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On the Distribution of Ideals in Cubic Number Fields

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Abstract. Let K be a cubic number field. Denote by $A_K(x)$ the number of ideals with ideal norm $\leqslant x$, and by $Q_K(x)$ the corresponding number of squarefree ideals. The following asymptotics are proved. For every $\varepsilon > 0$

$$A_K(x) = c_1 x + O(x^{43/96+\varepsilon}),$$

$$Q_K(x) = c_2 x + O(x^{1/2} \exp\{-c(\log x)^{3/5}(\log\log x)^{-1/5}\}) .$$

Here c_1, c_2 and c are positive constants. Assuming the Riemann hypotheses for the Dedekind zeta function ζ_K, the error term in the second result can be improved to $O(x^{53/116+\varepsilon})$.

1. Introduction. Let K be a number field of degree $n = [K:\mathbb{Q}]$ over $\mathbb{Q}$. Denote by $A_K(x)$ the ideal function, e.g. the function which counts the number of ideals with norm $\leqslant x$, and by $Q_K(x)$ the corresponding function counting the number of squarefree ideals with norm $\leqslant x$. Using LANDAU's classical estimate ([5], p. 135)

$$A_K(x) = \varrho_K x + O(x^{(n-1)/(n+1)}) , \qquad (1)$$

where ϱ_K denotes the residue of the Dedekind zeta-function ζ_K at $s = 1$, elementary considerations prove that

$$Q_K(x) = \frac{\varrho_K}{\zeta_K(2)} x + O(x^{\lambda}) , \qquad (2)$$

with $\lambda = (n-1)/(n+1)$ for $n \geqslant 3$ and $\lambda = 1/2$ for $n \leqslant 2$. In the case of a quadratic number field Ω, the evaluation of $A_\Omega(x)$ can be reduced to a twodimensional divisor problem. As indicated by W. G. NOWAK [7], the new investigations of Mozzochi and Iwaniec in the divisor problem lead to

$$A_\Omega(x) = \varrho_\Omega x + O(x^{7/22+\varepsilon})$$

for every $\varepsilon > 0$. With some $c > 0$ NOWAK [7] obtained the estimate

$$Q_\Omega(x) = \frac{\varrho_\Omega}{\zeta_\Omega(2)} x + O(x^{1/2} \exp\{- c(\log x)^{3/5} (\log\log x)^{-1/5}\}) .$$

Assuming the Riemann hypothesis for ζ_Ω, he sharpened this result to

$$Q_\Omega(x) = \frac{\varrho_\Omega}{\zeta_\Omega(2)} x + O(x^{15/38+\varepsilon}) .$$

For cubic number fields the general estimates (1) and (2) seem to be the best published ones. In this paper the following theorems are proved.

Theorem 1. *For every cubic number field K over* $\mathbb{Q}$ *and every* $\varepsilon > 0$ *one has*

$$A_K(x) = \varrho_K x + O(x^{43/96+\varepsilon}) , \tag{3}$$

where ϱ_K *is the residue of* ζ_K *at* $s = 1$.

Theorem 2. *For every cubic number field K over* $\mathbb{Q}$ *there is a constant* $c > 0$ *such that*

$$Q_K(x) = \frac{\varrho_K}{\zeta_K(2)} x + O(x^{1/2} \exp\{- c(\log x)^{3/5} (\log\log x)^{-1/5}\}) . \tag{4}$$

If ζ_K *has no zeros in the half plane* $\mathrm{Re}\, s > 1/2$, *then for every* $\varepsilon > 0$

$$Q_K(x) = \frac{\varrho_K}{\zeta_K(2)} x + O(x^{53/116+\varepsilon}) . \tag{5}$$

Remarks. 1) The O-constants in both theorems may depend on ε and K. 2) The generating Dirichlet series for $Q_K(x)$ is $\zeta_K(s)/\zeta_K(2s)$. Therefore the exponent $1/2$ in Theorem 2 cannot be improved without further assumptions on the zeros of ζ_K.

2. Auxiliary results. Lemma 1 collects the algebraic properties of cubic number fields which are used in the proof of Theorem 1.

Lemma 1. *Let K be a cubic number field over* $\mathbb{Q}$ *and* $D = df^2$ *(d squarefree) its discriminant; then*

i) *K is a normal extension if and only if* $D = f^2$. *In this case*

$$\zeta_K(s) = \zeta(s) L(s, \chi_1) \overline{L(s, \chi_1)} ,$$

where $\zeta(s)$ is the Riemann zeta function and $L(s,\chi_1)$ is an ordinary Dirichlet series (over $\mathbb{Q}$) corresponding to a primitive character χ_1 modulo f.

ii) *If K is not a normal extension, then $d \neq 1$ and*

$$\zeta_K(s) = \zeta(s)\, L(s,\chi_2)\ ,$$

where $L(s,\chi_2)$ is a Dirichlet L-series over the quadratic field $\Omega = \mathbb{Q}(\sqrt{d})$:

$$L(s,\chi_2) = \sum_{a} \chi_2(a)\, N_\Omega(a)^{-s}, \quad (\operatorname{Re} s > 1)\ .$$

Here summation is taken over all ideals $a \neq 0$ in Ω and N_Ω denotes the (absolute) ideal norm in Ω. To describe the character χ_2 let H be the ideal group in Ω according to which the normal extension $K(\sqrt{d})$ is the class field. Then H divides the set A^f of all ideals $a \subseteq \Omega$ with $(a,f) = 1$ into three classes $A^f = H \cup C \cup C'$, and $(\omega := e^{2\pi i/3})$

$$\chi_2(a) = \begin{cases} 1 & a \in H \\ \omega & a \in C \\ \bar{\omega} & a \in C' \\ 0 & (a,f) \neq 1\ . \end{cases}$$

The substitution $\tau = (\sqrt{d} \to -\sqrt{d})$ in Ω maps C onto C'.

Proof. Denote by $B \subset \mathbb{C}$ the normal closure of K and by $G = \operatorname{Gal}(B \mid \mathbb{Q})$ its Galois group. Then G is a subgroup of S_3 and 3 divides $[B:\mathbb{Q}] = \operatorname{ord}(G)$, e.g. G is cyclic of degree 3 or $G \simeq S_3$. In the first case $K = B$ is an abelian extension and K is the class field with respect to an ideal group H_1 of index 3 (the ideal class group is isomorphic to G). Hence by [2], p. 33, Theorem 14

$$\zeta_K(s) = \zeta(s)\, L(s,\chi_1)\, L(s,\bar{\chi}_1)\ ,$$

and via the discriminant formula ([2], p. 38)

$$D = \prod_{\chi} f_\chi = f_{\chi_1}^2,$$

where f_χ denotes the conductor of the character χ, $D = f^2$ and $f_{\chi_1} = f$. The case of a non-normal extension K was studied in [3]. There all the statements of ii) can be found up to $\zeta_K(s) = \zeta(s)\, L(s,\chi_2)$. To prove this, the factorization

$$\prod_{\mathscr{P}\mid q}(1 - N_K(\mathscr{P})^{-s}) = (1 - q^{-s})\prod_{p\mid q}(1 - \chi(p)\,N_\Omega(p)^{-s})$$

has to be established for all rational primes q (here $\mathscr{P}$ denotes prime ideals in K and p prime ideals in Ω). But this can be checked easily using the prime ideal factorization of q in K and Ω as it is summerized in [2], p. 568.

Lemma 2. *Let K be any number field over* $\mathbb{Q}$ *and denote by* μ_K *the Möbius function of K, then*

$$M_K(x) := \sum_{N_K(a)\leqslant x}\mu_K(a) = O(x\exp\{-c(\log x)^{3/5}(\log\log x)^{-1/5}\}),$$

with some positive constant c. Here summation is taken over all ideals $a \neq 0$ *in K with (absolute) ideal norm* $N_K(a) \leqslant x$.

Proof. For $K = \mathbb{Q}$ this is a classical result due to Walfisz. For the quadratic case see [7]. The proof of WALFISZ [11], p. 191, can be transfered to the general situation using the known zero free region of ζ_K ([6], p. 246), which is (up to the constants involved) the same as those for ζ. Following Walfisz, one first proves that

$$\sum_{N_K(a)\leqslant x}\mu_K(a)\log\frac{x}{N_K(a)} = O(x\,\delta^2(x)),$$

where $\delta(x) = \exp\{-c(\log x)^{3/5}(\log\log x)^{-1/5}\}$ is used as abbreviation. To get rid of the factor $\log(x/N_K(a))$, observe that furthermore

$$\sum_{N_K(a)\leqslant(1+\delta(x))x}\mu_K(a)\log\left(\frac{x}{N_K(a)}(1+\delta(x))\right) \ll x\,\delta^2(x)$$

holds; hence

$$M_K(x)\log(1+\delta(x)) + \sum_{N_K(a)\leqslant x}\mu_K(a)\log\left(\frac{x}{N_K(a)}\right) +$$
$$+ \sum_{x<N_K(a)\leqslant(1+\delta(x))x}\mu_K(a)\log\frac{x(1+\delta(x))}{N_K(a)} \ll x\,\delta^2(x).$$

The first sum is of the order $O(x\,\delta^2(x))$ and the second one (using a weak version of (1)) is less than

$$\sum_{x<N_K(a)\leqslant(1+\delta(x))x}\log(1+\delta(x)) =$$
$$= \log(1+\delta(x))\,(A_K((1+\delta(x))\,x) - A_K(x)) \ll x\,\delta^2(x).$$

Hence $M_K(x) \ll x\delta^2(x)/\log(1+\delta(x)) \ll x\delta(x)$, which proves the Lemma.

3. Proof of Theorem 1. Denote by $F(n)$ the number of ideals in K with norm equal to n. First a method of ATKINSON [1] is used to establish

$$A_K(x) = \varrho_K x + c x^{1/3} \sum_{n \leqslant X} F(n) n^{-2/3} f\left(\frac{6\pi}{\sqrt[3]{|D|}}(nx)^{1/3}\right) + O(x^{2/3+\varepsilon} X^{-1/3}) \quad (6)$$

for every $\varepsilon > 0$. Here c is a real constant, $f \in \{\sin, \cos\}$ and $x^{1/2} < X < x$ is a free real parameter. Since $F(n) \ll n^{\varepsilon}$, it sufficies to consider x to be half an odd integer. For every $\varepsilon > 0$ and $T := (xX)^{1/3} < x^{2/3}$ the truncated version of Perron's formula (e.g. [9], p. 376) yields

$$A_K(x) = \frac{1}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \zeta_K(s)\, x^s s^{-1}\, ds + O(x^{1+\varepsilon} T^{-1}),$$

where $\zeta_K(s) = \sum_{n=1}^{\infty} F(n)\, n^{-s}$, $\operatorname{Re} s = \sigma > 1$, is the zeta function of K. Next the line of integration is shifted to $\sigma = -\varepsilon$. The residue at $s = 1$ contributes $\varrho_K x$ to the integral. Since

$$\zeta_K(s) \ll |t|^{\frac{3}{2}(1-\sigma+\varepsilon)}$$

(uniformly in $-\varepsilon \leqslant \sigma \leqslant 1+\varepsilon$, $|t| \geqslant t_0$, cf. [10], p. 200), the integrals along the horizontal lines $[-\varepsilon \pm iT, 1+\varepsilon \pm iT]$ are of order $O(x^{1+\varepsilon} T^{-1})$; hence

$$A_K(x) = \varrho_K x + \frac{1}{2\pi i} \int_{-\varepsilon-iT}^{-\varepsilon+iT} \zeta_K(s)\, x^s s^{-1}\, ds + O(x^{1+\varepsilon} T^{-1}).$$

Now the functional equation $\zeta_K(1-s) = Z(s)\,\zeta_K(s)$ with

$$Z(s) = 8\,|D|^{-1/2} (8\pi^3/|D|)^{-s} \left(\cos\frac{\pi s}{2}\right)^{r} \left(\sin\frac{\pi s}{2}\right)^{r_2} \Gamma^3(s)$$

is used (cf. [5], p. 76). Here $(r, r_2) = (3, 0)$ or $(r, r_2) = (2, 1)$ according as all embeddings of K in $\mathbb{C}$ are real or not. One obtains

$$A_K(x) = \varrho_K x + \frac{x}{2\pi i} \int_{1+\varepsilon-iT}^{1+\varepsilon+iT} Z(s)\,\zeta_K(s) \frac{x^{-s}}{1-s}\, ds + O(x^{1+\varepsilon} T^{-1}).$$

By Stirling's formula for $s = 1 + \varepsilon + it$

$$\frac{\Gamma^3(s)}{1-s} = -\Gamma^2(s)\Gamma(s-1) = -\sqrt{3}\,2\pi 3^{2-3s}\Gamma(3s-2)(1 + O(|s|^{-1})) =$$

$$= O(|t|^{\frac{1}{2}+3\varepsilon} e^{-\frac{3\pi}{2}|t|}) + O(1)\,,$$

and with some $f \in \{\sin, \cos\}$

$$4\left(\cos\frac{\pi s}{2}\right)^{r}\left(\sin\frac{\pi s}{2}\right)^{r_2} = \pm f\left(\frac{3\pi}{2}s\right)(1 + O(e^{-|t|})) = O(e^{\frac{3\pi}{2}|t|})\,.$$

Hence

$$A_K(x) = \varrho_K x + c_1 x \frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} \zeta_K(s)\Gamma(3s-2)3^{2-3s} f\left(\frac{3\pi}{2}s\right)\left(\frac{8\pi^3 x}{|D|}\right)^{-s} ds +$$

$$+ O(x^{-\varepsilon}\int_1^T t^{-\frac{1}{2}+3\varepsilon}dt) + O(x^{1+\varepsilon}T^{-1})\,.$$

The first O-term is less than the second one. The substitution $3s - 2 \to s$ yields

$$A_K(x) - \varrho_K x =$$

$$= c_2 x^{1/3}\frac{1}{2\pi i}\int_{1+3\varepsilon-3iT}^{1+3\varepsilon+3iT} \zeta_K\left(\frac{1}{3}s + \frac{2}{3}\right)\Gamma(s) f\left(\frac{\pi}{2}s + \frac{\pi}{2}\right)\left(\frac{6\pi}{\sqrt[3]{|D|}}x^{1/3}\right)^{-s} ds +$$

$$+ O(x^{1+\varepsilon}T^{-1}) = \tag{7}$$

$$= c_3 x^{1/3}\sum_{n=1}^{\infty} F(n)\,n^{-2/3} I_n + O(x^{1+\varepsilon}T^{-1}),$$

where

$$I_n := \frac{1}{2\pi i}\int_{1+3\varepsilon-iT}^{1+3\varepsilon+iT} \Gamma(s) f_1\left(\frac{\pi}{2}s\right)\left(\frac{6\pi}{\sqrt[3]{|D|}}(nx)^{1/3}\right)^{-s} ds$$

and $f_1 \in \{\sin, \cos\}$. This integral has been studied by ATKINSON [1] in his Lemma 1 and Lemma 2 for $f_1 = \cos$. Replacing cos by sin, his arguments remain unaffected. Also the summation of (7) can be taken over from [1], where the same sum is evaluated with $d_3(n)$ in place of $F(n)$. This all together proves (6).

The trivial estimate of the exponential sum in (6) proves (1) with an extra factor x^{ε}. To improve this, Lemma 1 is used. The factorization of ζ_K gives

$$F(n) = \sum_{d \mid n} q(d)\ ,$$

where in the case of a normal extension $q(d) = \sum_{xy=d} \chi_1(x)\overline{\chi_1(y)}$. Otherwise $q(d)$ is equal to the number of ideals $a \in H$ with $N_\Omega(a) = d$ minus two times the number of ideals $a \in C$ with $N_\Omega(a) = d$. In both cases $|q(d)| \ll d^{\varepsilon}$.

Let $N \leqslant N' \leqslant 2N \leqslant X := x^{21/32}$, then partial summation yields

$$\sum_{N<n\leqslant N'} F(n)\, n^{-2/3} f(c'(xn)^{1/3}) \ll N^{-2/3} \Big| \sum_{N<n\leqslant N_1} F(n)\, e(c'(xn)^{1/3}) \Big|\ , \quad (8)$$

where $N_1 \leqslant N'$ and $e(t) := e^{2\pi i t}$. Hence it suffices to estimate

$$\sum_{N\leqslant nm\leqslant N_1} q(m)\, e(c'(xnm)^{1/3})\ .$$

The range of summation is divided into domains of the type

$$\mathscr{D}_1 = \{(n,m) \mid N \leqslant nm \leqslant N_1,\, M_1 \leqslant m \leqslant M_2 \leqslant M_1(1+\varepsilon),$$
$$X_1 \leqslant n \leqslant X_2 \leqslant X_1(1+\varepsilon)\}\ .$$

The arising sums

$$S_1 := \Big| \sum_{(n,m)\in\mathscr{D}_1} q(m)\, e(c'(xnm)^{1/3}) \Big|$$

have been estimated by G. KOLESNIK [4], p. 240—246 (for an arbitrary function $q(n)$ satisfying $|q(n)| \ll n^{\varepsilon}$). Assuming $x^{11/32} \leqslant N \leqslant X$ he obtained

$$S_1 \ll \begin{cases} N^{2/3}\, t^{11/96+\varepsilon} & \text{for } X_1 \leqslant N^{16/35}, \\ \{(tN)^{1/6} N^{-8/35} + N^{27/35} + (tN)^{1/12} N^{3/70} + \\ + (tN)^{1/9} N^{18/35} + (tN)^{1/6}\, t^{-1/20} N^{49/100} + N(tN)^{-1/6}\}\, t^{\varepsilon} & \text{else.} \end{cases}$$

In the remaining case the trivial estimate

$$\sum_{n\leqslant x^{11/32}} F(n)\, n^{-2/3} f(c_1(xn)^{1/3}) \ll x^{11/93+\varepsilon}$$

is sufficient. Using Kolesnik's estimate and (8), summation over N yields the same bound for the sum on the right side of (6). This completes the prove of Theorem 1.

4. Proof of Theorem 2. The following elementary convolution argument is used to prove part one of the theorem. For the sake of brevity set $N(a) := N_K(a)$, $Y := (x\,\delta(x))^{1/2}$, and $\delta(x) := \exp\{-c([\log x)^{3/5}(\log\log x)^{1/5}\}$:

$$Q_K(x) = \sum_{N(a)\leqslant x} \mu_K^2(a) = \sum_{N(a)\leqslant x}\sum_{b^2|a} \mu_K(b) = \sum_{N(b)^2 N(c)\leqslant x} \mu_K(b) =$$

$$= \sum_{N(b)\leqslant Y} \mu_K(b)\, A_K\left(\frac{x}{N(b)^2}\right) +$$

$$+ \sum_{N(c)\leqslant x Y^{-2}} \left(M_K\left(\sqrt{\frac{x}{N(c)}}\right) - M_K(Y)\right) =: S_1 + S_2\,.$$

By Theorem 1, $A_K(x) = \varrho_K x + O(x^\lambda)$ with $\lambda < 1/2$, hence

$$S_1 = \varrho_K x \sum_{b\neq 0} \mu_K(b)\, N(b)^{-2} - \varrho_K x \sum_{N(b)>Y} \mu_K(b)\, N(b)^{-2} +$$

$$+ O(x^\lambda \sum_{N(b)\leqslant Y} N(b)^{-2\lambda})\,.$$

In the second sum partial summation is used together with Lemma 2; this yields

$$S_1 = \frac{\varrho_K}{\zeta_K(2)} x + O(x\, Y^{-1}\delta(x)) + O(x^\lambda\, Y^{1-2\lambda}) =$$

$$= \frac{\varrho_K}{\zeta_K(2)} x + O(x\,\delta(x)^{\frac{1}{2}-\lambda})\,.$$

Using Lemma 2 again

$$S_2 = \sum_{N(c)\leqslant x Y^{-2}} O(x^{1/2} N(c)^{-1/2}\delta'(x)) = O(x\, Y^{-1}\delta'(x)) = O(x\,\delta''(x))\,,$$

where $\delta'(x)$ and $\delta''(x)$ are defined as $\delta(x)$ with suitable positive constants c' and c''. This proves (4). To establish the conditional result (5), a refined convolution method (going back to Montgomery and Vaughan) is used in its general formulation due to W. G. NOWAK and M. SCHMEIER [8]. By Theorem 1 the required assumptions are satisfied with $\lambda = 43/96$, $h = 1/2$, $a = 2$, $b = 1$ (in the notation of [8]) and (5) follows immediately.

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