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The Skew-Hyperbolic Motion Group of the Quaternion Plane

By

Markus Stroppel, Stuttgart

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Abstract. Up to conjugation, there exist three different polarities of the projective plane $P_2\mathbb{H}$ over Hamilton's quaternions $\mathbb{H}$. The skew hyperbolic motion group of $P_2\mathbb{H}$ is introduced as the centralizer of a polarity "of the third kind". According to a result of R. Löwen, the quaternion plane is characterized among the eight-dimensional stable planes by the fact that it admits an effective action of the centralizer of a polarity of the first or second kind (i.e., the elliptic or the hyperbolic motion group). In the present paper, we prove the analogous result for the skew hyperbolic case.

1. Polarities of the Quaternion Plane

This section collects basic information about polarities of the projective plane over Hamilton's quaternions. It seems that most of the contents of this section is folklore. As I could not find adequate references, I combine the introduction of the elliptic, hyperbolic and skew hyperbolic polarities with a proof that they represent the conjugacy classes of polarities of the quaternion plane.

A polarity of a projective plane $(\mathscr{P}, \mathscr{L})$ is an involutory correlation, i.e., a mapping $\kappa: \mathscr{P} \cup \mathscr{L} \to \mathscr{L} \cup \mathscr{P}$ such that κ^2 is the identity mapping, and that κ interchanges points and lines, but preserves incidence.

In this first section, we are going to classify the polarities of the projective plane $P_2\mathbb{H} = (u_1(\mathbb{H}^3), u_2(\mathbb{H}^3))$ over Hamilton's quaternions $\mathbb{H}$. Here $u_n(\mathbb{H}^3)$ denotes the set of all n-dimensional vector subspaces of the left vector space $\mathbb{H}^3$ consisting of rows $\boldsymbol{x} = (x_1, x_2, x_3)$. If $\boldsymbol{x} = (x_1, x_2, x_3) \neq (0,0,0)$, then the one-dimensional subspace spanned by $\boldsymbol{x}$ will be denoted by $[\boldsymbol{x}] = [x_1, x_2, x_3]$.

Notation 1.1. The skew field of Hamilton's quaternions is defined as the (associative) $\mathbb{R}$-algebra $\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k = \mathbb{C} + \mathbb{C}j$, subject to the rules $ij = k = -ji, jk = i = -kj, ki = j = -ik$, and $i^2 = j^2 = -1 = k^2$. The mapping

$$\rho: \mathbb{H} \to \mathbb{H}, h = h_1 + h_2 i + h_3 j + h_4 k \mapsto \bar{h} := h_1 - h_2 i - h_3 j - h_4 k$$

is called *conjugation*, it is an anti-automorphism of $\mathbb{H}$. The eigenspaces of ρ are $\mathbb{R}$ and $\mathrm{Pu}\mathbb{H} = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$.

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We will need the following compact subgroups of the multiplicative group.

$$\mathbb{T} = \{t \in \mathbb{C} \mid t\bar{t} = 1\}, \quad \mathbb{U} = \{u \in \mathbb{H} \mid u\bar{u} = 1\}.$$

Note that the inverse of $h \in \mathbb{H} \setminus \{0\}$ is $h^{-1} = \bar{h}/h\bar{h}$; in particular, we have that $u^{-1} = \bar{u}$ if $u \in \mathbb{U}$. For every matrix $A = (a_{\mu\nu})_{1 \leqslant \mu,\nu \leqslant 3} \in \mathbb{H}^{3\times 3}$ and every anti-automorphism α of $\mathbb{H}$, we write $A^\alpha := (a^\alpha_{\nu\mu})_{1 \leqslant \mu,\nu \leqslant 3}$ (i.e., the matrix A^α is obtained from A by transposing and applying α to the entries). This convention applies also to $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{H}^3$; we have $\boldsymbol{x}^\alpha = \begin{pmatrix} x_1^\alpha \\ x_2^\alpha \\ x_3^\alpha \end{pmatrix}$.

In order to classify the polarities, we need to know all involutory anti-automorphisms of $\mathbb{H}$.

Lemma 1.2. *Assume that α is an anti-automorphism of $\mathbb{H}$ such that $\alpha^2 = \mathrm{id}_{\mathbb{H}}$. Then there exists $u \in \mathbb{U}$ such that $u^2 \in \{1, -1\}$, and $h^\alpha = \bar{u}\bar{h}u$ for every $h \in \mathbb{H}$.*

Proof. Every automorphism of $\mathbb{H}$ is of the form $\iota_u = (h \to \bar{u}hu)$ for some $u \in \mathbb{U}$, see [18, Prop. 10.20, Prop. 10.25], and the group of all automorphisms and anti-automorphisms of $\mathbb{H}$ is the direct product of the group of all automorphisms and the group generated by ρ. For every anti-automorphism α of $\mathbb{H}$, we obtain that $\rho\alpha = \iota_u$ for some $u \in \mathbb{U}$, and $\alpha^2 = \mathrm{id}_{\mathbb{H}}$ implies that $\mathrm{id}_{\mathbb{H}} = \iota_u^2 = \iota_{u^2}$. This means that u^2 is contained in the center of $\mathbb{U}$, which is $\{1, -1\}$. □

Theorem 1.3. *Up to a change of basis, every polarity κ of $\mathrm{P}_2\mathbb{H}$ is described by a semi-bilinear form $f(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}S\bar{\boldsymbol{w}}$ according to $[\boldsymbol{v}]^\kappa = \{\boldsymbol{w} \mid f(\boldsymbol{v}, \boldsymbol{w}) = 0\}$, where S is one of the following diagonal matrices:*

$$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad \begin{pmatrix} i & & \\ & i & \\ & & i \end{pmatrix}.$$

Proof. According to [1, IV.3 Thm. 1, IV.2 Thm.], every polarity κ of $\mathrm{P}_2\mathbb{H}$ is described by a symmetrical α-form f, where α is an anti-automorphism of $\mathbb{H}$ such that $\alpha^2 = \mathrm{id}_{\mathbb{H}}$. The term "symmetrical" means that $f(\boldsymbol{v}, \boldsymbol{w}) = f(\boldsymbol{w}, \boldsymbol{v})^\alpha$. From [1, IV.4 Prop. 3] we infer that there exists a basis of $\mathbb{H}^3$ such that the form is described by a diagonal matrix $S = \mathrm{diag}(s_1, s_2, s_3)$, according to the formula $f(\boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{v}S\boldsymbol{w}^\alpha$. The fact that f is symmetrical (with respect to α) implies that

$$S^\alpha = \begin{pmatrix} s_1^\alpha & & \\ & s_2^\alpha & \\ & & s_3^\alpha \end{pmatrix} = S.$$

From 1.2 we know that there exists $u \in \mathbb{U}$ such that $u^2 \in \{1, -1\}$, and $s_\mu^\alpha = \bar{u}\bar{s}_\mu u$ for $\mu \in \{1, 2, 3\}$. If $u^2 = 1$, we obtain that $s_\mu \in \mathbb{R}$, and a basis transformation involving $\sqrt{|s_\mu|}$ transforms S to a matrix with diagonal entries from $\{1, -1\}$. Replacing f by

$-f$, if necessary, we obtain that

$$S \in \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \right\}.$$

If $u^2 = -1$, we have that $\bar{u} = -u$, and $s_\mu = s_\mu^\alpha = -u\bar{s}_\mu u$ implies that $-s_\mu u = \overline{s_\mu u}$. This means that $s_\mu u \in \mathrm{Pu}\mathbb{H}$. Now

$$-f(v, w)u = v \begin{pmatrix} s_1\bar{u} & & \\ & s_2\bar{u} & \\ & & s_3\bar{u} \end{pmatrix} \bar{w}.$$

Transforming the basis by a diagonal matrix with entries $\sqrt{|s_\mu \bar{u}|}$, we obtain that the polarity κ is described by a semi-bilinear form

$$f'(v, w) = v \begin{pmatrix} s_1' & & \\ & s_2' & \\ & & s_3' \end{pmatrix} \bar{w},$$

where $s_\mu' \in \mathbb{U} \cap \mathrm{Pu}\mathbb{H}$. Since the group $\mathbb{U}$ acts transitively on the set $\mathbb{U} \cap \mathrm{Pu}\mathbb{H}$ via $x \mapsto \bar{u}xu$ (see [18, Prop. 10.22]), we find a diagonal matrix which transforms f' to the form $s(v, w) = v \begin{pmatrix} i & & \\ & i & \\ & & i \end{pmatrix} \bar{w}$. $\square$

Note that the form s is not a symmetrical ρ-form, but rather a "skew-symmetrical" one.

Remark 1.4. Our arguments remain valid for polarities of the projective space $\mathrm{P}_n\mathbb{H}$, if $n \geqslant 3$. Passing to the factor space modulo the radical of the semi-bilinear form, one may further extend the classification of semi-bilinear forms on $\mathbb{H}^n$ to the degenerate case, where an interpretation as polarities is no longer possible. See [24] for a study of actions of the corresponding unitary groups on 8-dimensional stable planes.

Definition 1.5. The polarities described by the forms

$$e(v, w) = v \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix} \bar{w}, \quad h(v, w) = v \begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix} \bar{w}, \quad s(v, w) = v \begin{pmatrix} i & & \\ & i & \\ & & i \end{pmatrix} \bar{w}$$

are called the *ellipic*, *hyperbolic*, and *skew hyperbolic* polarity of $\mathrm{P}_2\mathbb{H}$, respectively.

2. The Skew-Hyperbolic Motion Group

In this section, we will study the centralizer of the polarity κ of $\mathrm{P}_2\mathbb{H}$ that is described by the form s. This centralizer is the so-called skew-hyperbolic group, it is induced by the unitary group with respect to s.

Definitions 2.1. If $\mathbb{F}\in\{\mathbb{R},\mathbb{C}\}$ and r and n are natural numbers such that $r\leqslant n$, we write $\mathrm{O}_n(\mathbb{F},r)$ for the orthogonal group with respect to the bilinear form

$$q((x_\mu)_{1\leqslant\mu\leqslant n},(y_\mu)_{1\leqslant\mu\leqslant n})=-\sum_{\mu=1}^{r}x_\mu y_\mu+\sum_{\mu=r+1}^{n}x_\mu y_\mu$$

on $\mathbb{F}^n$. The unitary group $\mathrm{U}_n(\mathbb{C},r)$ is defined with respect to the form

$$f((x_\mu)_{1\leqslant\mu\leqslant n},(y_\mu)_{1\leqslant\mu\leqslant n})=-\sum_{\mu=1}^{r}x_\mu\bar{y}_\mu+\sum_{\mu=r+1}^{n}x_\mu\bar{y}_\mu$$

on $\mathbb{C}^n$. Finally, we define the anti-unitary group $\mathrm{U}_n^\alpha(\mathbb{H})$ by the form

$$g((x_\mu)_{1\leqslant\mu\leqslant n},(y_\mu)_{1\leqslant\mu\leqslant n})=\sum_{\mu=1}^{n}x_\mu i\bar{y}_\mu$$

on $\mathbb{H}^n$. Note that

$$\mathrm{U}_n^\alpha(\mathbb{H})=\{A\in\mathbb{H}^{n\times n}\,|\,Ai\bar{A}=i\mathbb{1}\}.$$

The prefix S in $\mathrm{SO}_n(\mathbb{F},r)$ or $\mathrm{SU}_n(\mathbb{C},r)$ denotes the subgroup of matrices of determinant 1; the prefix P indicates the factor group modulo the center. Note that, if $n\geqslant 2$, the center of $\mathrm{SO}_n(\mathbb{F},r)$, $\mathrm{SU}_n(\mathbb{C},r)$, and $\mathrm{U}_n^\alpha(\mathbb{H})$ is $\langle-\mathbb{1}\rangle$, compare [2, Theorem 2]. We have the natural epimorphism

$$\pi:\mathrm{U}_n^\alpha(\mathbb{H})\to\mathrm{PU}_n^\alpha(\mathbb{H}):A\mapsto\{A,-A\}.$$

We will write $\Gamma:=\mathrm{PU}_3^\alpha(\mathbb{H})$, and call Γ the *skew-hyperbolic motion group* of $\mathrm{P}_2\mathbb{H}$.

Theorem 2.2. *For $n\geqslant 3$, the group $\mathrm{PU}_n^\alpha(\mathbb{H})=\mathrm{U}_n^\alpha(\mathbb{H})/\langle-\mathbb{1}\rangle$ is simple.*

Proof. By T_n, we denote the subgroup of $\mathrm{U}_n^\alpha(\mathbb{H})$ that is generated by the transvections in $\mathrm{U}_n^\alpha(\mathbb{H})$. If $n\geqslant 3$, then the center of T_n is $\langle-\mathbb{1}\rangle$, see [2, Theorem 2], and $T_n/\langle-\mathbb{1}\rangle$ is simple [2, Theorem 1]. In order to show that $T_n=\mathrm{U}_n^\alpha(\mathbb{H})$, we use the results of G. E. WALL [32], see [3, II §5, p. 47]. The set of fixed elements of ρ in the multiplicative group $\mathbb{H}^\times$ is just $\mathbb{R}^\times$; this set corresponds to Σ in the notation of [3, II §5, p. 47]. If e_μ denotes the μ-th standard basis element, we have that e_1 and e_2 span a "hyperbolic plane" in $\mathbb{H}^n$, and $[e_3]$ is orthogonal to this plane. We have that $\{g(he_3,he_3)\,|\,h\in\mathbb{H}\}=\mathrm{Pu}\mathbb{H}$. This set coincides with the set $\{h-\bar{h}\,|\,h\in\mathbb{H}^\times\}$. Therefore, the subgroup Ω_a of [3, II §5, p. 47] equals $\mathbb{H}^\times$ in our case. The commutator subgroup of $\mathbb{H}^\times$ is $\mathbb{U}$, see [18, Prop. 10.24]. Now Wall's theorem asserts that the factor group $\mathrm{U}_n^\alpha(\mathbb{H})/T_n$ is isomorphic to $\mathbb{H}^\times/(\mathbb{R}^\times\mathbb{U})$, and is therefore trivial. □

Since $-\mathbb{1}$ is contained in the connected subgroup $\{t\mathbb{1}\,|\,t\in\mathbb{T}\}$ of $\mathrm{U}_n^\alpha(\mathbb{H})$, we obtain the following.

Corollary 2.3. *For $n\geqslant 3$, the groups* $\mathrm{U}_n^\alpha(\mathbb{H})$ and $\mathrm{PU}_n^\alpha(\mathbb{H})$ *are connected.*

Remarks 2.4. The group $\mathrm{PU}_n^\alpha(\mathbb{H})$ is a real form of the simple complex Lie group of type D_n. It is called $\mathrm{S}\alpha\mathrm{U}_n\mathbb{H}$ in [31]; its Lie algebra is denoted by $\mathfrak{u}_n^*\mathbb{H}$ in [16], or $\mathrm{D}_n^{\mathbb{H}}$ in [31]. Note that $\mathrm{PU}_3^\alpha(\mathbb{H})\cong\mathrm{PSU}_4(\mathbb{C},1)$, see [3, IV §8 10), p. 113]. The group $\mathrm{PU}_2^\alpha(\mathbb{H})$ is not simple, but isomorphic to the direct product of $\mathbb{U}/\langle-\mathbb{1}\rangle\cong\mathrm{SO}_3(\mathbb{R},0)$

and $\mathrm{PSL}_2(\mathbb{R})$, see [2, §15, p. 379–380] or [3, IV §8 11), p. 113] and the description of Λ in Theorem 2.11 below.

Notation 2.5. We set

$$I=\begin{pmatrix} i & 0 & 0\\ 0 & i & 0\\ 0 & 0 & i\end{pmatrix},\quad J=\begin{pmatrix} 0 & 0 & -1\\ 0 & i & 0\\ 1 & 0 & 0\end{pmatrix},\quad K=\begin{pmatrix} -i & 0 & 0\\ 0 & i & 0\\ 0 & 0 & i\end{pmatrix},\quad L=\begin{pmatrix} i & 0 & 0\\ 0 & 0 & -1\\ 0 & 1 & 0\end{pmatrix}$$

and

$$P=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & -j\\ 0 & \sqrt{2} & 0\\ -i & 0 & -k\end{pmatrix}=\bar{P}^{-1},\quad Q=\begin{pmatrix} j & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}=\bar{Q}^{-1},$$

$$R=\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2} & 0 & 0\\ 0 & 1 & -j\\ 0 & -i & -k\end{pmatrix}=\bar{R}^{-1}.$$

These matrices will be used in order to obtain different descriptions of the skew hyperbolic motion group; this will ease some computations.

In order to understand certain subgroups of the skew hyperbolic motion group, we will need the following.

Lemma 2.6. *For* $r\in\{0,1\}$, *the group* $\mathrm{U}_3(\mathbb{C},r)$ *is isomorphic to* $\mathrm{U}_3(\mathbb{C},r)/\langle -\mathbb{1}\rangle$.

Proof. The group $\mathrm{U}_3(\mathbb{C},r)$ is obtained from the direct product $\mathrm{SU}_3(\mathbb{C},r)\times\mathbb{T}$ via identification of the (central) elements of order 3. This is realized by a surjective group homomorphism α with kernel $A=\{(a^{-1}\mathbb{1},a)\,|\,a\in\mathbb{T},a^3=1\}$. The group $\mathrm{U}_3(\mathbb{C},r)/\langle -\mathbb{1}\rangle$ is isomorphic to the factor group of $\mathrm{SU}_3(\mathbb{C},r)\times\mathbb{T}$ modulo the normal subgroup $B=\{(b^2\mathbb{1},b)\,|\,b\in\mathbb{T},b^6=1\}$. The corresponding natural epimorphism will be denoted by β. The mapping $\gamma\colon(S,t)\mapsto(S,t^2)$ is a surjective endomorphism of $\mathrm{SU}_3(\mathbb{C},r)\times\mathbb{T}$ with kernel $C=\{(\mathbb{1},c)\,|\,c\in\{1,-1\}\}$. Since C is contained in B, we obtain a factorization $\beta=\gamma\delta$, where the kernel of δ is $D=B^\gamma=\{(b^2\mathbb{1},b^2)\,|\,b^6=1\}=\{(a\mathbb{1},a)\,|\,a^3=1\}$. Since the mapping $(S,t)\mapsto(S,t^{-1})$ is an automorphism of the group $\mathrm{SU}_3(\mathbb{C},r)\times\mathbb{T}$, the assertion follows. □

The following lemma will be used several times; recall that $\mathbb{U}\cong\mathrm{SU}_2(\mathbb{C},0)$.

Lemma 2.7. *If* $\mathrm{SU}_2(\mathbb{C},0)$ *acts linearly on* $\mathbb{R}^n$, *then it leaves invariant some positive definite bilinear form on* $\mathbb{R}^n$. *In particular, the action is completely reducible. If the action is not trivial, we have that* $n\geqslant 3$. *If the action is effective, then even* $n\geqslant 4$.

Proof. The invariant bilinear form is obtained by Weyl's trick, see [6, II.4.14] or [16, Ch. 3, §4]. Since $\mathrm{SU}_2(\mathbb{C},0)$ equals its commutator group, every linear action on $\mathbb{R}^2$ is trivial; recall that $\mathrm{O}_2(\mathbb{R},0)$ is solvable. The centralizer of an involution in $\mathrm{SO}_3(\mathbb{R},0)$ is also solvable. Therefore, the group $\mathrm{SU}_2(\mathbb{C},0)$ cannot act effectively on $\mathbb{R}^3$, since it has a central involution. □

Theorem 2.8. (1) *The set U of absolute points of κ is homeomorphic to the sphere $\mathbb{S}_5$.*

(2) *The compact subgroup*

$$\mathrm{E} := \{A \in \mathbb{C}^{3\times 3} \mid AI\bar{A} = I, \det_{\mathbb{C}} A = 1\}^{\pi} \cong \mathrm{SU}_3(\mathbb{C}, 0)$$

of Γ acts transitively on U, while the group Γ even acts 2-transitively on U.

(3) *The stabilizer of the absolute point $[1,0,-j] = [1,0,0]^P$ is*

$$\Phi := \Gamma_{[1,0,0]^P} = \left\{ \begin{pmatrix} ru & 0 & 0 \\ h & t & 0 \\ xu + \frac{u\bar{h}ih}{2r} & \frac{u\bar{h}ti}{r} & \frac{u}{r} \end{pmatrix} \middle| \begin{array}{c} r > 0, u \in \mathbb{U}, h \in \mathbb{H} \\ t \in \mathbb{T}, x \in \mathbb{R} \end{array} \right\}^{P\pi},$$

the stabilizer of the two absolute points $[1,0,-j]$ and $[i,0,k] = [0,0,1]^P$ is

$$\Phi_{[0,0,1]^P} = \left\{ \begin{pmatrix} ru & & \\ & t & \\ & & \frac{u}{r} \end{pmatrix} \middle| r > 0, u \in \mathbb{U}, t \in \mathbb{T} \right\}^{P\pi}.$$

Proof. (i) After conjugation by P, the form s is described by the matrix J. Now one computes easily that the stabilizer of the absolute point $[1,0,0]^P$ is Φ, and that $\Phi_{[0,0,1]^P}$ has the form described in (3).

(ii) Let $[v]$, $[w]$ be two absolute points. Then $h := vI\bar{w} \neq 0$, and we may assume that $h = 1$. According to Witt's Theorem (see [17] or [3, I §11 p. 21]), there exists an element of Γ that maps $[v]$ to $[0,0,1]^P$ and $[w]$ to $[1,0,0]^P$. Thus Γ acts 2-transitively on U. Since Γ is connected by 2.3, we infer that U is connected.

(iii) Since $\dim \Phi = 10$, we infer from the dimension formula [5] (compare [27, 1.14]) that

$$\dim U = \dim(\Phi/(\Phi_{[i,0,k]})) = \dim \Phi - \dim(\Phi_{[i,0,k]}) = 5.$$

The stabilizer of the absolute point $[1, j, 0]$ in E is

$$\mathrm{E}_{[1,j,0]} = \left\{ \begin{pmatrix} a & b & \\ -\bar{b} & \bar{a} & \\ & & 1 \end{pmatrix} \middle| a, b \in \mathbb{C}, (a\bar{a} + b\bar{b}) = 1 \right\}^{\pi} \cong \mathrm{SU}_2(\mathbb{C}, 0).$$

Thus the E-orbit of $[1, j, 0]$ has dimension 5 as well. From [4, 1.8.10] we infer that this orbit has non-empty interior in U. This implies that it is open. On the other hand, it is compact, and we conclude that E acts transitively, since U is connected.

From the usual action of $\mathrm{E} \cong \mathrm{SU}_3(\mathbb{C}, 0)$ on $\mathbb{C}^3$, we infer that $\mathrm{E}/(\mathrm{E}_{[1,j,0]})$ is homeomorphic to $\mathbb{S}_5$. □

Corollary 2.9. *The group Γ has dimension* 15.

Proof. Since $\dim U = 5$ and $\dim \Phi = 10$, the assertion follows from the dimension formula again. □

Proposition 2.10. *The subgroups* ET, *where* $\mathrm{T} = \{t\mathbb{1} \,|\, t \in \mathbb{T}\}^{\pi}$, *and*

$$\Xi = \left\{ \left. \begin{pmatrix} r & 0 & 0 \\ h & 1 & 0 \\ x + \dfrac{\bar{h}ih}{2r} & \dfrac{\bar{h}i}{r} & \dfrac{1}{r} \end{pmatrix} \right| r > 0, h \in \mathbb{H}, x \in \mathbb{R} \right\}^{P\pi} .$$

form a polar decomposition of Γ; *i.e.*, $\Gamma = \Xi\mathrm{ET}$, *and* $\Xi \cap \mathrm{ET} = \{\mathbb{1}\}$. *The group* Ξ *is homeomorphic* (*though not isomorphic*) *to* $\mathbb{R}^6$, *and* ET *is a maximal compact subgroup of* Γ.

Proof. The given parametrization of Ξ by r, h, x shows that Ξ is homeomorphic to $\mathbb{R}^6$. By the Theorem of Malcev and Iwasawa [7, §4, Th. 13], every locally compact connected group G is homeomorphic to $\mathbb{R}^n \times C$ for some natural number n and any maximal compact subgroup C of G. As a nontrivial compact group is not contractible, we infer that the group Ξ has no non-trivial compact subgroup. Thus we obtain that Ξ and ET have trivial intersection. The quotient space Γ/Ξ has dimension 9, and the orbit of Ξ under ET in this quotient space satisfies $\dim((\Xi\mathrm{ET})/\Xi) = \dim((\mathrm{ET})/(\mathrm{E} \cap \Xi)) = \dim \mathrm{ET} = 9$. From [4, 1.8.10] we infer that this orbit is open. On the other hand, it is compact, and we conclude that ET acts transitively, since Γ, and therefore also Γ/Ξ, is connected. □

Theorem 2.11. (1) *With respect to the usual action, the group* Γ *has two orbits in the point space* (*and, dually, in the line space*) *of* $\mathrm{P}_2\mathbb{H}$. *The stabilizer of a non-absolute point* (*or line*) *is a conjugate of*

$$\Lambda := \Gamma_{[1,0,0]} = \left\{ \left. \begin{pmatrix} t & \\ & uD \end{pmatrix} \right| t \in \mathbb{T}, u \in \mathbb{U}, D \in \mathrm{SL}_2(\mathbb{R}) \right\}^{R\pi},$$

and the stabilizer of an absolute point (*or line*) *is a conjugate of* Φ. *Moreover, the stabilizer of two points is both a conjugate of a subgroup of* Λ *and a conjugate of a subgroup of* Φ.

(2) *The group* Λ *has two orbits in the pencil of lines through* $[1,0,0]$; *namely, the sets of absolute and non-absolute lines, respectively. Dually, there are two orbits on the line* $[1,0,0]^{\kappa} = \{[w] \,|\, w_1 = 0\}$ *under* Λ.

(3) *The group* Φ *has two orbits on the pencil of lines through* $[1,0,-j]$; *namely, the sets consisting of the absolute line* $[1,0,-j]^{\kappa}$, *and the non-absolute lines, respectively. Dually, there are two orbits on the line* $[1,0,-j]^{\kappa}$ *under* Φ.

Proof. After conjugation by R, the form s is described by the matrix L. Now it is easy to verify that the stabilizer of $[1,0,0] = [1,0,0]^R$ is Λ. For every anisotropic vector x, the length $s(x,x)$ belongs to $\mathrm{Pu}\,\mathbb{H}$. Therefore, there exists a scalar multiple of x of length i. Thus transitivity on the set of non-absolute points follows from Witt's Theorem. The last part of assertion (1) follows from the fact that two absolute points are always joined by a non-absolute line, cf. 2.8. Assertions (2) and (3) follow from Witt's Theorem, since two absolute (or two non-absolute) lines are always isometric. □

Putting $\iota = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{\pi} \in \Gamma$, we obtain that the lines $[1,0,0]^{\kappa\iota}$ $= [1,0,0]^{\iota\kappa} = [0,1,0]^{\kappa}$ and $[1,0,-j]^{\kappa\iota} = [1,0,-j]^{\iota\kappa} = [0,1,-j]^{\kappa}$ are representatives for the Γ-orbits in the set of all lines, and at the same time, for the Λ-orbits in the set of all lines through the point $[1,0,0]$.

We are now in a position to apply the result of [21], where the familiar description of flag-homogeneous incidence structures by means of homogeneous spaces has been generalized. For the reader's convenience, I briefly state this result.

Lemma 2.12. *Let (P,L,I) be an incidence structure, and let G act as a group of automorphisms of (P,L,I); that is, as a group of bijections of $P \cup L$ preserving the relation $I \subseteq P \times L$. Assume that there exists a point $p \in P$ and a set $R \subseteq L$ such that the following hold:*

(1) *The set $\{p\} \times R$ is contained in I.*
(2) *The group G acts transitively on P.*
(3) *The set R forms a set of representatives for the G-orbits on L.*
(4) *The set R also forms a set of representatives for the G_p-orbits on $L_p := \{l \in L \mid (p,l) \in I\}$.*

(5) *For different representatives $r, s \in R$, the stabilizers G_r and G_s are different. Then the incidence structure (P,L,I) is isomorphic to $(G/G_p, \bigcup_{r\in R} G/G_r, J)$, where $J = \{(G_p g, G_r g) \mid g \in G, r \in R\}$. (That is, two cosets are incident iff they have nonempty intersection.)*

If P is locally compact and G is a σ-compact locally compact group acting continuously on P then the Open Mapping Theorem yields that G/G_p is homeomorphic to P.

Corollary 2.13 *The incidence structure that is induced on the (open) orbit $[1,0,0]^{\Gamma}$ is reconstructible from the action of Γ by the method of [21]. That is, this incidence structure is isomorphic to $(\Gamma/\Lambda, \Gamma/(\Lambda') \cup \Gamma/(\Phi'))$; where two cosets are incident if, and only if, they have non-empty intersection.*

Proposition 2.14 *The groups Λ and Φ are maximal subgroups of Γ.*

Proof. According to 2.8, the group Γ acts 2-transitively on the set of absolute points. Thus we have that Φ is a maximal subgroup.

The group Λ centralizes the involution $\sigma = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}^{\pi}$, a reflection with center $[1,0,0]$. We conclude that Λ coincides with the centralizer of σ in Γ. Since σ is the only involution in the center of Λ, we infer that the normalizer of Λ centralizes σ, whence it coincides with Λ. As a consequence, every subgroup X of Γ that properly contains Λ satisfies $\dim \mathrm{X} > \dim \Lambda$. Via the adjoint action, the subgroup $\Sigma = \left\{ \begin{pmatrix} 1 & \\ & h\mathbb{1} \end{pmatrix} \middle| h \in \mathbb{U} \right\}^{R\pi}$ acts therefore effectively on the Lie algebra of X, and this algebra is the sum of the Lie algebra of Λ and a vector subspace of dimension at least 4, cf. 2.7. This implies that $\dim(\Gamma/X) \leqslant 4$. As the group Γ is simple, it acts effectively on Γ/X, and so does the maximal compact subgroup $\mathrm{ET} \cong \mathrm{U}_3\mathbb{C}$, in contradiction to [15, Th. 1]. $\square$

Proposition 2.15. (1) *There are exactly* 3 *conjugacy classes of involutions in* Γ, *respresented by*

$$\tau = I^\pi = \begin{pmatrix} i & & \\ & i & \\ & & i \end{pmatrix}^\pi, \quad \sigma = \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}^\pi, \quad \sigma\tau = \begin{pmatrix} i & & \\ & -i & \\ & & -i \end{pmatrix}^\pi.$$

(2) *The centralizers are the following*:

$$\begin{aligned} &\mathrm{C}_\Gamma(\tau) = \{A \in \mathbb{C}^{3\times 3} \mid AI\bar{A} = I\}^\pi = \mathrm{ET} \cong \mathrm{U}_3(\mathbb{C}, 0), \\ &\mathrm{C}_\Gamma(\sigma) = \Lambda = \mathrm{Z}\Sigma\Psi \cong \mathrm{U}_1^\alpha(\mathbb{H}) \times \mathrm{U}_2^\alpha(\mathbb{H}), \\ &\mathrm{C}_\Gamma(\sigma\tau) = \{A \in \mathbb{C}^{3\times 3} \mid AK\bar{A} = K\}^{Q\pi} = \Upsilon\Theta \cong \mathrm{U}_3(\mathbb{C}, 1), \end{aligned}$$

where

$$\mathrm{E} = \mathrm{SU}_3(\mathbb{C}, 0)^\pi, \quad \mathrm{T} = \{t\mathbb{1} \mid t \in \mathbb{T}\}^\pi,$$

$$\mathrm{Z} = \left\{ \begin{pmatrix} t & & \\ & 1 & \\ & & 1 \end{pmatrix} \middle| \, t \in \mathbb{T} \right\}^\pi,$$

$$\Sigma = \left\{ \begin{pmatrix} 1 & \\ & u\mathbb{1} \end{pmatrix} \middle| \, u \in \mathbb{U} \right\}^{R\pi} = \left\{ \begin{pmatrix} 1 & & \\ & a & b \\ & -\bar{b} & \bar{a} \end{pmatrix} \middle| \, \begin{matrix} a, b \in \mathbb{C} \\ a\bar{a} + b\bar{b} = 1 \end{matrix} \right\}^\pi,$$

$$\Psi = \left\{ \begin{pmatrix} 1 & \\ & D \end{pmatrix} \middle| \, D \in \mathrm{SL}_2(\mathbb{R}) \right\}^{R\pi},$$

$$\Upsilon = \mathrm{SU}_3(\mathbb{C}, 1)^{Q\pi}, \quad \Theta = \mathrm{T}^{Q^\pi}.$$

(3) *We also record the following relation to the stabilizer of an absolute point. Let* $H := \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. *Then*

$$\Sigma^{\bar{R}HP} Z^{HP} = \left\{ \begin{pmatrix} u & & \\ & t & \\ & & u \end{pmatrix} \middle| \, u \in \mathbb{U}, t \in \mathbb{T} \right\}^{P\pi} \leqslant \Phi,$$

in fact, the group $\Sigma^{\bar{R}HP}$ *is a Levi-complement to the solvable radical* $\mathrm{Z}^{HP}\Xi$ *of* Φ, *where* Ξ *is defined as in* 2.10.

(4) *The groups*

$$\Psi\cap \mathrm{ET}=\left\{\begin{pmatrix}1&\\&t\mathbb{1}\end{pmatrix}\middle| t\in\mathbb{T}\right\}^{\pi} \quad \textit{and}$$

$$\Sigma\Psi\cap \mathrm{ET}=\left\{\begin{pmatrix}1&\\&A\end{pmatrix}\middle| A\in \mathrm{U}_2(\mathbb{C},0)\right\}^{\pi}$$

are maximal compact subgroups of Ψ *and* $\Sigma\Psi$, *respectively.*

Proof. The decompositions $\mathrm{U}_3(\mathbb{C},0)^{\pi}=\mathrm{ET}$, $\Lambda=\mathrm{Z}\Sigma\Psi$, and $\mathrm{U}_3(\mathbb{C},1)^{Q\pi}=\Upsilon\Theta$ are verified by easy computations. Obviously, the involutions τ, σ, and $\sigma\tau$ belong to Γ, and the groups ET, Λ, and $\Upsilon\Theta$ are contained in the respective centralizer. The centralizer of τ in $\mathrm{PGL}_3\mathbb{H}$ is $\Pi:=\{j^{\varepsilon}A\,|\,A\in\mathbb{C}^{3\times 3},\varepsilon\in\{0,1\}\}^{\pi}$. We infer that $\mathrm{C}_{\Gamma}(\tau)\leqslant\Pi\cap\Gamma=\mathrm{ET}$. Since $\sigma\tau=K^{\pi}=I^{Q\pi}=\tau^{Q\pi}$, we have that $\mathrm{C}_{\Gamma}(\sigma\tau)\leqslant \leqslant\Pi^{Q^{\pi}}\cap\Gamma=\Upsilon\Theta$. According to 2.14, the group Λ is a maximal subgroup of Γ, and $\Lambda=\mathrm{C}_{\Gamma}(\sigma)$. This completes the proof of assertion (2).

Regarding assertion (3), we observe that $\Sigma^{\bar{R}HP}$ has trivial intersection with $\mathrm{Z}^{HP}\Xi$. It is easy to see that $\mathrm{Z}^{HP}\Xi$ is a solvable closed normal subgroup of Φ, and that $\Phi=\Sigma^{\bar{R}HP}\mathrm{Z}^{HP}\Xi$. As $\Sigma^{\bar{R}HP}$ is almost simple, we conclude that $\mathrm{Z}^{HP}\Xi$ is the radical of Φ, and that $\Sigma^{\bar{R}HP}$ is a Levi-complement.

The equations in (4) are obtained by easy computations. It is well known that $\Psi\cap\mathrm{ET}\cong\mathrm{SO}_2(\mathbb{R},0)$ is maximal compact in $\Psi\cong\mathrm{SL}_2\mathbb{R}$. This yields that $\Sigma\Psi\cap\mathrm{ET}=\Sigma(\Psi\cap\mathrm{ET})$ is maximal compact in $\Sigma\Psi$.

It remains to prove assertion (1). Since they have non-isomorphic centralizers, the involutions σ,τ, and $\sigma\tau$ belong to different conjugacy classes. The subgroup ET is isomorphic to $\mathrm{U}_3(\mathbb{C},0)$ by 2.6. In this group, there are exactly three conjugacy classes of involutions; namely, those represented by the diagonal matrices

$$\begin{pmatrix}-1&&\\&-1&\\&&-1\end{pmatrix},\begin{pmatrix}1&&\\&-1&\\&&-1\end{pmatrix},\text{ and }\begin{pmatrix}1&&\\&1&\\&&-1\end{pmatrix}.$$

Finally, every involution in Γ has a conjugate in ET, since the latter is a maximal compact subgroup of Γ by 2.10. $\square$

3. Actions of the Skew-Hyperbolic Motion Group

Definition 3.1. A *stable plane* is a linear space $\mathbb{M}=(M,\mathcal{M})$, where the point space M and the line space $\mathcal{M}$ are endowed with locally compact topologies such that

- The mappings $\vee$ (joining points) and $\wedge$ (intersecting lines) are continuous.
- The set $\mathcal{D}_{\wedge}$ of pairs of intersecting lines is open in $\mathcal{M}\times\mathcal{M}$ (axiom of stability).
- The point space M has positive and finite (topological) dimension.

If this is convenient, we will tacitly identify a line of a stable plane with the set of points that are incident with it. The *pencil* of all lines that are incident with a point p will be denoted by $\mathcal{M}_p$.

General information about stable planes can be found in the work of R. LÖWEN; in particular, see [8] and [11]. Most important is the deep result [11, Th. 1] that $\dim M = \dim \mathcal{M} \in \{2, 4, 8, 16\}$. I will also need the fact that $\dim M \in \{2, 4\}$ implies that M is a manifold, and that $\mathcal{M}_p$ is homeomorphic to a sphere of dimension $\frac{1}{2} \dim M$, see [8]. For recent developments of the theory, compare also [28].

Endowed with the compact-open topology derived from the action on M (or on $\mathcal{M}$), the group $\mathrm{Aut}(\mathbb{M})$ of all continuous collineations of $\mathbb{M}$ is a locally compact transformation group both on M and on $\mathcal{M}$. An *action* of a topological group G on $\mathbb{M}$ is a continuous group homomorphism from G to $\mathrm{Aut}(\mathbb{M})$. Let $\mathbb{A} = (A, \mathcal{A})$ and $\mathbb{B} = (B, \mathcal{B})$ be stable planes, and assume that $\alpha: G \to \mathrm{Aut}(\mathbb{A})$ and $\beta: H \to \mathrm{Aut}(\mathbb{B})$ are actions of topological groups G and H. If $\gamma: G \to H$ is a continuous homomorphism of groups, and $\lambda: A \to B$ is a continuous mapping that preserves collinearity (a so-called *lineation*), such that $g^\alpha \lambda = \lambda g^\beta$ for every $g \in G$, then we say that (γ, λ) is a *morphism of actions*. If both γ and λ are injections, we call (γ, λ) an *embedding of actions*. See [28, Chapter 3] and [23] for a discussion of these concepts.

The study of involutions in $\mathrm{Aut}(\mathbb{M})$ plays a crucial role. This is due to the fact that the possible actions are well understood. Every involution $\tau \in \mathrm{Aut}(\mathbb{M})$ has a set $\mathcal{M}_\tau = \{x^\tau x \mid x \in M, x \neq x^\tau\}$ of fixed lines. For each point $x \neq x^\tau$ we find that a suitable neighborhood of $x^\tau x$ in $\mathcal{M}_\tau$ is homeomorphic to some neighborhood of x in any line through x different from $x^\tau x$, compare [9, 1.1]. We will frequently use the following results from [25].

Lemma 3.2. (1) *If α is an involution of a stable plane $\mathbb{M} = (M, \mathcal{M})$ (that is, a non-trivial automorphism such that $\alpha^2 = \mathrm{id}_{\mathbb{M}}$), then one of the following cases occurs:*

– *The involution α fixes no point; such an involution is called free.*

– *The involution α is planar; i.e., the fixed points and lines form a subplane $\mathbb{B} = (B, \mathcal{B})$ such that $\dim M = 2 \dim B$. In this case, the space B is locally homeomorphic to any line (or line pencil) in $\mathbb{M}$.*

– *The involution α has a center c or an axis A; i.e., there exists a point c such that α acts trivially on $\mathcal{M}_c$, or a line A such that α acts trivially on A. If α has both center and axis, then α is called a reflection.*

(2) *If α and β are commuting involutions with the same axis, then $\alpha = \beta$.*

(3) *If three commuting involutions have a common fixed point p, and if none of the involutions is planar, then p is the center of at least one of the involutions.*

(4) *Axis and center of a reflection are never incident.*

(5) *If an involution α with center c fixes a line L then L passes through c, or L is an axis of α.*

(6) *If an involution α with center c fixes a point $a \neq c$ then α has also an axis A, and $a \in A$.*

(7) *If an involution α with axis A fixes a point c outside A then c is a center of α.*

It is easy to see that the center or axis of a non-trivial automorphism α are uniquely determined, and therefore fixed by the centralizer of α in $\mathrm{Aut}(\mathbb{M})$.

In the sequel, we study actions of the skew hyperbolic motion group $\Gamma = \mathrm{PU}_3^\alpha(\mathbb{H})$ on 8-dimensional stable planes. We will use the information that we have collected in the previous section.

Lemma 3.3. *Assume that* Γ *acts non-trivially on a stable plane* $\mathbb{M} = (M, \mathcal{M})$ *such that* $\dim M = 8$. *Then every point stabilizer contains a subgroup which is isomorphic to* $\mathbb{T}^2$. *Moreover, we have the following.*

(1) *The involution* τ *is planar, its centralizer* ET *induces the elliptic motion group* $\mathrm{E}/\langle\tau\rangle \cong \mathrm{PSU}_3(\mathbb{C}, 0)$ *on the subplane* $\mathbb{E} = (E, \mathcal{E})$ *of fixed elements, and* $\mathbb{E}$ *is isomorphic to* $\mathrm{P}_2\mathbb{C}$.

(2) *The involution* $\sigma\tau$ *is planar, its centralizer* $\Upsilon\Theta$ *induces the hyperbolic motion group* $\Upsilon/\langle\sigma\tau\rangle \cong \mathrm{PSU}_3(\mathbb{C}, 1)$ *on the subplane* $\mathbb{F} = (F, \mathcal{F})$ *of fixed elements, and* $\mathbb{F}$ *is isomorphic to some* $\Upsilon/\langle\sigma\tau\rangle$*-invariant subplane of* $\mathrm{P}_2\mathbb{C}$.

(3) *The involution* σ *is a reflection, and every point in* M *lies on the axis of some conjugate of* σ.

Proof. (i) According to [26, 2.10], there is no 8-dimensional orbit of E in M, whence $\dim \mathrm{E}_p > 0$ for every point $p \in M$. Since every involution in $\mathrm{E} \cong \mathrm{SU}_3(\mathbb{C}, 0)$ is a conjugate of σ, we infer that every point p is fixed by some conjugate of σ. The centralizer Λ of σ contains a semi-simple group which is not almost simple. From [22, 6.5] we infer that σ is not planar. Therefore, every point either is the center or belongs to the axis of a conjugate of σ.

(ii) If the involution τ fixes a point x, then it is planar, since its centralizer acts neither on a line pencil nor on a line of $\mathbb{M}$, see [26, 2.10]. According to [14], the subplane $\mathbb{E}$ of fixed elements of τ is isomorphic to $\mathrm{P}_2\mathbb{C}$, and $\mathrm{E} \cong \mathrm{SU}_3(\mathbb{C}, 0)$ induces the usual action. In particular, the involution σ has both a center and an axis in $\mathbb{E}$, and therefore is a reflection of $\mathbb{M}$. Now σ and $\sigma\tau$ induce the same reflection on $\mathbb{E}$. Since commuting reflections with the same axis are equal by 3.2(2), the involution $\sigma\tau$ is planar.

In its usual action on $\mathrm{P}_2\mathbb{C}$, the group E has discrete centralizer in the automorphism group of $\mathrm{P}_2\mathbb{C}$. This yields that T acts trivially on $\mathbb{E}$, and that every point of $\mathbb{E}$ is fixed by a subgroup of ET which is a conjugate of ΣZT.

We have thus proved that the assertions (1)–(3) hold if τ fixes a point. In the sequel, we are going to reduce all cases to step (ii).

(iii) I claim that $\sigma\tau$ has no center. Indeed, a center of $\sigma\tau$ is fixed by τ, and we infer that $\sigma\tau$ and σ induce the same reflection on $\mathbb{E}$, compare step (ii). Since σ is not planar, this implies that $\sigma\tau$ and σ are commuting reflections with the same axis and center, in contradiction to 3.2(2).

(iv) If the involution σ has a center c, then τ fixes c, and from (ii) we know that σ is a reflection, and that $\sigma\tau$ is planar.

(v) If $\sigma\tau$ is planar, then Υ induces on the subplane $\mathbb{F}$ of fixed elements the complex hyperbolic motion group $\Upsilon/\langle\sigma\tau\rangle \cong \mathrm{PSU}_3(\mathbb{C}, 1)$, and we infer from [14] that $\mathbb{F}$ is isomorphic to some open subplane of $\mathrm{P}_2\mathbb{C}$, and that Υ acts as usual on $\mathbb{F}$. The involutions σ and $\sigma' := \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}^{\pi}$ represent the two conjugacy classes of reflections in the complex hyperbolic motion group $\Upsilon/\langle\sigma\tau\rangle$; namely, those with interior and exterior center, respectively. We infer that at least one of these involutions has a center in $\mathbb{F}$. Since σ' and σ are conjugates in Γ, we obtain that σ has a center in $\mathbb{M}$, and step (ii) applies.

The next step establishes the existence of a subgroup isomorphic to $\mathbb{T}^2$ in each stabilizer. Afterwards, we complete the proof of assertions (1)–(3).

(vi) The maximal compact subgroup ET of Γ has dimension 9. According to [26, 2.10], there is no 8-dimensional orbit of ET in M. Therefore, every stabilizer in ET has dimension at least 2. Aiming for a contradiction, we assume that Γ_x does not contain a subgroup isomorphic to $\mathbb{T}^2$. As the connected group Γ leaves the connected components of M invariant, we may also assume that M is connected.

Every maximal compact subgroup X of the connected component of Γ_x is a compact connected Lie group of rank at most 1 and dimension at least 2, and therefore locally isomorphic to $\mathrm{SU}_2(\mathbb{C}, 0)$. Since ET is a maximal compact subgroup of Γ, we may assume that $\mathrm{X} < \mathrm{ET}$. Since X equals its commutator subgroup, we have that $\mathrm{X} \leqslant \mathrm{E}_x$. As step (ii) would imply the existence of a subgroup isomorphic to $\mathbb{T}^2$ in Γ_x, we have that the point x is moved by τ, and E_x is contained in the stabilizer of the line $x^\tau x$. According to step (i), the involution σ has an axis. From step (iv) we infer that σ has no center.

We consider the action of E on $\mathcal{M}_\tau$. This space is connected (since we assumed that M is connected) and locally homeomorphic to a line. According to [26, 2.14], the group E does not act trivially on $\mathcal{M}_\tau$. Applying [15, Th. 2] to a nontrivial E-orbit in $\mathcal{M}_\tau$, one sees that the action of E on this orbit is equivalent to the usual action on $\mathrm{P}_2\mathbb{C}$. In particular, the orbit has dimension 4 and is therefore open in $\mathcal{M}_\tau$. As the orbit is also compact and $\mathcal{M}_\tau$ is connected, this implies that E acts transitively on $\mathcal{M}_\tau$, and that T acts trivially on $\mathcal{M}_\tau$. In particular, we obtain that the stabilizer $(\mathrm{ET})_{x^\tau x} = \mathrm{E}_{x^\tau x}\mathrm{T}$ is a conjugate of $\mathrm{TZ}\Sigma$, and we may assume that $\mathrm{X} = \Sigma$.

Up to isomorphism, the only linear semi-simple Lie groups that contain $\mathrm{SU}_2(\mathbb{C}, 0)$, but no subgroup isomorphic to $\mathbb{T}^2$ are $\mathrm{SU}_2(\mathbb{C}, 0)$ and $\mathrm{SL}_2(\mathbb{C})$, see [31] or [16]. Since $\mathrm{SL}_2(\mathbb{C})$ is not contained in Γ, we conclude that Γ_x^1 is the product of Σ and some solvable, compact-free normal subgroup Ω such that $\dim \Omega \geqslant 4$. From Lie's Theorem (see [6, II.2.11] or [16, Ch. 1, §4, Th. 5]) it follows that the group Ω fixes some point of $\mathrm{P}_2\mathbb{H}$ in the usual action. Since Λ does not contain a compact-free subgroup of dimension greater than 3, we conclude that Ω fixes exactly one point of $\mathrm{P}_2\mathbb{H}$, and that this point is absolute with respect to κ, see 2.11. Being the unique fixed point of a normal subgroup, this point is also fixed by Γ_x, and we may assume that $\Gamma_x \leqslant \Phi$; note that every subgroup of Φ that is isomorphic to $\mathrm{SU}_2(\mathbb{C}, 0)$ is a conjugate of $\Sigma^{\bar{R}HP}$ in Φ. Since Ω is not contained in the centralizer of $\sigma^{\bar{R}HP}$, we have that $\Sigma^{\bar{R}HP}$ acts effectively on the Lie algebra of Ω via the adjoint action. Under the action of $\Sigma^{\bar{R}HP}$, the Lie algebra of Φ splits into the sum of the Lie algebras of

$$\Sigma^{PR}, \quad \left\{ \begin{pmatrix} r & & \\ & 1 & \\ & & \frac{1}{r} \end{pmatrix} \middle| r > 0 \right\}^{P\pi}, \quad \left\{ \begin{pmatrix} 1 & & \\ & t & \\ & & 1 \end{pmatrix} \middle| t \in \mathbb{T} \right\}^{P\pi},$$

$$\text{and } \Delta = \left\{ \begin{pmatrix} 1 & & \\ h & 1 & \\ x + \frac{\bar{h}ih}{2} & \bar{h}i & 1 \end{pmatrix} \middle| h \in \mathbb{H}, x \in \mathbb{R} \right\}^{P\pi}.$$

Of these components, only the Lie algebra of Δ splits further into the Lie algebra of the commutator group

$$\Delta' = \left\{ \begin{pmatrix} 1 & & \\ & 1 & \\ x & & 1 \end{pmatrix} \middle| x \in \mathbb{R} \right\}^{P\pi}$$

and its Σ^{PR}-invariant complement, the tangent space of

$$D = \left\{ \begin{pmatrix} 1 & & \\ h & 1 & \\ \frac{\bar{h}ih}{2} & \bar{h}i & 1 \end{pmatrix} \middle| h \in \mathbb{H} \right\}^{P\pi}.$$

Note that D is not a group. The centralizer of $\sigma^{\bar{R}HP}$ in Φ is

$$C_\Phi(\sigma^{\bar{R}HP}) = \left\{ \begin{pmatrix} ur & & \\ 0 & t & \\ x & 0 & \frac{u}{r} \end{pmatrix} \middle| u \in \mathbb{U}, r > 0, t \in \mathbb{T}, x \in \mathbb{R} \right\}^{P\pi},$$

and D is the eigenspace of -1 under the adjoint action of $\sigma^{\bar{R}HP}$ on the Lie algebra of Φ. Thus our observation that $\Sigma^{\bar{R}HP}$ acts effectively on the Lie algebra of Ω implies that D, and therefore $\Delta = \langle D \rangle$, is contained in Ω.

We consider the action of Γ_x on the orbit of x under Φ. Being a normal subgroup of Φ, the group Δ acts trivially on this orbit. Since $\dim \Delta = 5$, we infer from [22, 7.3] that x^Φ is contained in a line L. Since σ has no center, we know from step (i) that L is the axis of σ, and we conclude that $\Gamma = \langle \Phi \cup \Lambda \rangle$ fixes L. This contradicts the fact that E cannot fix a line of $\mathbb{M}$, see [26, 2.10].

We have proved that Γ_x contains a subgroup that is isomorphic to $\mathbb{T}^2$. We now turn to a study of the possible triplets of commuting involutions in such a subgroup.

(vii) If three commuting conjugates of σ fix a point x, then x is the center of one of these involutions by step (i) and 3.2(3), and step (ii) applies.

(viii) Assume that Γ_x contains three commuting conjugates of $\sigma\tau$. If $\sigma\tau$ had an axis, this would imply that $\sigma\tau$ has a center, as in step (vii). This is impossible by step (iii). Therefore, the involution $\sigma\tau$ is planar, and step (v) applies, leading to (ii).

(ix) There remains the case that Γ_x contains σ, α, and $\sigma\alpha$, where α is a conjugate of $\sigma\tau$ that commutes with σ. Then the involution α has no center by step (iii). Conjugates of τ cannot have a center by step (ii). If σ, α, and $\sigma\alpha$ are all axial, we obtain that one of them has center x by 3.2(3). If $\sigma\alpha$ has a center, we therefore know that $\sigma\alpha$ is a conjugate of σ. Thus σ has a center as well, and step (ii) applies. If α or $\sigma\alpha$ is planar, we obtain that $\sigma\tau$ or τ is planar, since no conjugate of σ is planar by step (i). Using step (v) in the first case, we arrive at (ii).

(x) We have proved that σ is a reflection, and that τ and $\sigma\tau$ are planar. Moreover, every point p of $\mathbb{M}$ either is the center or belongs to the axis of some conjugate of σ. The center c of σ belongs to the axis of σ', cf. step (v). □

The existence of planar involutions implies the following, see [22, 3.7, 3.8].

Corollary 3.4. *If* Γ *acts non-trivially on a stable plane* $\mathbb{M}=(M,\mathscr{M})$ *such that* $\dim M=8$, *then each line pencil is homeomorphic to the sphere* $\mathbb{S}_4$.

Theorem 3.5. *Assume that* Γ *acts non-trivially on a stable plane* $\mathbb{M}=(M,\mathscr{M})$ *such that* $\dim M=8$. *Then the open subplane* $\mathbb{D}=(D,\mathscr{D})$ *induced on the orbit* D *of the center of* σ *is isomorphic to the open subplane of* $\mathrm{P}_2\mathbb{H}$ *which is induced on the set of non-absolute points. The action of* Γ *on* $\mathbb{D}$ *is equivalent to the usual one.*

Proof. (i) According to 3.3(2), the involution σ has a center c. The stabilizer Γ_c contains the centralizer Λ of σ. Since Λ is a maximal subgroup, we obtain that either $\Gamma_c=\Lambda$ or $\Gamma_c=\Gamma$. In the latter case, the simple group Γ would act trivially on $\mathscr{M}_c$, since $\mathbb{T}^3$ does not act effectively on $\mathbb{S}_4$ by [19, 3.4]. But a trivial action of Γ on a line pencil is impossible by [25, 11b)]. Thus $\dim c^\Gamma=\dim(\Gamma/\Lambda)=8$, and c^Γ is open in M by [11, Th. 11]. We denote the subplane induced on $D:=c^\Gamma$ by $\mathbb{D}$.

(ii) We recall from 2.15 that $\Lambda=\mathrm{Z}\Sigma\Psi$. The group Σ centralizes $\tau=L^{R\pi}$, and Ψ centralizes $I^{R\pi}=\begin{pmatrix} i & & \\ & i & \\ & & -i\end{pmatrix}^{\pi}\in(\sigma\tau)^\Gamma$. Both involutions are planar by 3.3, and Σ resp. Ψ acts effectively on the subplane $\mathbb{E}$ resp. $\mathbb{F}'$ of fixed elements. Since neither Σ nor Ψ acts trivially on the pencil $\mathscr{E}_c$ resp. $\mathscr{F}'_c$ by [25, 10], we obtain that $\Sigma\Psi$ acts almost effectively on $\mathscr{M}_c$. Applying [19] to a maximal torus (for instance, to $\mathrm{T}\Theta\mathrm{Z}\cong\mathbb{T}^3$) in Λ, we infer that Λ cannot act almost effectively on $\mathscr{M}_c$. This yields that $\Lambda_{[c]}=\mathrm{Z}$. From 2.15(4) we know that

$$\Sigma\Psi\cap\mathrm{ET}=\left\{\begin{pmatrix}1 & \\ & A\end{pmatrix}\middle|\,A\in\mathrm{U}_2(\mathbb{C},0)\right\}^{\pi}$$

is a maximal compact subgroup of $\Sigma\Psi$. Thus the maximal compact subgroups of $\Lambda/\mathrm{Z}\cong(\Sigma\Psi)/\langle\sigma\rangle$ are isomorphic to $\mathrm{SO}_3(\mathbb{R},0)\times\mathrm{SO}_2(\mathbb{R},0)$. Applying [19] to the effective action of such a compact subgroup, we obtain that it acts in the usual way (as a subgroup of $\mathrm{SO}_5(\mathbb{R},0)$) on $\mathscr{M}_c\approx\mathbb{S}_4$. In particular, the group Σ has a subset $\mathscr{L}\approx\mathbb{S}_1$ of fixed lines. This set is of course invariant under $\Lambda=\Sigma\Psi\mathrm{Z}$, and Ψ acts transitively on $\mathscr{L}$. We infer that there exists a line $S\in\mathscr{L}$ such that $\Lambda_S=(\mathrm{Z}\Sigma)\Psi_S$, and

$$\Psi_S=\left\{\begin{pmatrix}1 & & \\ & u & \\ & v & u^{-1}\end{pmatrix}\middle|\,0\neq u\in\mathbb{R},v\in\mathbb{R}\right\}^{R\pi}.$$

As $\Sigma\Psi\cap\mathrm{ET}$ acts as usual on $\mathscr{M}_c$, it is easy to see that for each $T\in M_c\setminus\mathscr{L}$ the stabilizer Σ_T is a conjugate of $\left\{\begin{pmatrix}1 & & \\ & s & \\ & & s\end{pmatrix}\middle|\,s\in\mathbb{T}\right\}^{\pi}$ in Σ. Choose T such that Σ_T coincides with this group. Then $(\Sigma\mathrm{Z})_T=\Sigma_T\mathrm{Z}=\left\{\begin{pmatrix}t & & \\ & s & \\ & & s\end{pmatrix}\middle|\,t,s\in\mathbb{T}\right\}^{\pi}$. In particular, the line T is fixed by τ. According to 3.3(1) the subplane $\mathbb{E}=(E,\mathscr{E})$ of fixed elements of

T is isomorphic to $P_2\mathbb{C}$, and E induces the usual action. This means that there is an involution in $E_c = E \cap \Lambda$ that fixes each point of $T \cap E$. Thus T is fixed by the centralizer of that involution of E, and therefore by a maximal torus in E. This maximal torus contains three commuting reflections. We infer that there is a reflection $\alpha \in \Gamma$ with axis T, and $\Lambda_T = C_\Lambda(\alpha)$ (by maximality). As α fixes exactly two lines through c and as only one of them is the axis, we infer that axis and stabilizer determine each other. Since the candidates for such reflections form a single conjugacy class in Λ, we conclude that Λ acts transitively on $\mathcal{M}_c \setminus \mathcal{L}$. The stabilizer Γ_T contains $C_\Gamma(\alpha)$, which is a conjugate of Λ. As Λ is a maximal subgroup of G, we have $\Gamma_T = C_\Gamma(\alpha)$. Since no element of $\mathcal{L}$ is the axis of a reflection, the group Γ has exactly two orbits on $\mathcal{M}$, represented by S and T.

(iii) Applying the results of step (ii) to the incidence geometry $(\{c\}, \mathcal{M}_c)$ we obtain that the action of $\Lambda = \Gamma_c$ on the pencil $\mathcal{M}_c$ is always equivalent to the usual one, cf. 2.12. Moreover, we know already that the stabilizer Γ_T of every line $T \in \mathcal{M}_c \setminus \mathcal{L}$ is the centralizer of the involution in Λ that has axis T. In order to apply the reconstruction method 2.12, it remains to determine the stabilizer Γ_S for $S \in \mathcal{L}$. Since $\dim \Lambda_S < \dim \Gamma - 8$, we have that $\dim \Gamma_S > \dim \Lambda_S$. In particular, the involution σ, and (a fortiori) the group Σ do not centralize Γ_S. Therefore, the adjoint action of $\Sigma \cong SU_2(\mathbb{C}, 0)$ on the Lie algebra of Γ_S is effective, and splits this Lie algebra into the Lie algebra of Λ_S and some vector space complement of dimension at least 4, cf. 2.7. From $\Gamma_{S,c} = \Lambda_S$ we infer that $\dim \Gamma_S = \dim \Lambda_S + 4 = 10$.

The semi-simple Lie groups of dimension 10 are known, see [31] or [16]: such a group is always almost simple, and locally isomorphic to $SO_5(\mathbb{R}, r)$ for some $r \in \{0, 1, 2\}$. None of these groups contains a subgroup that is isomorphic to Λ_S. Therefore, the connected component of Γ_S is not semi-simple, but contains some minimal abelian closed connected normal subgroup Ω, see [27, 7.3, 7.4].

If σ does not centralize Ω, then Σ acts effectively on Ω via conjugation. In this case, the group Ω is isomorphic to $\mathbb{R}^n$ for $n \geqslant 4$, see [27, 7.4] and 2.7. Combining Lie's Theorem (see [6, II.2.11] or [16, Ch. 1, §4, Th. 5]) and 2.11, we infer that Ω fixes exactly one (absolute) point in the usual action on $P_2\mathbb{H}$. This means that the group $\Omega\Sigma$ is contained in a conjugate Φ^γ of Φ. Now $\Sigma^{\gamma^{-1}}$ is a Levi complement of Φ, and thus a conjugate of $\Sigma^{\bar{R}HP}$ in Φ by 2.15(3). Considering the $\Sigma^{\bar{R}HP}$-invariant decomposition of the Lie algebra of Φ as in step (vi) of the proof of 3.3, we arrive at the contradiction that Φ has no subgroup isomorphic to $\Omega\Sigma$.

There remains the case that $\Omega \leqslant C_{\Gamma_S}(\sigma) = \Lambda_S$. Being a normal subgroup, the group Ω acts trivially on the orbit c^{Γ_S}. In particular, we have that $\Omega \cap Z$ is trivial; recall from step (ii) that Z has the same center and axis as σ. Since Ω is contained in the solvable radical $\Psi_S Z$ of Λ_S and $\Psi_S Z$ is not abelian, we obtain that $\dim \Omega = 1$. Since Z is the unique maximal compact subgroup of $\Psi_S Z$, we infer that $\Omega \cong \mathbb{R}$. If $\Omega \cap \Psi_S$ were trivial, we would obtain that $\Psi_S Z \cong \Omega \times \Psi_S$. This contradicts the fact that $\Psi_s Z$ is not homeomorphic to $\mathbb{R}^3$. Thus $\Omega \cap \Psi_S$ is a non-trivial closed normal subgroup of Ψ_S. There exists only one proper non-trivial closed normal subgroup of Ψ_S, namely, the commutator subgroup $\Psi'_S = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & v & 1 \end{pmatrix} \middle| v \in \mathbb{R} \right\}^{R\pi}$, and we conclude that $\Omega = \Psi'_S$. In the usual action on $P_2\mathbb{H}$, the group Ψ'_S is just the group of all translations in Γ with axis K^R, where $K = \{[v] \mid v_3 = 0\}$. Therefore, the normalizer of Ψ'_S in Γ is the stabilizer Γ_{K^R}, viz., a conjugate of Φ. Since Φ is a connected maximal subgroup of Γ and $\dim \Phi = 10 = \dim \Gamma_S$, this means that $\Gamma_S = \Gamma_{K^R}$. Now the geometry $\mathbb{D}$ can be reconstructed by the method of [21]. □

Theorem 3.6. *Every non-trivial action of* Γ *on an* 8*-dimensional stable plane* $\mathbb{M}$ *is embeddable into the usual action of* Γ *on* $P_2\mathbb{H}$. *If* $\mathbb{M}$ *is not isomorphic to* $P_2\mathbb{H}$, *then* $\mathbb{M}$ *is the subplane induced on the open* Γ*-orbit.*

Proof. From 3.5, we know the subplane $\mathbb{D} = (D, \mathscr{D})$ that is induced on the open orbit c^Γ, where c is the center of σ.

(i) The set $\mathscr{D}$ is open in $\mathscr{M}$, since it consists of all lines that meet the open set D. On the other hand, the space $\mathscr{D}$ is compact, since no line of $P_2\mathbb{H}$ is entirely contained in the unital U. Since $\mathscr{M}$ is connected [28, 1.4d], we obtain that $\mathscr{D} = \mathscr{M}$.

(ii) If there exists a point $p \in M \setminus D$, then we may assume that p belongs to the axis A of σ, see 3.3(3). According to step (ii) of the proof of 3.5, the centralizer Λ of σ acts with two orbits on the pencil $\mathscr{M}_c$. As D is even invariant under the action of Γ, we infer that Λ has at least two orbits on A. This implies that every line through c meets the axis A, and A is homeomorphic to $\mathscr{M}_c$ via projection. Every compact line in a stable plane is a projective line (i.e., it meets every other line), see [8, 1.15]. Let L be the line that joins p and c. If a is a point in $A \cap D$, then investigation of $\mathbb{D}$ shows that L intersects every element of $\mathscr{M}_a \setminus \{A\}$ in a point of D. Since p is the intersection of A and L, we obtain that L is a projective line as well. Since L represents the Γ-orbit $\mathscr{M} \setminus A^\Gamma$, this implies that $\mathbb{M}$ is a projective plane. Now [10, §3] says that the action of Γ on $\mathbb{M}$ embeds into the usual action of Γ on $P_2\mathbb{H}$, and of course $\mathbb{M} \cong P_2\mathbb{H}$. □

Theorem 3.7. *Assume that* Δ *is a locally compact, connected, almost simple group such that the center factor group is isomorphic to* Γ. *If* Δ *acts non-trivially on a stable plane* $\mathbb{M} = (M, \mathscr{M})$ *such that* $\dim M = 8$, *then the center acts trivially (and the effective action embeds into the usual action of* Γ *on* $P_2\mathbb{H}$*).*

Proof. The group Δ contains a covering group of E. Since E is simply connected, this covering is trivial, and step (i) of the proof of 3.3 yields that every point of $\mathbb{M}$ either is the center or belongs to the axis of a conjugate of σ. The stabilizer of the center or of the axis of σ is the centralizer of σ in Δ, which has dimension 7. Therefore, the center of Δ acts trivially on either an open set of points (the orbit of the center of σ) or an open set of lines (the orbit of the axis of σ). Thus the center acts trivially on $\mathbb{M}$, and the assertion follows from 3.6. □

4. The Symplectic Group

In several respects, the structure of $PSp_6\mathbb{R}$ is similar to that of the skew hyperbolic motion group. Therefore, we will use the acquaintance we have got by now in order to prove that no group with center factor group $PSp_6\mathbb{R}$ acts non-trivially on any 8-dimensional stable plane. This contributes to the project of determining all actions of almost simple groups on stable planes [30], compare [28, Section 9] and [20, Kap. 9]. In [29] the results of the present paper are used in order to determine "almost all" actions of almost simple groups on 8-dimensional projective planes.

Notation 4.1. We set

$$I=\begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & & & & 0 & 1 \\ & & & & -1 & 0 \end{pmatrix}, \quad J=\begin{pmatrix} 0 & 1 & & & & \\ -1 & 0 & & & & \\ & & 0 & 1 & & \\ & & -1 & 0 & & \\ & & & & 0 & -1 \\ & & & & 1 & 0 \end{pmatrix},$$

$$Q=\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & -1 \end{pmatrix},$$

and define $\mathrm{Sp}_6\mathbb{R}=\{A\in\mathbb{R}^{6\times 6} \mid AIA'=I\}$. The image of $\mathrm{Sp}_6\mathbb{R}$ under the natural epimorphism $\pi\colon A\mapsto\{A,-A\}$ is $\Delta:=\mathrm{PSp}_6\mathbb{R}$. Note that $J=I^Q$.

Lemma 4.2. (1) *The conjugacy classes of involutions in* $\mathrm{PSp}_6\mathbb{R}$ *are represented by*

$$\tau=I^\pi,\quad \sigma=\begin{pmatrix} 1 & & & & & \\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & -1 & \\ & & & & & -1 \end{pmatrix}^\pi, \textit{ and } \sigma\tau=J^\pi.$$

(2) *The centralizer of* τ *is the maximal compact subgroup*

$$\mathrm{C}_\Delta(\tau)=\{A \mid AIA'=I, AI=IA\}^\pi.$$

Putting $\mathrm{E}=\mathrm{C}_\Delta(\tau)'\cong\mathrm{SU}_3(\mathbb{C},0)$ *and* $\mathrm{T}=\{c\mathbb{1}+sI \mid c^2+s^2=1\}^\pi\cong\mathbb{T}$, *we have that* $\mathrm{C}_\Delta(\tau)=\mathrm{ET}\cong\mathrm{U}_3(\mathbb{C},0)$, *and* $\sigma\in\mathrm{E}$.

(3) *The centralizer of* σ *is the maximal subgroup*

$$\mathrm{C}_\Delta(\sigma)=\left\{\begin{pmatrix} C & \\ & D\end{pmatrix} \middle| C\in\mathrm{Sp}_4\mathbb{R}, D\in\mathrm{Sp}_2\mathbb{R}\right\}^\pi.$$

Putting

$$\Phi=\left\{\begin{pmatrix} \mathbb{1} & \\ & D\end{pmatrix} \middle| D\in\mathrm{Sp}_2\mathbb{R}\right\}^\pi \textit{ and } \Psi=\left\{\begin{pmatrix} C & \\ & \mathbb{1}\end{pmatrix} \middle| C\in\mathrm{Sp}_4\mathbb{R}\right\}^\pi,$$

we have that $\mathrm{C}_\Delta(\sigma)=\Phi\Psi$.

(4) *The centralizer of* $\sigma\tau$ *is the group*

$$\mathrm{C}_\Delta(\sigma\tau) = \{A \mid AIA' = I, AJ = JA\}^\pi = \{B \mid BJB' = J'BI = IB\}^{Q\pi}.$$

Putting $\Upsilon = \mathrm{C}_\Delta(\sigma\tau)' \cong \mathrm{SU}_3(\mathbb{C},1)$ *and* $\Theta = \mathrm{T}^{Q^\pi}$, *we have that* $\mathrm{C}_\Delta(\sigma\tau) = \Upsilon\Theta \cong \mathrm{U}_3(\mathbb{C},1)$.

Proof. The centralizers are easily computed, using the representation of $\mathbb{C}$ by similitudes of $\mathbb{R}^2$. In particular, we have that $I = i\mathbb{1}$ and $J = \begin{pmatrix} i & & \\ & i & \\ & & -i \end{pmatrix}$ in $\mathbb{C}^{3\times 3}$.

Since ET is a maximal compact subgroup of $\Delta = \mathrm{PSp}_6\mathbb{R}$, cf. [31] or [16], we have that every involution in Δ is a conjugate of some involution in ET. Obviously, the involutions τ, σ, and $\sigma\tau$ represent the conjugacy classes of involutions in ET. Since the centralizers are not isomorphic, no fusion of conjugacy classes takes place in Δ. By arguments that are analogous to the proof of 2.14, one obtains that the centralizer of σ is a maximal subgroup of Δ. □

Lemma 4.3. *If a subgroup of* Δ *is isomorphic to* $\mathrm{SU}_2(\mathbb{C},0)$, *then it is a conjugate of*

$$\Sigma := \left\{ \begin{pmatrix} A & B & \\ -B' & A' & \\ & & \mathbb{1} \end{pmatrix} \middle| \begin{matrix} A, B \in \mathbb{R}^{2\times 2}, AA' + BB' = \mathbb{1}, \\ AI = IA, \quad BI = IB \end{matrix} \right\}^\pi.$$

Regarding the centralizers of Σ, *we obtain that* $\mathrm{C}_\Delta(\Sigma) = \Phi\mathrm{T}$, *and that*

$$\mathrm{Z} := \mathrm{C}_\Psi(\Sigma) = \left\{ \begin{pmatrix} A & & \\ & A & \\ & & \mathbb{1} \end{pmatrix} \middle| \begin{matrix} AA' = \mathbb{1} \\ \det A = 1 \end{matrix} \right\}^\pi \cong \mathbb{T}.$$

Proof. Let Ξ be a subgroup of Δ that is isomorphic to $\mathrm{SU}_2(\mathbb{C},0)$. Then Ξ is compact and semi-simple, and we may assume that $\Xi < \mathrm{E}$. The central involution ξ of Ξ is a conjugate of σ, and we obtain that Ξ coincides with the commutator subgroup of the centralizer of ξ in E. This means that Ξ is a conjugate of Σ, which is the commutator subgroup of the centralizer of σ in E. The decomposition of the centralizer of Σ is verified by an easy computation, using 4.2(3). □

Lemma 4.4. *If* Δ *acts non-trivially on a stable plane* $\mathbb{M} = (M, \mathcal{M})$ *such that* $\dim M = 8$, *then no involution in* Δ *has a center, and* τ *acts freely.*

Proof. (i) If τ fixes a point, then one proceeds as in step (ii) of the proof of 3.3 to show that τ is planar, the centralizer induces the elliptic motion group on the subplane of fixed elements, and σ is a reflection.

(ii) If an involution in Δ has center p, then a commuting conjugate of τ fixes p, and σ has a center by step (i).

(iii) If σ has center c, then τ is planar, and σ is a reflection by step (i). The stabilizer Δ_c coincides with $\Phi\Psi$, compare step (i) of the proof of 3.5. From [22, 3.7, 3.8] we infer that $\mathcal{M}_c$ is homeomorphic to the sphere $\mathbb{S}_4$. The maximal compact subgroups of $\Phi\Psi$

are conjugates of $\Sigma Z\Theta$. According to [19], no maximal compact subgroup of $\Phi\Psi$ acts almost effectively on $\mathcal{M}_c$. Since Ψ cannot act trivially on $\mathcal{M}_c$ by [25, 11], we obtain that Φ is the kernel of the restriction of $\Phi\Psi$ to $\mathcal{M}_c$. The maximal compact subgroup ΣZ of Ψ induces an effective action of $\mathrm{SO}_3(\mathbb{R},0) \times \mathrm{SO}_2(\mathbb{R},0)$ on $\mathcal{M}_c$, which is equivalent to the usual action on $\mathbb{S}_4$ by [19]. In particular, the group Σ fixes a line $L \in \mathcal{M}_c$. Denoting the stabilizer of L by Λ, we obtain that $\Lambda_c = \Lambda \cap \Delta_c$ is the centralizer of σ in Λ. This implies that, via the adjoint action, the reductive group $\Phi\Sigma$ acts effectively on a vector space complement V of the Lie algebra of Λ_c in the Lie algebra of Λ. Since the orbit c^Λ is contained in L, we obtain that $\dim V \leqslant 4$. This implies that Σ acts irreducibly on V by 2.7, and Φ is contained in the multiplicative group of a skew field by Schur's Lemma. As a closed subalgebra of the real algebra $\mathrm{End}(V)$, this skewfield is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$, and its multiplicative group cannot contain a group isomorphic to Φ. This is a contradiction. $\square$

Proposition 4.5. *There is no nontrivial action of* $\Delta = \mathrm{PSp}_6\mathbb{R}$ *on a stable plane* $\mathbb{M} = (M, \mathcal{M})$ *such that* $\dim M = 8$.

Proof. Assume that Δ acts non-trivially on an 8-dimensional stable plane $\mathbb{M}$. As in step (i) of the proof of 3.3, we obtain that every point of $\mathbb{M}$ is fixed by some conjugate of σ, and that σ is not planar. Combining 4.4 and 3.2(1), we infer that σ has an axis A, but no center. The group Ψ contains the commuting involutions σ and

$$\alpha = \begin{pmatrix} 1 &&&&& \\ & 1 &&&& \\ && -1 &&& \\ &&& -1 && \\ &&&& 1 & \\ &&&&& 1 \end{pmatrix}^{\pi}.$$

As σ, α and $\alpha\sigma$ are three commuting involutions in $\sigma^\Gamma \cap \Psi$, the group Ψ cannot fix any point of A by 3.2(3).

We consider the action of the compact group $\mathrm{K} = \Sigma Z\Theta \leqslant \Lambda$ on A. Let a be an arbitrary point of A. Then $\dim \mathrm{K}_a \leqslant 1$ would imply that $\dim a^{\mathrm{K}} \geqslant 4$, and that K acts transitively on A, in contradiction to [26, 2.10]. As the almost simple group Ψ acts nontrivially on A, there exists $a \in A$ such that $\Sigma \not\leqslant \mathrm{K}_a$. We conclude that K_a contains a subgroup isomorphic to $\mathbb{T}^2$. This means that there are three commuting involutions σ, β, and $\sigma\beta$ in K_a. By 3.2(2), only one of these involutions has axis A. According to 4.4, none of them has a center. Therefore, we may assume that β is planar. Since τ acts freely and σ has an axis, we obtain that β is a conjugate of $\sigma\tau$. As in step (v) of the proof of 3.3, we infer that σ has a center, a contradiction. $\square$

Proceeding analogously as in the proof of 3.7, we obtain:

Theorem 4.6. *If* $\tilde{\Delta}$ *is a locally compact, connected, almost simple group such that the center factor group is isomorphic to* $\mathrm{PSp}_6\mathbb{R}$, *then there is no non-trivial action of* $\tilde{\Delta}$ *on a stable plane* $\mathbb{M} = (M, \mathcal{M})$ *such that* $\dim M = 8$.

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M. Stroppel
Mathematisches Institut B/1
Universität Stuttgart
D-70550 Stuttgart
Germany