (E)-cocyle, then $\mathscr{M}$ will be called a Conze-Lesigne type G-module. If H is irreducible, LESIGNE (see [9, pages 184–186]) showed that if there exist measurable $\lambda_t: G \to U(1)$ and (not necessarily measurable) $F: G \times G \to U(k)$ such that (3) is true, then H is an (E)-cocycle. Recently, RUDOLPH proved that any irreducible (E)-cocycle on the Kronecker factor must be a one-dimensional cocycle. This result significantly reduced the complexity of our work. A proof can be found in [12, Corollary 2.3].

Theorem 2.1 (Rudolph). *Let G be the Kronecker factor of an ergodic measure preserving system $(X, \mathscr{B}, \mu, R)$ and let $H: G \to U(d)$ be an irreducible (E)-cocycle induced by R on an irreducible G-module. Then H must be a one-dimensional cocycle (i.e. $d = 1$).*

From now on, we only consider one-dimensional (E)-cocycles. Finally we would like to state a classical result due to Van der Corput. A proof can be found in [1].

Lemma 2.2 (Van der Corput). *Let $\{u_n; n \in \mathbf{N}\}$ be a bounded family of elements in a Hilbert space $\mathscr{H}$. If*

$$\lim_{M \to \infty} \lim_{N \to \infty} \frac{1}{M} \frac{1}{N} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \langle u_n, u_{n+m} \rangle = 0,$$

then

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} u_n = 0.$$

3. Conze and Lesigne Algebra

The following proposition is due to Lesigne. (see [9, pages 179–183])

Proposition 3.1 (Lesigne). *Let ρ_α be an ergodic rotation on a compact abelian group G. Then any (E)-cocycle $H(x)$ with (E)-pair $\{F(\cdot,\cdot), 1\}$ is cohomologous to a constant.*

We can have the following immediate corollary.

Corollary 3.2. *Let $\alpha \in G$ such that $|G/\overline{\langle \alpha \rangle}| < \infty$. Then any (E)-cocyle $H(x)$ with (E)-pair $\{F(\cdot,\cdot), 1\}$ is cohomologous to a ρ_α-invariant function on G.*

Let $V \subset G$ be an open neighborhood of the identity. A function $\lambda(\cdot,\cdot): V \times V \to U(1)$ is called a l.m.b. (locally multiplicative bilinear) form if

i) for $t, s \in V$, $\lambda(t,t) = 1$ and $\lambda(s,t) = \overline{\lambda(t,s)}$ and
ii) for any t_1, t_2 and $s \in V$ with $t_1 + t_2 \in V$, $\lambda(s, t_1 + t_2) = \lambda(s, t_1)\lambda(s, t_2)$.

A correct proof of the following result, without the assumption that G is connect, can be found in [9, pages 188–192].

Proposition 3.3 (Conze and Lesigne). *Let $H(x)$ be an (E)-cocyle with respect to an ergodic rotation ρ_α on a compact abelian group G. Then there is an (E)-pair $\{F(\cdot,\cdot), \lambda(\cdot)\}$ and a l.m.b. form $\lambda(s,t): V \times V \to U(1)$ (where $V \subset G$ is open) such that*

i) for every $t \in V$, $\lim_{s\to 0}\lambda(t,s) = 1$ and
ii) for $s, t \in V$, we have that

$$F(s, x+t)F(t,x) = \lambda(s,t)F(t, x+s)F(s,x). \tag{4}$$

From now on, such a l.m.b. form will be called a *l.m.b. form induced by (E)-cocycle $H(x)$*. Using Lesigne's method, one can have the following property for a l.m.b. form.

Lemma 3.4. *Let $\lambda(\cdot,\cdot)$ be a l.m.b. form on V and for any $t \in V$, $\lim_{s\to 0}\lambda(t,s) = 1$. Then there is a neighborhood $V_0 \subset V$ of the identity of G, a k-dimensional torus $\mathbf{T}^k$, a finite abelian group F, surjective homomorphisms with the following commutative diagram*

$$\begin{array}{ccc} G & \xrightarrow{\;p_1\;} & \mathbf{T}^k \times F \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle p_2} & \\ \mathbf{T}^k & & \end{array}$$

and a locally multiplicative bilinear form $\tilde{\lambda}(\cdot,\cdot)$ defined on a neighborhood U of the identity of $\mathbf{T}^k$ such that

i) $\ker(p_1) \subset V_0$ *and* $p_1(V_0) \subset \mathbf{T}^k \times \{0\}$;
ii) $|\ker(p)/\ker(p_1)| < \infty$;
iii) $p(V_0) \subset U$ *and for* $s, t \in V_0$, $\tilde{\lambda}(p(s), p(t)) = \lambda(s,t)$.

Proof. Let $G_1 \subset V$ be a closed subgroup such that $G/G_1 \cong \mathbf{T}^i \times F_1$, where F_1 is a finite abelian group. For any $s \in V$, $\lambda(s,\cdot)$ is a character of G_1. Since $\lim_{s\to 0}\lambda(s,t) = 1$, there is a neighborhood $V_1 \subset V$ such that for any $s \in V_1$, $\lambda(s,\cdot) = 1$. Let $V_2 \subset V_1$ such that $V_2 + V_2 \subset V_1$. Then there is a closed subgroup G_2 of G_1 such that $G_2 \subset V_2 \cap G_1$ and $G_1/G_2 \cong \mathbf{T}^{i'} \times F_2$, where F_2 is a finite abelian group. It is clear that G_2 is a closed subgroup of G and $G/G_2 \cong \mathbf{T}^k \times F$, where F is a finite group.

Let $p_1: G \to \mathbf{T}^k \times F$, $p_2: \mathbf{T}^k \times F \to \mathbf{T}^k$ be the natural homomorphisms and let $p = p_2 \circ p_1$.

Letting $V_0 = V_2 \cap p_1^{-1}(\mathbf{T}^k \times \{0\})$, we have that $\ker(p_1) = G_2 \subset V_0$ and $|\ker(p)/\ker(p_1)| = |F|$. Since $\ker(p_1) \subset G_1$, we have that for any $s \in V_1$ and $t \in \ker(p_1)$, $\lambda(s,t) = 1$. Noticing that $V_0 + \ker(p) \subset V_0 +$

$+V_0 \subset V_1$ and $\lambda(s,t) = \overline{\lambda(t,s)}$, one can define a l.m.b. form $\tilde{\lambda}(\cdot,\cdot): p_1(V_0) \times p_1(V_0) \to \mathbf{T}$ by $\tilde{\lambda}(p_1(s), p_1(t)) = \lambda(s,t)$. Since $p_2^{-1}: \mathbf{T}^k \to \mathbf{T}^k \times F$, defined by $p_2^{-1}(u) = (u, 0)$, is an injective homomorphism, $\tilde{\lambda} = \lambda \circ p_2^{-1}$ is a locally multiplicative bilinear function on $p(V_0) \times p(V_0)$.

Let $U = p(V_0)$. Then U is open because p is open. Now the lemma follows. □

The torus $\mathbf{T}^k$, described in Lemma 3.4, will be called an *essential torus* for $\lambda(\cdot,\cdot)$. We still use the same notations as Lemma 3.4. It is clear that there is a bilinear form $B(\cdot,\cdot)$ on $\mathbf{R}^k \times \mathbf{R}^k$ such that

$$\lambda(s,t) = \tilde{\lambda}(\mathbf{u}, \mathbf{v}) = e^{2\pi i B(\mathbf{u},\mathbf{v})}.$$

for $\mathbf{u} = p(s)$, $\mathbf{v} = p(t)$ and $s, t \in V$. If $B(\mathbf{n}, \mathbf{m}) \in \mathbf{Z}$ for $\mathbf{n}, \mathbf{m} \in \mathbf{Z}^k$, we will say $\lambda(\cdot,\cdot)$ is *k-dimensional expandable or expandable.*

Proposition 3.5. *Let $H(x)$ be an (E)-cocycle with respect to the ergodic rotation ρ_α on a compact abelian group G and let $\lambda(\cdot,\cdot)$ be a l.m.b. form induced by $H(x)$. Then $\lambda(\cdot,\cdot)$ is expandable.*

Proof. We still use the same notations as in Lemma 3.4. Let $B(\cdot,\cdot)$ be the local bilinear form on $\mathbf{T}^k \times \mathbf{T}^k$ induced by $\lambda(\cdot,\cdot)$. It is clear that $B(\cdot,\cdot)$ can be extended to a bilinear form on $\mathbf{R}^k \times \mathbf{R}^k$ and we still denote this bilinear form by $B(\cdot,\cdot)$.

Let

$$\mathbf{v}_i = (\underbrace{0,\ldots,0,}_{i} 1/L, 0, \ldots, 0) \in U.$$

We can choose a sufficiently large integer L so that for any $1 \leqslant i \leqslant k$, there exists $t_i \in V_0$ with $p(t_i) = \mathbf{v}_i$. Let

$$\tilde{F}_i(x) = \frac{\prod_{j=0}^{L-1} F(t_i, x + jt_i)}{F(Lt_i, x)}.$$

Noticing that $p(Lt_i) = 0$ and $t_i \in V_0$, we have that $p_1(Lt_i) = 0$ which implies that $Lt_i \in V_0$. By Proposition 3.5, for any $s \in V_0$ and $j = 0, 1, \ldots, L-1$,

$$F(Lt_i, x+s) = \lambda(Lt_i, s) F(s, x + Lt_i) F(Lt_i, x)/F(s, x)$$

and

$$F(t_i, x + jt_i + s) = \lambda(t_i, s) F(s, x + (j+1)t_i) F(t_i, x + jt_i)/F(s, x + jt_i).$$

Therefore one has $\widetilde{F}_i(x+s)=(\lambda(s,t_i))^L\lambda(s,Lt_i)^{-1}\widetilde{F}_i(x)$. Since $Lt_i\in V_0$ and $p(Lt_i)=0$, $\lambda(Lt_i,s)=1$ for any $s\in V_0$. Thus $\widetilde{F}_i(x+s)=(\lambda(s,t_i))^L\widetilde{F}_i(x)$. Now for any i', since $t_{i'}\in V_0$,

$$\begin{aligned}\widetilde{F}_i(x+Lt_{i'}) &= (\lambda(t_{i'},t_i))^L\widetilde{F}_i(x+(L-1)t_{i'})= \\ &= (\lambda(t_{i'},t_i))^{2L}\widetilde{F}_i(x+(L-2)t_{i'})=\cdots=(\lambda(t_{i'},t_i))^{L^2}\widetilde{F}_i(x).\end{aligned}$$

Same as t_i, $Lt_{i'}\in V_0$. So $\lambda(Lt_{i'},s)=0$ for all $s\in V_0$ and

$$\widetilde{F}_i(x+Lt_{i'})=(\lambda(Lt_{i'},t_i))^L\widetilde{F}_i(x)=\widetilde{F}_i(x).$$

Now we have $(\lambda(t_{i'},t_i))^{L^2}=1$ which implies that $B(\mathbf{v}_{i'},\mathbf{v}_i)\in\mathbf{Z}$. Let B be the matrix such that $B(\mathbf{v}_{i'},\mathbf{v}_i)=\mathbf{v}_{i'}B\mathbf{v}_i^{\tau}$. Then the coefficients of B must be in $\mathbf{Z}$. □

Definition 3.1. Let G be a compact abelian group with a rotation ρ_α (not necessarily ergodic). A complex-valued function $f(x)$ on G is a nil-boundary with respect to α if there is a finite dimensional simply connected order 2 nilpotent group N with a lattice Γ, a homomorphism $p:G\to N/\Gamma[N,N]$, an $\hat{\alpha}\in N$ with $p(\alpha)=\hat{\alpha}\Gamma[N,N]$ and a bounded function $b:N/\Gamma\to U(1)$ such that, for a.e. $x\in G$, a.e. $\mathbf{v}\in N$ with $p(x)=\mathbf{v}\Gamma[N,N]$

$$f(x)=b(\hat{\alpha}\mathbf{v}\Gamma)\overline{b(\mathbf{v}\Gamma)}.$$

A $U(1)$-valued function $M(x)$ is an (E)-boundary if there is a nil-boundary $f(x)$, a character $\gamma(x)$ of G and an α-invariant $U(1)$-valued function $D(x)$ such that $M(x)$ is cohomologous to $\gamma(x)f(x)D(x)$.

Proposition 3.6. *Let $H(x)$ be an (E)-cocycle with respect to an ergodic rotation ρ_α on a compact abelian group G. Then there exists an integer d such that $H^{(d)}(x)$ is an (E)-boundary with respect to $d\alpha$.*

Proof. Let $\lambda(\cdot,\cdot)$ be the l.m.b. form induced by $H(x)$ and let $\mathbf{T}^k$ be the essential torus. By Lemma 3.4, there are surjective homomorphisms p, p_1 and p_2 such that the following diagram commutes.

$$\begin{array}{ccc} G & \xrightarrow{p_1} & \mathbf{T}^k\times F \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle p_2} & \\ \mathbf{T}^k & & \end{array}$$

Since $\lambda(\cdot,\cdot)$ is expendable, there is $B(\cdot,\cdot)$ on $\mathbf{R}^k\times\mathbf{R}^k$ and an open set $V_0\subset G$ such that

i) $\ker(p_1) \subset V_0 \subset p_1^{-1}(\mathbf{T}^k \times 0)$,
ii) for $\mathbf{n}, \mathbf{m} \in \mathbf{Z}^k$, $B(\mathbf{n}, \mathbf{m}) \in \mathbf{Z}$ and
iii) for any $s, t \in V_0$, $\lambda(s, t) = \exp\{2\pi i B(p(s), p(t)\}$.

Therefore there is an antisymmetric matrix $B \in gl(n, Z)$ such that $B(\mathbf{v}_1, \mathbf{v}_2) = \mathbf{v}_1 B \mathbf{v}_2^\tau$. Let B' be a triangle matrix such that $B = B' - (B')^\tau$. Then one can define a bilinear form $B'(\mathbf{z}', \mathbf{z}) = \mathbf{z}' B' \mathbf{z}^\tau$ such that $B'(\mathbf{n}, \mathbf{m}) \in \mathbf{Z}$ for $\mathbf{n}, \mathbf{m} \in \mathbf{Z}^k$. Next, we will use $B'(\cdot, \cdot)$ to define a nilpotent group.

For $\mathbf{z}, \mathbf{z}' \in \mathbf{R}^k$ and $s, s' \in \mathbf{R}$, define

$$(\mathbf{z}, s) * (\mathbf{z}', s') = (\mathbf{z} + \mathbf{z}', s + s' - B'(\mathbf{z}, \mathbf{z}')).$$

Then $(\mathbf{R}^{k+1}, *)$ is a step 2 nilpotent group. Let $\Gamma = \{(\mathbf{z}, s); \mathbf{z} \in \mathbf{Z}^k$ and $s \in \mathbf{Z}\}$ and let

$$b(\mathbf{z}, s) = \exp\{-2i\pi(s + B'(\mathbf{z}, [\mathbf{z}]))\}.$$

Then b is a function defined on N/Γ and $f(\mathbf{z}', \mathbf{z}) = b((\mathbf{z}', 0) * (\mathbf{z}, s))\overline{b(\mathbf{z}, s)}$ is a function on $N/\Gamma[N, N] = \mathbf{T}^k$. By calculation, one also can show that, for any $\mathbf{z}, \mathbf{z}', \mathbf{z}'' \in \mathbf{R}^k$,

$$f(\mathbf{z}', \mathbf{z} + \mathbf{z}'') f(\mathbf{z}'', \mathbf{z}) = \exp\{2\pi i B(\mathbf{z}'', \mathbf{z}')\} f(\mathbf{z}'', \mathbf{z} + \mathbf{z}') f(\mathbf{z}', \mathbf{z}). \qquad (5)$$

Let $d = |F|$, $\alpha_1 = p_1(\alpha)$ and $\alpha_0 = p(\alpha) = p_2(\alpha_1)$. Then $\alpha_1 = (\alpha_0, h)$ for some $h \in F$ and $d\alpha_1 = (d\alpha_0, 0)$. Choose $\varepsilon \in V_0$ such that $Lp_1(\varepsilon) = d\alpha_1$ for some integer L and let

$$\hat{F}(x) = F(d\alpha - L\varepsilon, x + L\varepsilon) F(\varepsilon, x + (L-1)\varepsilon) \cdots F(\varepsilon, x).$$

Then

$$\hat{F}(x + t) F(t, x) = \lambda(d\alpha - L\varepsilon, t)(\lambda(\varepsilon, t))^L F(t, x + d\alpha)\hat{F}(x).$$

Since $p_1(d\alpha - L\varepsilon) = 0$, $d\alpha - L\varepsilon \in \operatorname{Ker}(p_1) \subset V_0$. Thus $\lambda(d\alpha - L\varepsilon, t) = 1$ for $t \in V_0$. So

$$\lambda(d\alpha - L\varepsilon, t)(\lambda(\varepsilon, t))^L = \exp\{2\pi i p(t) B(d\alpha_0)^\tau\}.$$

From (5) we immediately have

$$\begin{aligned} & f(d\alpha_0, p(x+t))\hat{F}(x+t) f(p(t), p(x)) F(t, x) = \\ & \quad = f(p(t), p(x + d\alpha)) F(t, x + d\alpha) f(d\alpha_0, p(x))\hat{F}(x). \end{aligned}$$

Let $M(x) = f(d\alpha_0, p(x))\hat{F}(x)$ and let $F'(t, x) = f(p(t), p(x))F(t, x)$. Then for $t \in V$ and $x \in G$

$$M(x + t) F'(t, x) = F'(t, x + d\alpha) M(x).$$

Since ρ_α is ergodic, $|\overline{G/\langle d\alpha\rangle}| < \infty$. Therefore, by Corollary 3.2, $M(x)$ is cohomologous to a $d\alpha$-invariant function $D_0(x)$.

Considering the relation between $\hat{F}(x)$ and $H(x)$, one can find out that

$$H^{(d)}(x+d\alpha)\hat{F}(x) = \lambda(d\alpha - L\varepsilon)\lambda^L(\varepsilon)\hat{F}(x+d\alpha)H^{(d)}(x).$$

Let $\lambda = \lambda(d\alpha - L\varepsilon)\lambda^L(\varepsilon)$ and $D(x) = \hat{F}(x)^* H^{(d_0)}(x)$. Then $D(x+d\alpha) = \lambda D(x)$. So $H^{(d)}(x) = f^{-1}(d\alpha_0, p(x))D_0(x)D(x)$. Noticing that $D(x)$ is a product of a character and a $d\alpha$-invariant function, we have the lemma. □

4. Properties for (E)-Cocycles

The following result belongs to Lesigne. A proof can be found in [10].

Theorem 4.1 (Lesigne). *Let Γ be a lattice of a simply connected finite dimensional step 2 nilpotent Lie group N and let c be a continuous function on N/Γ. Then for any $\mathbf{a}, \mathbf{v} \in N$, any real polynomial Q, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \exp(2i\pi Q(n))c(\mathbf{a}^n\mathbf{v}\Gamma)$$

exists.

Now we have an immediate corollary and readers can figure out a proof by considering a direct product of nilpotent groups. A proof also can be found in [10].

Corollary 4.2. *Let $c_1, \ldots, c_m$ be continuous functions on N/Γ. For any $\mathbf{a}_1, \ldots, \mathbf{a}_m$ and $\mathbf{v} \in N$, and for any $z_1, z_2 \in \mathbf{C}$ with $|z_1| = |z_2| = 1$, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} z_1^n z_2^{n(n-1)/2} \prod_{k=1}^{m} c_k(\mathbf{a}_k^n\mathbf{v}\Gamma)$$

exists.

We will use $C(Y)$ to denote the space of all continuous functions on a topological space Y.

Lemma 4.3. *Let Y be a compact space with probability measure ν and measure preserving transformations $S_1, \ldots, S_m$ and let Ω be a set of sequences of complex numbers with norm less than 1. Let*

$\Lambda_i \subset L^1(Y) \cap L^\infty(Y)$ for $i=1,\dots,m$. If for any $c_i \in \Lambda_i$, $i=1,\dots,m$, there is a set A with full measure such that for $y \in A$, for any $\omega=\{\omega_n\}$ the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \omega_n \prod_{k=1}^{m} c_k(S_k^n y)$$

exists, then for any $f_i \in \bar{\Lambda}_i(\text{in } L^1(Y)) \cap L^\infty(Y)$ $i=1,\dots,m$, this is also true, i.e., there is a set A with full measure such that for $y \in A$, for any $\omega=\{\omega_n\}\in\Omega$, the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \omega_n \prod_{k=1}^{m} f_k(S_k^n y)$$

exists.

Proof. Let $B_1(Y)=\{f\in L^\infty(Y); |f|\leqslant 1\}$. Without loss of generality, we assume that $f_1,\dots,f_m \in B_1(Y)$.

For any $q\in\mathbf{Z}^+$, there exist $c_{qi}\in\Lambda_i$, $i=1,\dots,m$, such that for $1\leqslant k\leqslant m$,

$$\int |c_{qk}(y)-f_k(y)|\,dv < \frac{1}{4^q}.$$

Let

$$A_{qk}=\left\{y\in Y; \sup_{N\geqslant 1} \frac{1}{N}\sum_{n=0}^{N} |c_{qk}(S_k^n y)-f_k(S_k^n y)| \geqslant \frac{1}{2^q}\right\}.$$

Then, by the Maximal Ergodic Theorem,

$$v(A_{qk}) \leqslant 2^q \int |c_{qk}(y)-f_k(y)|\,dv < \frac{1}{2^q}.$$

By assumption of the lemma, there is a set A with full measure such that for $y\in A$, for any $\omega=\{\omega_n\}$ and any q the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N-1} \omega_n \prod_{k=1}^{m} c_{qk}(S_k^n y)$$

exists. Now for any $y\in A\setminus\bigcup_{q'\geqslant q}\left(\bigcup_{k=1}^m A_{q'k}\right)$, for any $\omega\in\Omega$, we can choose $N>0$ such that when $N_1, N_2 > N$,

$$\left|\frac{1}{N_1}\sum_{n=0}^{N_1-1}\omega_n\prod_{k=1}^{m} c_{qk}(S_k^n y) - \frac{1}{N}\sum_{n=0}^{N_2-1}\omega_n\prod_{k=0}^{m} c_{qk}(S_k^n y)\right| < \frac{1}{2^q}.$$

Therefore

$$\left|\frac{1}{N_1}\sum_{n=0}^{N_1-1}\omega_n\prod_{k=1}^{m}f_k(S_k^n y)-\frac{1}{N_2}\sum_{n=0}^{N_2-1}\omega_n\prod_{k=0}^{m}f_k(S_k^n y)\right|\leqslant$$

$$\leqslant\sum_{k=1}^{m}\frac{1}{N_1}\sum_{n=0}^{N_1-1}|f_k(S_k^n y)-c_{qk}(S_k^n y)|+$$

$$+\left|\frac{1}{N_1}\sum_{n=0}^{N_1-1}\omega_n\prod_{k=1}^{m}c_{qk}(S_k^n y)-\frac{1}{N_2}\sum_{n=0}^{N_2-1}\omega_n\prod_{k=0}^{m}c_{qk}(S_k^n y)\right|+$$

$$+\sum_{k=1}^{m}\frac{1}{N_2}\sum_{n=0}^{N_2-1}|f_k(S_k^n y)-c_{qk}(S_k^n y)|\leqslant$$

$$\leqslant\frac{2m+1}{2^q}$$

Since $v\left(\bigcup_{q'\geqslant q}\left(\bigcup_{k=1}^{m}A_{q'k}\right)\right)\to 0$, the lemma follows. □

Proposition 4.4. *Let $f_1(x),\ldots,f_m(x)$ be nil-boundaries on a compact abelian group G with respect to rotations $\rho_{\alpha_1},\ldots,\rho_{\alpha_m}$ respectively. Then for a.e. $x\in G$, for any $z_1,z_2\in\mathbf{C}$ with $|z_1|=|z_2|=1$, the limit*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}z_1^n z_2^{n(n-1)/2}\prod_{k=1}^{m}f_k^{(n)}(x)$$

exists. Here $f_k^{(n)}(x)=f_k(x+(n-1)\alpha_k)\cdots f_k(x)$ for $k=1,\ldots,m$.

Proof. Suppose that $f_1,\ldots,f_m$ set in tori $\mathbf{T}_1,\ldots,\mathbf{T}_m$ respectively and let $p_1,\ldots,p_m$ be the corresponding surjective homorphisms from G to $\mathbf{T}_1,\ldots,\mathbf{T}_m$ respectively. Define $p:G\to\mathbf{T}_1\times\cdots\times\mathbf{T}_m$ by

$$p(x)=(p_1(x),\ldots,p_m(x)).$$

Then $G_0=p(G)$ is a compact abelian group. Let $\Gamma_1,\ldots,\Gamma_m$ be lattices of simply connected step 2 nilpotent groups $N_1,\ldots,N_m$ respectively such that $\mathbf{T}_k\cong N_k/\Gamma_k[N_k,N_k]$ for $k=1,\ldots,m$. Let $\hat{\mathbf{a}}_{kj}\in N_k$ such that $\hat{\mathbf{a}}_{kj}\Gamma_k[N_k,N_k]=p_k(\alpha_j)$ for $k=1,\ldots,m$. By definition, for any $k=1,\ldots,m$, there is a function b_k on N_k/Γ_k with $|b_k|=1$ such that

$$f_k(x)=b_k(\mathbf{a}_{kk}\mathbf{v}_k\Gamma_k)b_k^*(\mathbf{v}_k\Gamma_k)$$

where $\mathbf{v}_k\Gamma_k[N_k,N_k]=p_k(x)$. We first assume that $b_1,\ldots,b_m$ are continuous. Let $N=N_1\times\cdots\times N_m$ and let $\Gamma=\Gamma_1\times\cdots\times\Gamma_m$. It is clear that

$$[N,N]=[N_1,N_1]\times\cdots\times[N_m,N_m],$$

$$N/\Gamma=N_1/\Gamma_1\times\cdots\times N_m/\Gamma_m$$

and

$$N/\Gamma[N,N]=N_1/\Gamma_1[N_1,N_1]\times\cdots\times N_m/\Gamma_m[N_m,N_m].$$

For $k=1,\dots,m$, let

$$\tilde{\mathbf{a}}_k=(\hat{\mathbf{a}}_{1k},\dots,\hat{\mathbf{a}}_{mk}).$$

Since N_k/Γ_k, $k=1,\dots,m$, are factors of N/Γ, b_k can be respectively considered as functions on N/Γ and G_0. Therefore for any $\mathbf{v}=(\mathbf{v}_1,\dots,\mathbf{v}_m)\in N$ with $p(x)=\mathbf{v}\Gamma[N,N]$,

$$f_k(x)=b_k(\tilde{\mathbf{a}}_k\mathbf{v}\Gamma)b_k^*(\mathbf{v}\Gamma).$$

Then

$$\frac{1}{N}\sum_{n=0}^{N-1}z_1^n z_2^{n(n-1)/2}\prod_{k=1}^m f_k^{(n)}(x)=\frac{1}{N}\sum_{n=0}^{N-1}z_1^n z_2^{n(n-1)/2}\prod_{k=1}^m b_k(\tilde{\mathbf{a}}_k^n\mathbf{v}\Gamma)b_k^*(\mathbf{v}).$$

Since $b_1,\dots,b_m$ are continuous functions, by Corollary 4.2, the limit exists for any $x\in G$.

Now we consider the convergence of

$$\frac{1}{N}\sum_{n=0}^{N-1}z_1^n z_2^{n(n-1)/2}\prod_{k=1}^m b_k(\tilde{\mathbf{a}}_k^n\mathbf{v}) \tag{6}$$

for general b_k.

Let $N_0=\{\mathbf{v}\in N,\mathbf{v}\Gamma[N,N]\in G_0\}$ and $Y=\{\mathbf{v}\Gamma\in N\Gamma;\mathbf{v}\Gamma[N,N]\in G_0\}$. Then $Y/[N,N]=G_0$, $N_0/\Gamma=Y$ and Y is a compact space. Since $N_0\supset\Gamma$, there is a unique invariant probability measure ν on Y (see [11, page 23]). It is clear that ν is induced by the Haar measure on N_0. Let $\tilde{p}\colon Y\to G_0$ be the homomorphism $\tilde{p}(y)=y[N,N]$. Then $\tilde{p}^*(\nu)$ is the Haar measure on G_0. These tell us that G_0 is a factor of N_0/Γ. Since $\tilde{\mathbf{a}}_1,\dots,\tilde{\mathbf{a}}_m\in N_0$, maps $\mathbf{v}\Gamma\to\tilde{\mathbf{a}}_k\mathbf{v}\Gamma$, $k=1,\dots,m$, are measure preserving on Y.

Now let $p_k\colon N/\Gamma\to N_k/\Gamma_k$ be the natural projection. Then one can check that $p_k(N_0/\Gamma)=N_k/\Gamma_k$. Since the invariant measure is unique, N_k/Γ_k is a factor of N_0/Γ. Then b_k, $k=1,\dots,m$, can be considered as functions on N_0/Γ. Let Λ_k be the set of continuous functions on N_k/Γ_k

for $k=1,\ldots,m$. Then for any $c_k \in \Lambda_k$, $k=1,\ldots,m$ and any $z_1, z_2 \in U(1)$, the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} z_1^n z_2^{n(n-1)/2} \prod_{k=1}^{m} c_k(\tilde{\mathbf{a}}_k^n y)$$

exists for any $y \in Y$. By Lemma 4.3, for any $b_k \in L^\infty(N_k/\Gamma_k)$ with $k=1,\ldots,m$, there is a set $A \subset Y$ with full measure such that for $y \in Y$ and for any $z_1, z_2 \in U(1)$, (6) converges. This will give the proposition.

□

Lemma 4.5. *Let $f_1,\ldots,f_m$ be nil-boundaries on a compact abelian group G with respect to rotations defined by $\alpha_1,\ldots,\alpha_m$ respectively. Then for any bounded functions $\phi_1,\ldots,\phi_m$ on G, there is a set A with full measure such that for $x \in A$, for any ε with $|\varepsilon|=1$, for any $d_1(x),\ldots, d_m(x): G \to U(1)$ and for any characters $\gamma_1,\ldots,\gamma_m$ on G, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \varepsilon^n \prod_{k=1}^{m} \phi_k(x+n\alpha_k)\gamma_k(x)^n \gamma_k(\alpha_k)^{n(n-1)/2} f_k^{(n)}(x) d_k^n(x) \qquad (7)$$

exists.

Proof. By Proposition 4.4, there is a set A_0 with full measure such that the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} z_1^n z_2^{n(n-1)/2} \prod_{k=1}^{m} f_k^{(n)}(x)$$

exists for all z_1, $z_2 \in \mathbf{C}$ with $|z_1|=|z_2|=1$ and $x \in A_0$. We first assume that $\phi_1,\ldots,\phi_m$ are characters of G. Letting

$$z_1 = \varepsilon \prod_{k=1}^{m} \phi_k(\alpha_k)\gamma_k(x)d_k(x)$$

and

$$z_2 = \prod_{k=1}^{m} \gamma_k(\alpha_k),$$

we have that the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \varepsilon^n \prod_{k=1}^{m} \phi_k(x+n\alpha_k)\gamma_k(x)^n \gamma_k(\alpha_k)^{n(n-1)/2} f_k^{(n)}(x)_k^n(x) =$$

$$= \lim_{N\to\infty} \frac{1}{N} \left(\sum_{n=0}^{N-1} z_1^n z_2^{n(n-1)/2} \prod_{k=1}^{m} f_k^{(n)}(x) \right) \prod_{k=1}^{m} \phi_k(x)$$

exists for all $x \in A_0$.

Since G is compact, the set of all linear combinations of characters is dense in $C(G)$ with respect to the norm: $\|\phi\|_c = \max_{x\in G}|\phi(x)|$. Therefore (7) converges for all continuous $\phi_1, \ldots, \phi_m$ and all $x \in A_0$. Now let

$$\Omega = \left\{ \varepsilon^n \prod_{k=1}^{m} \gamma_k(x)^n \gamma_k(\alpha_k)^{n(n-1)/2} f_k^{(n)}(x) d_k^n(x);\ \text{for all } x \in A_0 \textit{ and } \gamma_k \in G^* \right\}.$$

Then the theorem follows from Lemma 4.3. □

The following corollary follows from the above proposition immediately.

Corollary 4.6. *Let G be a compact abelian group and let $M_1, \ldots, M_m$ be (E)-boundaries with respect to $\alpha_1, \ldots, \alpha_m$ respectively. Then for any bounded functions $\Phi_1, \ldots, \Phi_m$, there is a set G_M with full measure such that for any $x \in G_M$, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \prod_{k=1}^{m} \Phi_k(x + n\alpha_i) M_k^{(n)}(x)$$

exists.

Now we give a certain type of convergence theorem for (E)-cocycles.

Theorem 4.7. *Let $H_1, \ldots, H_m$ be (E)-cocycles on a compact abelian group G with respect to ergodic rotations $\rho_{\alpha_1}, \ldots, \rho_{\alpha_m}$, respectively, and let $\Phi_1, \ldots, \Phi_m$ be bounded functions on G. Then, for any integer $j_1, \ldots, j_m$, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \prod_{k=1}^{m} \Phi_k(x + nj_k\alpha_k) H_k^{(nj_k)}(x)$$

exists for a.e. $x \in G$.

Proof. By Proposition 3.6, there are positive integers $d_1, \ldots, d_m$ such that

$$H_1^{(d_1)}, \ldots, H_m^{(d_m)}$$

are (E)-boundaries. Let ℓ be the least common multiple of $d_1, \ldots, d_m$. Then

$$H_1^{(\ell j_1)}, H_2^{(\ell j_2)}, \ldots, H_m^{(\ell j_m)}$$

are (E)-boundaries with respect to $\ell j_1\alpha_1, \ell j_2\alpha_2, \ldots, \ell j_m\alpha_m$ respectively. By Corollary 4.6, for almost all $x \in G$, the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \prod_{k=1}^{m} \Phi_k(x + n\ell j_k) H_k^{(n\ell j_k)}(x)$$

exists. Therefore the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \prod_{k=1}^{m} \Phi_k(x + nj_k\alpha_k) H_k^{(nj_k)}(x) =$$

$$= \frac{1}{\ell} \sum_{j=0}^{\ell-1} \left(\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \prod_{k=1}^{m} \Phi_k(x + n\ell j_k\alpha_k + jj_k\alpha_k) H_k^{(n\ell j_k)}(x + jj_k\alpha_k) \right) \times$$

$$\times \prod_{k=1}^{m} H_k^{(jj_k)}(x)$$

exists. These give us the theorem. □

Finally we state an immediate corollary which will be used to prove Theorem 6.2

Corollary 4.8. *Let H be an (E)-cocycle with respect to an irrational rotation ρ_α. Then for any $\Phi_1, \ldots, \Phi_m$ and any integers $j_1, \ldots, j_m$, the limit*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \prod_{k=1}^{m} \Phi_k(x + nj_k\alpha) H^{(nj_k)}(x)$$

exists for a.e. $x \in G$.

5. Decomposition Based on Conze-Lesigne Algebra

The method used in this section is similar to the method CONZE and LESIGNE used in [4]. We begin with a proposition dealing with two commuting measure preserving transformations. For sake of completeness, we give a proof.

Proposition 5.1. *Let T, S be commuting (i.e. $ST = TS$) measure preserving transformations on a probability space $(X, \mathscr{B}, \mu)$ such that ST^{-1} has f.m.e.c. Let $E(S)$, $E(T)$ denote the closed subspaces spanned by all eigenfunctions of S and T, respectively. Then for any bounded functions f_1 and f_2 with either $f_1 \perp E(S)$ or $f_2 \perp E(T)$,*

$$\lim_{N\to\infty} \int \left| \frac{1}{N} \sum_{n=0}^{N-1} f_1(S^n x) f_2(T^n x) \right| d\mu = 0.$$

Proof. Let $e_1, \ldots, e_k$ denote an orthonormal basis of the ST^{-1}-invariant space. We first assume f_1 and f_2 are bounded and use the Van der Corput Lemma. Let $u_n = f_1(S^n x) f_2(T^n x)$. Then

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \langle u_n, u_{n+m} \rangle =$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int f_1(S^n x) f_2(T^n x) \overline{f_1(S^{n+m}x) f_2(T^{n+m}x)}\, d\mu =$$

$$= \int f_1(x) \overline{f_1(S^m x)} \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} (TS^{-1})^n (f_2(x) \overline{f_2(T^m x)})\, d\mu =$$

$$= \sum_{i=1}^{k} \int f_1(x) \overline{f_1(S^m x)} \langle f_2, T^m f_2 e_i \rangle e_i\, d\mu =$$

$$= \sum_{i=1}^{k} \langle f_1, S^m f_1 \bar{e}_i \rangle \langle f_2, T^m f_2 e_i \rangle.$$

Let P denote the orthogonal projection from $L^2(X \times X, \mathscr{B} \times \mathscr{B}, \mu \times \mu)$ to the $S \times T$-invariant subspace. Since the $S \times T$-invariant space is contained in the closed subspace spanned by $E(S) \times E(T)$, we have

$$\lim_{M\to\infty} \lim_{N\to\infty} \frac{1}{M} \frac{1}{N} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} \langle u_n, u_{n+m} \rangle =$$

$$= \sum_{i=1}^{k} \lim_{M\to\infty} \frac{1}{M} \sum_{m=0}^{M-1} \langle f_1 \times f_2, (S \times T)^m (f_1 \times f_2) \bar{e}_i \times e_i \rangle =$$

$$= \sum_{i=1}^{k} \langle f_1 \times f_2, P(f_1 \times f_2) \bar{e}_i \times e_i \rangle =$$

$$= 0.$$

By Lemma 2.2, $\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f_1(S^n x) f_2(T^n x) = 0$ in the L_2-norm which implies the theorem. □

If S and T are commuting measure preserving transformations on a probability space $(X, \mathscr{B}, \mu)$ such that S, T and ST^{-1} have f.m.e.c., then $E(S) = E(T)$ and there is a basis $\{e_n\}$ of $E(S)$ such that $e_1, e_2, \ldots$ are eigenfunctions of S, T and ST^{-1}.

Corollary 5.2. *Let S, T be commuting measure preserving transformations on a probability space $(X, \mathscr{B}, \mu)$ such that S, T* and ST^{-1} *have f.m.e.c. Let $\{e_n\}$ be a basis of $E(S)$ such that $e_1, e_2, \ldots$ are eigenfunctions of S and T. Let λ_n^S, λ_n^T be eigenvalues with respect to $\{e_n\}$ for S and T respectively. Then for any f_1, $f_2 \in L^\infty(X, \mathscr{B}, \mu)$,* the limit

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f_1(S^n x) f_2(T^n x) = \sum_{\lambda_n^S = \lambda_m^T} \langle f_1, e_n\rangle \langle f_2, \bar{e}_m\rangle e_n(x)\overline{e_m(x)} \quad (8)$$

exists in the L_1-norm.

Now we consider measure preserving transformations R, S and T on a probability space $(X, \mathscr{B}, \mu)$. Since our goal is to find conditions for the convergence of

$$\frac{1}{N} \sum_{n=0}^{N-1} f_1(R^n x) f_2(S^n x) f_3(T^n x),$$

without loss of any generality, we can assume that the abelian group $\langle R, S, T\rangle$ acts on X ergodicly. Therefore from now on, we always automatically assume the ergodicity of $\langle R, S, T\rangle$ and readers can generalize to nonergodic situation.

Define a measure ω on X^3: for any A, B and $C \in \mathscr{B}$

$$\omega(A \times B \times C) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int 1_A(R^n x) 1_B(S^n x) 1_C(T^n x) d\mu. \quad (9)$$

The existence of the above limit directly follows from a result of CONZE and LESIGNE (see [3, Théoréme 4, page 152]). It is clear that ω is a joining of (X, R), (X, S) and (X, T).

Proposition 5.3. *Let Z be the subspace of $L^2(X^3, \mathscr{B}^3, \omega)$ spanned by all $R \times S \times T$-invariant functions (with respect to ω) and let $P_Z: L^2(X^3, \mathscr{B}^3, \omega) \to Z$ be the projection. Then for any f_1, f_2, $f_3 \in L^\infty(X, \mathscr{B}, \mu)$ with $f_1 \otimes f_2 \otimes f_3 \perp Z$,*

$$\lim_{N\to\infty} \left\| \frac{1}{N} \sum_{n=0}^{N-1} f_1(R^n x) f_2(S^n x) f_3(T^n x) \right\|_1 = 0.$$

Proof. We will use the Van der Corput Lemma. Let

$$u_n = f_1(R^n x) f_2(S^n x) f_3(T^n x).$$

Then

$$\frac{1}{M}\frac{1}{N}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\langle u_n, u_{n+m}\rangle =$$

$$=\frac{1}{M}\frac{1}{N}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\times$$

$$\times\int f_1(R^n x)f_2(S^n x)f_3(T^n x)f_1(R^{n+m}x)f_2(S^{n+m}x)f_3(T^{n+m}x)d\mu=$$

$$=\frac{1}{M}\sum_{m=0}^{M-1}\left(\frac{1}{N}\sum_{n=0}^{N-1}\int R^n(f_1R^mf_1)S^n(f_2S^mf_2)T^n(f_3T^mf_3)d\mu\right).$$

Therefore

$$\lim_{N\to\infty}\frac{1}{M}\frac{1}{N}\sum_{m=0}^{M-1}\sum_{1}^{N-1}\langle u_n, u_{n+m}\rangle =$$

$$=\frac{1}{M}\sum_{m=0}^{M-1}\int(f_1R^mf_1)\otimes(f_2S^mf_2)\otimes(f_3T^mf_3)d\omega=$$

$$=\frac{1}{M}\sum_{m=0}^{M-1}\int(f_1\otimes f_2\otimes f_3)(R^m\times S^m\times T^m(f_1\otimes f_2\otimes f_3))d\omega.$$

This means

$$\lim_{M\to\infty}\lim_{N\to\infty}\frac{1}{M}\frac{1}{N}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\langle u_n, u_{n+m}\rangle =$$

$$=\int(f_1\otimes f_2\otimes f_3)P_Z(f_1\otimes f_2\otimes f_3)d\omega=0.$$

Now the theorem follows from the Van der Corput Lemma. □

If R, S, T and RS^{-1}, ST^{-1}, TR^{-1} have f.m.e.c., then

$$E(R)=E(S)=E(T)=E(RS^{-1})=E(ST^{-1})=E(TR^{-1}).$$

Let $\{e_n\}$ be the set of all characters of Kronecker group. Then $\{e_n\}$ is a basis of E. Let $\lambda_{1,n}$, $\lambda_{2,n}$, $\lambda_{3,n}$, $\lambda_{12,n}$, $\lambda_{23,n}$ and $\lambda_{31,n}$ be eigenvalues for e_n with respect to R, S, T, RS^{-1}, ST^{-1} and TR^{-1} respectively. It is clear that $\lambda_{12,n}=\lambda_{1,n}\lambda_{2,n}^{-1}$, $\lambda_{23,n}=\lambda_{2,n}\lambda_{3,n}^{-1}$ and $\lambda_{31,n}=\lambda_{3,n}\lambda_{1,n}^{-1}$. Let

$\lambda_{ij,n} = \bar{\lambda}_{ji,n}$. By Corollary 5.2,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \int f_1(R^n x) f_2(S^n x) f_2(T^n x) d\mu =$$
$$= \sum_{\lambda_{21,i}=\lambda_{31,j}} \langle f_1, e_i \rangle \langle f_2, \bar{e}_j \rangle \langle f_3, \bar{e}_i e_j \rangle. \tag{10}$$

The following theorem is due to H. FURSTENBERG (see [5, page 244]).

Theorem 5.4 (Furstenberg). *Let $(X_i, \mathscr{B}_i, \mu_i, R_i)$, $i = 1, \ldots, k$, be ergodic measure preserving systems and let $\mathscr{D}_1, \ldots, \mathscr{D}_k$ be factors of $\mathscr{B}_1, \ldots, \mathscr{B}_k$, respectively. Let ω be a joining such that*

$$\int h_1(x_1) \cdots h_k(x_k) d\omega = \int \mathbf{E}(h_1 | \mathscr{D}_1)(x_1) \cdots \mathbf{E}(h_k | \mathscr{D}_k)(x_k) d\omega$$

for $h_i \in L^\infty(X_i, \mathscr{D}_i, \mu_i)$, $i = 1, 2, \ldots, k$. Then the subspace of $L^2(\prod_{i=1}^k X_i, \prod_{i=1}^k \mathscr{B}_i, \omega)$ spanned by all of $R_1 \times \cdots \times R_k$-invariant functions (with respect to ω) is contained in $\bigotimes_{i=1}^k K(\mathscr{D}_i, R_i)$.

One can generalize this result without difficulty to the case where $R_1, \ldots, R_k$ are f.m.e.c. So we can have the following corollary.

Corollary 5.5. *Assume that R, S, T and RS^{-1}, ST^{-1}, TR^{-1} have f.m.e.c. Let G be the Kronecker factor of $(X, \{R, S, T\})$ and let ω be the measure defined by (9) on $(X^3, \mathscr{B}^3, \omega)$. Then the subspace of $L^2(X^3, \mathscr{B}^3, \omega)$ spanned by all of $R \times S \times T$-invariant functions (with respect to ω) is contained in $K(G, R) \otimes K(G, S) \otimes K(G, T)$.*

Proof. We need to check the conditions in Theorem 5.4. By (10),

$$\omega(f_1 \otimes f_2 \otimes f_3) = \sum_{\lambda_{21,i}=\lambda_{31,j}} \langle f_1, e_i \rangle \langle f_2, \bar{e}_j \rangle \langle f_3, \bar{e}_i e_j \rangle =$$
$$= \sum_{\lambda_{21,i}=\lambda_{31,j}} \langle P_G f_1, e_i \rangle \langle P_G f_2, \bar{e}_j \rangle \langle P_G f_3, \bar{e}_i e_j \rangle =$$
$$= \omega(P_G f_1 \otimes P_G f_2 \otimes P_G f_3).$$

Hence all conditions in Theorem 5.4 are satisfied. This implies the corollary. □

The following lemma is slightly different from a theorem in [3, 4]. For the sake of the completeness, we give a proof here.

Lemma 5.6. *Let R, S and T be measure preserving transformations with f.m.e.c. If* $f_k \in L^\infty(X, \mathscr{B}, \mu)$ *for* $k=1,2$ *and* 3 *with* $P_Z(f_1 \otimes f_2 \otimes f_3) \neq \neq 0$, *then for any sequence* $\{n_k\}$ *of integers there exist bounded functions u, v and w such that*

$$\limsup_{k\to\infty} \sup_i |\langle uR^{n_k} f_1, e_i\rangle| > 0,$$

$$\limsup_{k\to\infty} \sup_i |\langle vS^{n_k} f_2, e_i\rangle| > 0,$$

and

$$\limsup_{k\to\infty} \sup_i |\langle wT^{n_k} f_3, e_i\rangle| > 0,$$

Proof. First of all we claim that given $\{n_k\}$ there exist bounded functions u, v and w such that

$$\limsup_{k\to\infty} \left| \int (u \otimes v \otimes w)(R^{n_k} f_1 \otimes S^{n_k} f_2 \otimes T^{n_k} f_3) d\omega \right| > 0.$$

If this is not the case, then for any $\Psi \in L^2(X^3, \mathscr{B}^3, \omega)$,

$$\lim_{k\to\infty} \int \Psi(R^{n_k} f_1 \otimes S^{n_k} f_2 \otimes T^{n_k} f_3) d\omega = 0.$$

Hence for any $R \times S \times T$-invariant function Ψ, we have $\int \Psi(f_1 \otimes f_2 \otimes f_3) d\omega = 0$. This contradicts the condition $P_Z(f_1 \otimes f_2 \otimes f_3) \neq 0$.

Assume that for some u, v, $w \in L^\infty(X, \mathscr{B}, \mu)$,

$$\limsup_{k\to\infty} \left| \int (u \otimes v \otimes w)(R^{n_k} f_1 \otimes S^{n_k} f_2 \otimes T^{n_k} f_3) d\omega \right| > 0.$$

By (10) we have

$$\limsup_{k\to\infty} \left| \sum_{\lambda_{21,i}=\lambda_{31,j}} \langle uR^{n_k} f_1, e_i\rangle \langle vS^{n_k} f_2, \bar{e}_j\rangle \langle wT^{n_k} f_3, \bar{e}_i e_j\rangle \right| > 0.$$

Let K be an integer such that the dimensions of RS^{-1}, ST^{-1} and TR^{-1}-invariant spaces are all less than K. Then

$$\left| \sum_{\lambda_{21,i}=\lambda_{31,j}} \langle uR^{n_k} f_1, e_i\rangle \langle vS^{n_k} f_2, \bar{e}_j\rangle \langle wT^{n_k} f_3, \bar{e}_i e_j\rangle \right| \leqslant$$

$$\leqslant \left(\sum_{\lambda_{21,i}=\lambda_{31,j}} |\langle uR^{n_k} f_1, e_i\rangle| \, |\langle vS^{n_k} f_2, \bar{e}_j\rangle| \right) \sup_t |\langle wT^{n_k} f_3, e_t\rangle| \leqslant$$

$$\leqslant K\left(\sum_i |\langle uR^{n_k}f_1, e_j\rangle|^2\right)^{1/2}\left(\sum_j |vS^{n_k}f_2, e_j\rangle|^2\right)^{1/2} \sup_t |\langle wT^{n_k}f_3 e_t\rangle| \leqslant$$

$$\leqslant K \| uR^{n_k}f_1 \|_2 \| vS^{n_k}f_2 \|_2 \sup_t |\langle wT^{n_k}f_3, e_t\rangle| \leqslant$$

$$\leqslant K \| u \|_\infty \| v \|_\infty \| f_1 \|_2 \| f_2 \|_2 \sup_t |\langle wT^{n_k}f_3, e_t\rangle|.$$

We have that

$$\limsup_{k\to\infty} \sup_t |\langle wT^{n_k}f_3, e_t\rangle| > 0.$$

Since the positions of R, S and T are same, we can get the rest of the lemma by similar arguments. □

We use the following result of CONZE and LESIGNE to finish this section. A proof can be found in [3, pages 167–169] and [4].

Proposition 5.7 (Conze and Lesigne). *Let R be an ergodic measure preserving transformation on a probability space $(X, \mathscr{B}, \mu)$ and let G be the Kronecker group. Let $\mathscr{M}$ be an irreducible finite dimensional R-invariant G-module and let $f \in \mathscr{M} \cap L^\infty(X, \mathscr{B}, \mu)$ be a bounded function. If for any sequence $\{n_k\} \subset \mathbf{N}$ there exists a bounded function u such that*

$$\limsup_{k\to\infty} \sup_i |\langle uR^{n_k}f, e_i\rangle| > 0,$$

then the matrix-valued function H induced by R with respect to a basis of $\mathscr{M}$ is an (E)-cocycle on the Kronecker group G, i.e. $\mathscr{M}$ is of Conze-Lesigne type.

6. Existence of the Limits

Now we are in the position of summarizing our main results.

Theorem 6.1. *Let R, S and T be ergodic measure preserving transformations on $(X, \mathscr{B}, \mu)$ such that RS^{-1}, ST^{-1} and TR^{-1} are ergodic. Then for any f_1, f_2 and $f_3 \in L^\infty(X)$, the limit*

$$\lim_{n\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f_1(R^n x) f_2(S^n x) f_3(T^n x) \tag{11}$$

exists in the L^1-norm.

Proof. Let G be the Kronecker group. Let ω be the joining on $(X^3, \mathscr{B}^3)$ defined by (9) and let Z be the closed subspace of $L^2(X^3, \mathscr{B}^3, \omega)$ spanned by all $R \times S \times T$-invariant functions with respect to ω. By Corollary 5.5, we know that

$$Z \subset K(G,R) \otimes K(G,S) \otimes K(G,T).$$

Now we claim that if $f_1 \perp K(G,R) \cap L^\infty(X)$, then for any f_2 and $f_3 \in L^\infty(X)$, $f_1 \otimes f_2 \otimes f_3 \perp Z$. Actually we only need to show that for any $\varphi_1 \in K(G,R)$, $\varphi_2 \in K(G,S)$ and $\varphi_3 \in K(G,T)$,

$$\int (f_1 \otimes f_2 \otimes f_3)(\varphi_1 \otimes \varphi_2 \otimes \varphi_3) d\omega = 0. \tag{12}$$

Since $e_i \varphi_1 \in K(G,R)$ for any eigenfunction e_i, $\langle f_1 \varphi_1, e_i \rangle = 0$. By (10), Eq. (12) is true which gives our claim. By Proposition 5.3, we only need to show (11) for $f_1 \in K(G,R)$, $f_2 \in K(G,S)$ and $f_3 \in K(G,T)$.

Assume that $K(R,G) = \bigoplus_i \mathscr{L}_j$, $K(S,G) = \bigoplus_j \mathscr{M}_j$ and $K(T,G) = \bigoplus_k = \mathscr{N}_k$ where $\mathscr{L}_i$, $\mathscr{M}_j$ and $\mathscr{N}_k$ are respectively R, S and T-invariant irreducible finite dimensional G-modules. It follows from Lemma 5.6 and Proposition 5.7 that if one of $\mathscr{L}_i$, $\mathscr{M}_j$ and $\mathscr{N}_k$ is not a Conze-Lesigne type G-module then

$$P_Z(\mathscr{L}_i \otimes \mathscr{M}_j \otimes \mathscr{N}_k) = 0.$$

Therefore by Proposition 5.3, we only need to prove the theorem for f_1, f_2 and f_3 belonging to Conze-Lesigne type G-modules $\mathscr{L}_i$, $\mathscr{M}_j$ and $\mathscr{N}_k$ respectively. Since irreducible Conze-Lesigne type G-module must be one-dimensional, there are functions φ_{Ri}, φ_{Sj}, φ_{Tk} for $\mathscr{L}_i$, $\mathscr{M}_j$, $\mathscr{N}_k$ respectively and (E)-cocycles H_{Ri}, H_S, H_T with respect to R, S, T respectively such that

$$\varphi_{Ri}(Rx) = H_{Ri}(x)\varphi_{Ri}(x),$$
$$\varphi_{Sj}(Sx) = H_{Sj}(x)\varphi_{Sj}(x)$$

and $\varphi_{Tk}(Tx) = H_{Tk}(x)\varphi_{Tk}(x)$. Assume that

$$f_1(x) = \sum_i a_{1i}(x)\varphi_{Ri}(x), f_2(x) = \sum_j a_{1j}(x)\varphi_{Sj}(x) \text{ and } f_3(x) =$$
$$= \sum_i a_{3k}(x)\varphi_{Tk}(x),$$

where a_{1i}, a_{2j}, a_{3k} are G-measurable. Let ρ_{α_1}, ρ_{α_2} and ρ_{α_3} be the rotations on G with respect to measure preserving transformations R,

S and T on X respectively. Then

$$\frac{1}{N}\sum_{n=0}^{N-1} f_1(R^n x) f_2(S^n x) f_3(T^n x) =$$

$$= \sum_{i,j,k}\left(\frac{1}{N}\sum_{n=0}^{N-1} a_{1i}(x+n\alpha_1)a_{2j}(g+n\alpha_2)a_{3k}(g+n\alpha_3)\times\right.$$

$$\left.\times H_R^{(n)}(x)H_S^{(n)}(s)H_T^{(n)}(x)\right)\varphi_{Ri}(x)\varphi_{Sj}(x)\varphi_{Tk}(x).$$

Now the theorem follows from Theorem 4.7. □

The proof of the following known result is basically the same as the proof of Theorem 6.1. For sake of completeness, we sketch our proof again.

Theorem 6.2 (Conze and Lesigne, Furstenberg and Weiss). *Assume that R is a measure preserving transformation on $(X,\mathscr{B},\mu)$. For any integers ℓ_1, ℓ_2 and ℓ_3 and any f_1, f_2 and $f_3 \in L^\infty(X)$, the limit*

$$\lim_{n\to\infty}\frac{1}{N}\sum_{n=0}^{N-1} f_1(R^{\ell_1 n}x)f_2(R^{\ell_2 n}x)f_3(R^{\ell_3 n}x) \tag{13}$$

exists in the L^1-norm.

Proof. If we can prove the convergence for the ergodic measure preserving transformation R, the theorem will follow from the ergodic decomposition theorem. Here we only prove the theorem for $\ell_1 = 1$, $\ell_2 = 2$ and $\ell_3 = 3$. Our proof can be easily generalized for all integers ℓ_1, ℓ_2 and ℓ_3.

Let ω be the joining on $(X^3, \mathscr{B}^3)$ defined by

$$\omega(A\times B\times C) = \lim_{n\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\int 1_A(R^n x)1_B(R^{2n}x)1_C(R^{3n}x)$$

for any A, B and $C\in\mathscr{B}$ and let Z be the subspace of $L^2(X^3,\mathscr{B}^3,\omega)$ spanned by all of $R\times R^2\times R^3$-invariant function with respect to the measure ω. Let G be the Kronecker group of the system (X,R) (notice that $\langle R,R^2,R^3\rangle = \langle R\rangle$) and $e_1, e_2, \ldots$ are all characters. Since $K(R,G) = K(R^m,G)$ for any integer m, we have that, by Corollary 5.5.

$$Z \subset K(R,G)\otimes K(R,G)\otimes K(R,G).$$

It follows from Lemma 5.6 that if $P_Z(f_1 \otimes f_2 \otimes f_3) \neq 0$, then for any sequence $\{n_k\}$ there exist bounded functions u, v and w such that

$$\limsup_{k\to\infty} \sup_i |\langle uR^{n_k} f_1, e_i\rangle| > 0, \tag{14}$$

$$\limsup_{k\to\infty} \sup_i |\langle vR^{2n_k} f_2, e_i\rangle| > 0, \tag{15}$$

and

$$\limsup_{k\to\infty} \sup_i |\langle wR^{3n_k} f_3, e_i\rangle| > 0. \tag{16}$$

By Proposition 5.7, if f_1 is in a finite dimensional G-module, this G-module must be a Conze-Lesigne type G-module. We still have to show that f_2 and f_3 are also in Conze-Lesigne type G-modules. Actually, we only need to show that (15) and (16) imply (14).

For any sequence $\{n_k\}$, there must be an infinite subsequence of even numbers or odd numbers. Without loss of generality, we assume that there exists an infinite subsequence of odd numbers, i.e. there exists a sequence $\{m_j\}$ such that $\{2m_j + 1\}$ is a subsequence of $\{n_k\}$. Then there is a bounded function v such that

$$\limsup_{k\to\infty} \sup_i |\langle vR^{2m_j} f_2, e_i\rangle| > 0.$$

Since

$$|\langle vR^{2m_j} f_2, e_i\rangle| = |\langle RvR^{2m_j+1} f_2, Re_i\rangle| = |\langle RvR^{2m_j+1} f_2, e_i\rangle|,$$

we know that

$$\limsup_{k\to\infty} \sup_i |\langle RvR^{n_k} f_2, e_i\rangle| = \limsup_{j\to\infty} \sup_i |\langle RvR^{2m_j+1} f_2, e_i\rangle| > 0.$$

This tells us that f_2 must be in a Conze-Lesigne G-module. The same method can prove that f_3 is also in a Conze-Lesigne G-module. Therefore we only need to prove the theorem for f_1, f_2 and f_3 belonging to Conze-Lesigne type G-modules. Now, similarly as in the proof of Theorem 6.1, the theorem follows from Corollary 5.7.

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