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Electrons in Disordered Systems. Scaling Near the Mobility Edge

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Renormalization group arguments are applied to an ensemble of disordered electronic systems (without electron-electron interaction). The renormalization group procedure consists of a sequence of transformations of the length and the energy scales, and of orthogonal transformations of the electronic states. Homogeneity and power laws are obtained for various one and two-particle correlations and for the low-temperature conductivity in the vicinity of the mobility edge. Two types of fixed point ensembles are proposed, a homogeneous ensemble which is roughly approximated by a cell model, and an inhomogeneous ensemble.

1. Introduction and Results

In this paper we consider the correlations of wave-functions in an ensemble of disordered electronic systems. The system is described by the tight-binding hamiltonian

$$H=\sum_r \varepsilon_r |r\rangle \langle r| + \sum_{r\neq r'} f_{rr'} |r\rangle \langle r'| \tag{1.1}$$

with the electronic wave-functions $|r\rangle$ localized at sites r. The one-particle energies ε and/or the transfer matrix elements f are distributed at random with a given probability. It is assumed that the transfer matrix elements f decay rapidly with distance $|r-r'|$ and that the ensemble does not exhibit long-range correlations of the ε's and f's.

Systems of this type have received much attendance. We refer to the book by Mott and Davis [1] and the review articles by Elliott et al. [2], Thouless [3] and Economou et al. [4]. Anderson gave an argument in his fundamental paper [5] that within a certain energy region the eigenstates of the hamiltonian (1.1) are extended, whereas outside this regime the states are localized. Here we are concerned with the behaviour of the wave functions close to the mobility edge which separates localized from extended states.

To derive the correlation functions we will follow the renormalization group (RG) ideas as outlined by Licciardello and Thouless [6], compare Edwards and Thouless [7]. The basic idea is to perform successive orthogonal transformations which mix nearby states so that the transfer energies f decrease. It is expected that such a procedure converges for localized eigenstates. The limit of convergency will be the mobility edge. We add to this elimination process a transformation which rescales all distances and energies. We expect this procedure to converge to a fixed point distribution of the ensemble at the mobility edge. Two different types of fixed point distributions are proposed, a homogeneous and an inhomogeneous fixed point ensemble. We are not able to decide whether both or one or neither of these ensembles are approached starting from the ensemble of hamiltonians (1.1) described above. Both ensembles exhibit features expected in disordered systems near the mobility edge.

Homogeneous Fixed Point Ensemble. This ensemble is homogeneous in energy ε. It transforms into itself under the transformation $\varepsilon\to\varepsilon+$ constans. Since the density of states has to be constant during the RG procedure the scale change

$$r\to r/b \tag{1.2a}$$

implies

$$\varepsilon\to\varepsilon b^d. \tag{1.2b}$$

We assume one relevant perturbation to this ensemble which grows like

$$\tau \to \tau b^y. \tag{1.2c}$$

Depending on the sign of τ the perturbation produces localized and extended states, resp. This perturbation is added to the homogeneous fixed point ensemble in a strength increasing linearly in energy

$$\tau = c(E - E_c) \tag{1.3}$$

thus yielding extended states for $\tau > 0$ and localized states for $\tau < 0$. The theory yields scaling behaviour for $0 < y < d$ whereas for $y > d$ there would be a discontinuous change from localized to extended states.

Inhomogeneous Fixed Point Ensemble. In this ensemble the scale factors for lengths and energies are independent from each other, the ensemble is inhomogeneous in the energy,

$$r \to r/b \tag{1.4a}$$

$$\varepsilon \to \varepsilon b^y. \tag{1.4b}$$

It is assumed that there is no relevant perturbation to such an ensemble. The inhomogeneity allows for extended states in the region $E > 0$ and for localized states in the region $E < 0$. We choose $E_c = 0$.

Both ensembles yield power and homogeneity laws. The density of states ρ near the mobility edge obeys for the two ensembles

$$\rho_{\text{hom}} = \text{const.} \qquad \rho_{\text{inh}} \sim |E|^{(d-y)/y}. \tag{1.5}$$

The amplitudes $\psi(r)$ of the normalized real eigenfunctions

$$\varepsilon_r \psi(r) + \sum f_{rr'} \psi(r') = E\psi(r), \qquad \sum_r \psi^2(r) = 1 \tag{1.6}$$

transform under the RG transformation according to

$$\sqrt{V}\,|\psi(r)| \to \sqrt{V^{(1)}}\, b^{-\kappa/2}\, |\psi(r/b)| \tag{1.7a}$$

$$V\psi^2(r) \to V^{(1)}\, \psi^2(r/b) \tag{1.7b}$$

$$V^2\, \psi^4(r) \to V^{(1)\,2}\, b^{\kappa'}\, \psi^4(r/b) \tag{1.7c}$$

provided any other factors $\psi(r')$ are sufficiently far separated in distance. The volumes of the original system and the system after the RG procedure are denoted by V and $V^{(1)}$, respectively.

The scaling properties (1.2), (1.4), and (1.7) imply various homogeneity and power laws for correlation functions. At the mobility edge the correlations of the amplitudes $|\psi|$ decay as

$$V\,\overline{|\psi(0)\,\psi(r)|} \sim r^{-\kappa} \tag{1.8}$$

$$V^{2-\kappa'/d}\,\overline{\psi^2(0)\,\psi^2(r)} \sim r^{-\kappa'} \tag{1.9}$$

where the bar indicates the ensemble average. As a function of τ (for the inhomogeneous system take $\tau = E$) the correlation length and the mean fourth power follow the power laws

$$\xi \sim |\tau|^{-1/y} \tag{1.10}$$

$$V\overline{\psi^4(r)} \sim |\tau|^{(d-\kappa')/y}, \qquad \tau < 0. \tag{1.11}$$

Exponents for the power laws (1.8), (1.10) and (1.11) have been proposed by Abram and Edwards [8], Anderson [9], Freed [10], Last and Thouless [11], and on the basis of percolation theory (compare Ref. 3).

The low-temperature a.c. conductivity obtained from the Kubo-Greenwood formula obeys the homogeneity-relation

$$\sigma(\omega, \tau) = \begin{cases} b^{2-d}\, \sigma(\omega b^d, \tau b^y) & \text{hom. ens.} \\ b^{2-d}\, \sigma(\omega b^y, \tau b^y) & \text{inh. ens.} \end{cases} \tag{1.12}$$

in the limit of small ω and τ. It implies at the mobility edge the power law

$$\sigma(\omega, 0) \sim \begin{cases} \omega^{(d-2)/d} & \text{hom. ens.} \\ \omega^{(d-2)/y} & \text{inh. ens.} \end{cases} \tag{1.13}$$

and yields the d.c. conductivity in the region of the extended states

$$\sigma(0, \tau) \sim \tau^{(d-2)/y}, \qquad \tau > 0 \tag{1.14}$$

which agrees with the prediction of a finite conductivity in two-dimensional systems (Licciardello and Thouless [6]).

In Section 2 the RG approach for the homogeneous ensemble is considered. A simplified version of this model is the cell model introduced in Section 3. The correlation functions of this cell model are evaluated in Sections 4 and 5. In Section 6 the approximations of the cell model are abandoned and arguments of the scaling properties are given on a more general basis. A few modifications of the theory for the homogeneous ensemble allow the derivation of the scaling laws for the inhomogeneous system in Section 7.

2. Renormalization Group of the Homogeneous Ensemble

The renormalization group procedure consists of two parts, an elimination process and a scale transformation. We begin with the *elimination process.* To obtain the eigenstates of the hamiltonian (1.1) we apply repeated orthogonal transformations

$$|r^{(l)}\rangle_l = \sum U^{(l)}_{r^{(l)} r^{(l-1)}}\, |r^{(l-1)}\rangle_{l-1} \tag{2.1}$$

with

$$|r^{(0)}\rangle_0 \equiv |r\rangle, \tag{2.2}$$

so that the transfer matrix elements $f^{(l)}$ of the hamiltonian in terms of the new states $|r^{(l)}\rangle_l$ are smaller than the elements $f^{(l-1)}$. We require that the matrices U are localized, that is $U^{(l)}_{r^{(l)}r^{(l-1)}}$ decays as a function of $r^{(l-1)}$ for fixed $r^{(l)}$ more rapidly than any power in $r^{(l-1)}$, and similarly as a function of $r^{(l)}$ for fixed $r^{(l-1)}$. The iterative application of such transformations U leads to a total localized transformation from $|r\rangle$ to $|r^{(l)}\rangle_l$. An appropriate choice of the matrices U yields finally the localized eigenstates as limit of $|r^{(l)}\rangle_l$ for $l\to\infty$, but we cannot expect convergency for the extended states. We define the limit energy E_c of the convergency as the mobility edge.

Homogeneous Ensemble. Until now the states of the system are labelled only by the d components of the vector $r^{(l)}$ in the local space. It is essential to add (as in Ref. 6) the energy $\varepsilon^{(l)}$ as $(d+1)$th variable and to consider the states $|\rangle_l$ in this $(d+1)$ dimensional space. Assume that the one-particle energies ε are not restricted to some finite interval but allow for states with uniform probability density ρ spread over the unbounded $(d+1)$ dimensional space. The transfer energies f shall decay faster than any power of the distance or of the energy difference between both levels for large separations. The probability distribution of the transfer energies must not have any long-range correlations in the $(d+1)$ dimensional space, nor change under a uniform transfer of the $r^{(l)}$ or $\varepsilon^{(l)}$. We call such an ensemble shortly a homogeneous ensemble.

Scale Transformation. The elimination process will easily diminish those matrix elements to a negligible amount which connect states differing in ε by an amount large in comparison to characteristic values of f as is well-known from perturbation theory. Thus the matrix elements f will connect states of smaller and smaller energy difference. On the other hand one can hardly avoid that the orthogonal transformation increases matrix elements f connecting nearly degenerate states further and further apart in local space. This observation invites to a scale transformation

$$r^{(l)}=b^{-1}\,r^{(l-1)} \tag{2.3a}$$

$$\varepsilon^{(l)}=b^{d}\,\varepsilon^{(l-1)} \tag{2.3b}$$

so that the range of appreciable transfer energies f in local and energy space stays constant. The scale factors in the transformation (2.3) are chosen so that the density of states in (r,ε) space is conserved. (Besides the transformation (2.3b) the elimination transformation will shift the energies ε.) The transfer energies f will become smaller due to the elimination process, on the other hand the terms f are energies and have to be multiplied by the factor b^d.

Fixed Point. In the homogeneous ensemble one expects either all states or no states to be localized. Suppose there is a sufficient amount of disorder to produce localized states. In this case the RG procedure converges so that finally a diagonalized hamiltonian is reached. If, however, the disorder decreases we will reach a situation where the increase of f due to the scale transformation balances the decrease of the elements f due to the elimination process. Then the ensemble remains invariant under the RG procedure. We call such an ensemble fixed point ensemble. It will be characterized by some dimensionless quantity like

$$K=\rho\sum_{r'}|f_{rr'}|\,|r-r'|^d \tag{2.4}$$

which assumes its fixed point value

$$K=K^*. \tag{2.5}$$

If the matrix elements f are slightly different from those of the fixed point ensemble, then we expect that the difference

$$\tau=K-K^* \tag{2.6}$$

grows by some factor we call b^y during one RG step

$$\tau_l=b^y\,\tau_{l-1}. \tag{2.7}$$

We assume that there is only one relevant perturbation to the fixed point ensemble, that is, only one eigenperturbation yields $y>0$. We neglect all irrelevant perturbations.

In the *original disordered system* the one-particle levels are restricted to some finite region. We apply the RG procedure as described above. Due to the scale transformation (2.3b) the available energy region increases exponentially. After a few steps it becomes reasonable to introduce the slowly varying function $K(\varepsilon)$. Close to the mobility edge one has in linear approximation

$$\tau_l=c_l(\varepsilon^{(l)}-E_c^{(l)}). \tag{2.8}$$

After one RG step one has

$$\tau_l=b^y\,\tau_{l-1}=b^y\,c_{l-1}(\varepsilon^{(l-1)}-E_c^{(l-1)})=c_l(\varepsilon^{(l)}-E_c^{(l)}) \tag{2.9}$$

with

$$c_l=b^{y-d}\,c_{l-1}. \tag{2.10}$$

Thus if $y<d$ then the gradient

$$\mathrm{d}\tau_l/\mathrm{d}\varepsilon_l=c_l \tag{2.11}$$

vanishes in the limit $l\to\infty$ which implies that repeated application of the RG leads to an ensemble more and more similar to the homogeneous ensemble if considered in the vicinity of E_c.

In the following

$$0<y<d \tag{2.12}$$

is assumed so that these ideas apply. If y would exceed the dimensionality d then c_l would diverge for $l\to\infty$. There would be an abrupt transition from localized to extended states.

If we start from a system with extended states between two tails of localized states and increase the disorder until the energy interval of the extended states has reached zero, then we have the limit case

$$\tau_l = c_l(\varepsilon^{(l)} - E_c^{(l)})^2 \tag{2.13}$$

which yields

$$c_l = b^{y-2d}\, c_{l-1}. \tag{2.14}$$

In this case it is sufficient to have

$$0<y<2d \tag{2.15}$$

for convergency to the homogeneous fixed point ensemble. We restrict ourselves to the discussion of the case (2.8).

3. Cell Model

To make the RG procedure transparent we introduce in this section a cell model and discuss its implications in Sections 4–5. Divide a simple lattice into cells each of which contains n states $|r\rangle$. Then form larger cells out of n original cells etc. If this division into the hierarchy of cells is performed appropriately and

$$n = b^d \tag{3.1}$$

with integer b then each site can be numbered by a set of labels $i_1, i_2, i_3, \ldots$

$$|r\rangle = |i_1, i_2, i_3, \ldots\rangle \tag{3.2}$$

with

$$r = e_{i_1} + b\, e_{i_2} + b^2\, e_{i_3} + \cdots \tag{3.3}$$

where any i runs from 1 to n.

Diagonalization within the n states of a cell yields

$$|j_1; i_2, i_3 \ldots\rangle = \sum U_{j_1 i_1} |i_1, i_2, i_3 \ldots\rangle \tag{3.4}$$

with an orthogonal matrix U. We choose $j_1 = 1$ for the lowest lying state, $j_1 = 2$ for the next lowest state etc. The crucial assumption of our model is that all transfer matrix elements between states of different energy label j_1 will be neglected. Thus it is assumed that only the lowest lying states of the cells interact with each other, the second lowest states etc. Although this is a crude approximation we note that states which are separated by a large energy difference mix only weakly upon diagonalization whereas the interaction between nearly degenerate states is essential. Thus we keep an important part of the interaction. With this approximation the diagonalization within the cells of n^2 states is performed and so on. The l-th step of diagonalization yields

$$|\ldots j_{l-1}, j_l; i_{l+1}\ldots\rangle = \sum_{i_l} U_{j_l i_l} |\ldots j_{l-1}; i_l, i_{l+1}\ldots\rangle \tag{3.5}$$

The matrices U depend on l and $\ldots j_{l-2}, j_{l-1}, i_{l+1}\ldots$ which will not be denoted explicitly in most cases. It is assumed that different matrices U are independent of each other, that is, the probability distribution of several matrices U factorizes in the probability distribution of the single matrices U. Further it is assumed that the probability distribution is invariant with respect to any permutation of the labels j_l and any permutation of the labels i_l.

The states, Equation (3.5) are characterized by energy labels $j_1 \ldots j_l$ and site labels i_{l+1}, $i_{l+2}\ldots$. Since the length scale changes by a factor b^l within l RG steps the state $|\ldots j_l; i_{l+1}\ldots\rangle$ is localized around

$$r^{(l)} = e_{i_{l+1}} + b\, e_{i_{l+2}} + b^2\, e_{i_{l+3}} + \cdots. \tag{3.6}$$

Thus the original lattice (3.3) is reproduced. The energy labels yield an approximate value of the energy. If the first diagonalization yields a splitting ε_0 of the energy labels then apart from an additive constant one has $\varepsilon^{(1)} = \varepsilon_0 j_1$. Application of the scale transformation (2.3b) and consideration of the level spacing occurring at each step leads to the approximate energy level

$$\varepsilon^{(l)} = \varepsilon_0 (n^{l-1} j_1 + n^{l-2} j_2 + \cdots + j_l). \tag{3.7}$$

We introduce the notation

$$\psi_j^{(l)}(r^{(l)}) = \langle j_1 j_2 \ldots j_l; i_{l+1} \ldots | j_1 j_2 \ldots\rangle \tag{3.8}$$

for the wave-functions of the eigenstates $|j_1 j_2 \ldots\rangle$. In particular one has

$$\psi_j(r) = \psi_j^{(0)}(r) = \langle i_1 i_2 \ldots | j_1 j_2 \ldots\rangle. \tag{3.9}$$

Equation (3.5) yields the recurrence relation

$$\psi_j^{(l-1)}(r^{(l-1)}) = U_{j_l i_l}\, \psi_j^{(l)}(r^{(l)}) \tag{3.10}$$

which will be useful in the following sections.

The model can be generalized to noninteger b with

$$r = e_i^{(1)} + b\, e_{i_2}^{(2)} + b^2\, e_{i_3}^{(3)} + \cdots \tag{3.11}$$

and $e_i^{(l)} = e_i^{(l+d)}$ to allow for arbitrary integer $n \geqq 2$.

4. One-Particle Correlation

Three types of one-particle correlation functions are discussed in this section,

$$G_1(r, r', \tau) = \overline{\langle r' |\, \delta(H - E(\tau))\, | r\rangle}$$
$$= \overline{\sum_j \psi_j(r')\, \psi_j(r)\, \delta(E_j - E(\tau))}, \tag{4.1a}$$

$$C(r, r', \tau) = \overline{\sum_j |\psi_j(r')\, \psi_j(r)|\, \delta(E_j - E(\tau))}, \tag{4.1b}$$

and

$$D(r, r', \tau) = \overline{\sum_j \psi_j^2(r)\, \psi_j^2(r')\, \delta(E_j - E(\tau))}. \quad (4.1\,c)$$

where E and τ are related according to Equation (2.8) by

$$\tau = c(E - E_c) \quad (4.2)$$

for energies close to the mobility edge. If we call $\bar{A}^\tau$ the average of some function A over all eigenstates of the ensemble in the infinitesimal energy interval $E(\tau-0) \ldots E(\tau+0)$ then the correlation functions read

$$G_1(r, r', \tau) = \rho V \overline{\psi(r)\, \psi(r')}^\tau \quad (4.3\,a)$$

$$C(r, r', \tau) = \rho V \overline{|\psi(r)\, \psi(r')|}^\tau \quad (4.3\,b)$$

and

$$D(r, r', \tau) = \rho V \overline{\psi^2(r)\, \psi^2(r')}^\tau \quad (4.3\,c)$$

where we use that ρV is the density of states per energy.

To evaluate (4.3) one applies the recurrence relation (3.10) and obtains for G_1 (with $E_j^{(1)} = nE(\tau)$)

$$G_1(r, r', \tau) = \rho V \overline{U_{j_1 i_1}\, U_{j_1 i_1'}\, \psi_j^{(1)}(\tau'^{(1)})\, \psi_j^{(1)}(r^{(1)})}^{\tau_1}. \quad (4.4)$$

According to our assumption the matrices U which relate $\psi_j^{(1)}$ to $\psi_j^{(2)}$, $\psi_j^{(3)}$ etc. are independent of the matrices U in Equation (4.4). Therefore the expression (4.4) factorizes

$$G_1(r, r', \tau) = \rho V\, \overline{U_{j_1 i_1}\, U_{j_1 i_1'}}\; \overline{\psi_j^{(1)}(r'^{(1)})\, \psi_j^{(1)}(r^{(1)})}^{\tau_1}. \quad (4.5)$$

We note the relations $\tau_1 = b^y \tau$ and $V = V^{(1)} n$, and obtain the recurrence relation for G_1

$$G_1(r, r', \tau) = n\overline{U_{j_1 i_1}\, U_{j_1 i_1'}}\; G_1(r^{(1)}, r'^{(1)}, b^y \tau). \quad (4.6)$$

For general l one obtains the recurrence relation

$$G_1(r^{(l-1)}, r'^{(l-1)}, \tau_{l-1}) = \phi_G\, G_1(r^{(l)}, r'^{(l)}, \tau_l) \quad (4.7)$$

with

$$\phi_G = n\overline{U_{j_l i_l}\, U_{j_l i_l'}}. \quad (4.8\,a)$$

Similar recurrence relations are obtained for C and D with the renormalization factors

$$\phi_C = n\overline{|U_{j_l i_l}\, U_{j_l i_l'}|} \quad (4.8\,b)$$

and

$$\phi_D = n\overline{U_{j_l i_l}^2\, U_{j_l i_l'}^2}. \quad (4.8\,c)$$

For the evaluation of the factors ϕ, which are given in Table 1, three cases have to be distinguished:

α) the sites $r^{(l-1)}$ and $r'^{(l-1)}$ are in different cells: $r^{(l)} \neq r'^{(l)}$. Then the matrix elements U in Equations (4.8) come from different matrices. Therefore the expectation values factorize. We obtain the values ϕ in Table 1, with the notation

$$n\overline{U_{ji}}^2 \equiv x, \quad n\overline{|U_{ji}|}^2 \equiv c, \quad n\overline{U_{ji}^2}^2 = 1/n. \quad (4.9)$$

The last relation is obtained from the orthonormality relation

$$\sum_{j=1}^{n} U_{ji}\, U_{ji'} = \delta_{ii'}. \quad (4.10)$$

β) the sites $r^{(l-1)}$ and $r'^{(l-1)}$ are different, but in the same cell: $r^{(l)} = r'^{(l)}$, $r^{(l-1)} \neq r'^{(l-1)}$. Then the matrix elements U_{ji} and $U_{ji'}$ are from the same matrix but $i \neq i'$. The factors ϕ are

$$n\overline{U_{ji}\, U_{ji'}} = 0, \quad n\overline{|U_{ji}\, U_{ji'}|} \equiv c',$$
$$n\overline{U_{ji}^2\, U_{ji'}^2} = (n-\alpha)/(n^2-n). \quad (4.11)$$

The first relation is obtained from Equation (4.10), the third relation from

$$\sum_{i=1}^{n} U_{ji}\, U_{j'i} = \delta_{jj'} \quad (4.12)$$

with the definition

$$\alpha \equiv n^2\, \overline{U_{ji}^4}. \quad (4.13)$$

γ) the sites $r^{(l-1)}$ and $r'^{(l-1)}$ are identical. Then we obtain immediately from Equations (4.10) and (4.13) the values quoted in Table 1.

The quantities x, c, c', and α depend on τ. Particular for the mobility edge we introduce two critical exponents κ and κ' by

$$c(0) = b^{-\kappa} \quad (4.14\,a)$$

$$\alpha(0) = b^{\kappa'} \quad (4.14\,b)$$

By means of Schwarz's inequality one shows (Appendix A)

$$0 \leqq \kappa \leqq \kappa' \leqq d. \quad (4.14\,c)$$

We discuss the correlation functions: Application of the recurrence relation (4.7) for $r \neq r'$ yields factors $\phi_G = x$ until $r^{(l)}$ and $r'^{(l)}$ are in the same cell. Then the factor $\phi_G = 0$ leads to a zero correlation. Only for

Table 1. Renormalization factors ϕ for the recurrence relations of the one-particle correlation functions

	$r^{(l)} \neq r'^{(l)}$	$r^{(l)} = r'^{(l)}$ $r^{(l-1)} \neq r'^{(l-1)}$	$r^{(l-1)} = r'^{(l-1)}$
ϕ_G	x	0	1
ϕ_C	c	c'	1
ϕ_D	$1/n$	$(n-\alpha)/(n^2-n)$	α/n

$r=r'$ the function G_1 differs from zero

$$G_1(r, r', \tau)=\varepsilon_0^{-1}\,\delta_{rr'} \tag{4.15}$$

where the normalization factor can be easily derived from Equation (4.3a) and the normalization (1.5). This surprising result deserves an interpretation. We expect wave-functions with more nodes to have a higher energy, a property which leads at least to short-range correlations. In such a case $\overline{U_{ji}U_{ji'}}$, will depend on the energy levels whereas $\phi_G=0$ has been obtained by averaging over all energy levels j. If we apply the RG procedure so often that within an energy interval large in comparison to characteristic transfer energies f the variation of τ is negligible then the ensemble may be approximated by the homogeneous ensemble of Section 2. In this case Equation (4.15) holds exactly since no eigenvalue E of the eigenfunctions is distinguished and $\overline{\psi(r)\psi(r')}$ is obtained by averaging over *all* eigenfunctions. Since the statistical operator of all eigenfunctions is identical to that of all local states one obtains immediately Equation (4.15).

For $r=r'$ the correlation function C coincides with G_1. For $r\neq r'$ we define a number k by

$$r^{(k)}=r'^{(k)}, \qquad r^{(k-1)}\neq r'^{(k-1)} \tag{4.16}$$

from which we obtain the correlation function at the mobility edge

$$C(r, r', 0)=\begin{cases}\varepsilon_0^{-1} & r=r'\\ c'\,\varepsilon_0^{-1}\,b^{-\kappa(k-1)} & r\neq r'\end{cases} \tag{4.17}$$

since $(k-1)$ factors $c=b^{-\kappa}$ and one factor c' enter during the k RG steps which lead to $C(r^{(k)}, r'^{(k)}, 0)=\varepsilon_0^{-1}$. If we approximate the distance by

$$|r-r'|\approx a_0\,b^k \tag{4.18}$$

where a_0^d is the volume per site then the correlation function decays according to the power law

$$C(r, r', 0)\sim|r-r'|^{-\kappa}. \tag{4.19}$$

As long as $|r-r'|\gg a_0$ one obtains for general τ within the approximation (4.18) the homogeneity law

$$C(r, r', \tau)=b^{-\kappa}\,C(r\,b^{-1}, r'\,b^{-1}, \tau\,b^y) \tag{4.20}$$

and

$$\begin{aligned}C(r, r', \tau)&=|r-r'|^{-\kappa}\,C(0, 1, \tau|r-r'|^y)\\ &=|r-r'|^{-\kappa}\,\tilde{C}(|r-r'|/\xi(\tau))\end{aligned} \tag{4.21}$$

where the correlation length ξ diverges like

$$\xi(\tau)\sim|\tau|^{-1/y} \tag{4.22}$$

at the mobility edge.

For $r=r'$ the correlation function D transforms like

$$D(0, 0, \tau)=n^{-1}\,\alpha(\tau)\,D(0, 0, \tau\,b^y). \tag{4.23}$$

This factor $n^{-1}\,\alpha(\tau)$ is in contrast to the factor 1 for the functions G_1 and C. Far in the region of localized states the diagonalization will have nearly converged. Therefore one of the matrix elements U_{ji}, $i=1, \ldots n$ will be very close to unity whereas all other matrix elements vanish. Thus α, Equation (4.13) approaches n in the limit of large negative τ and we may expect that repeated iteration of Equation (4.23) leads to a finite value for $\tau<0$. Close to the mobility edge $\alpha(\tau)$ may be replaced by $b^{\kappa'}$ which yields

$$D(0, 0, \tau)=b^{\kappa'-d}\,D(0, 0, \tau\,b^y). \tag{4.24}$$

Thus the average $V\overline{\psi^4(0)}$ vanishes with the power law

$$V\overline{\psi^4(0)}\sim D\sim|\tau|^{(d-\kappa')/y} \tag{4.25}$$

on approaching the mobility edge from below. Bell and Dean [12] introduced this quantity as a measure for localization. For $|r-r'|\gg a_0$ one obtains for D a homogeneity law similar to that for C

$$\begin{aligned}&D(r, r', \tau)=b^{-d}\,D(r\,b^{-1}, r'\,b^{-1}, \tau\,b^y)\\ &=|\tau|^{(d-\kappa')/y}\,r^{-\kappa'}\,\tilde{D}(|r-r'|/\xi(\tau)).\end{aligned} \tag{4.26}$$

Since $V\overline{\psi^4(0)}$ vanishes at the mobility edge $V\overline{\psi^2(0)\psi^2(r)}$ vanishes, too. We note, however, that $V^{2-\kappa'/d}\overline{\psi^4(0)}$ remains invariant at the mobility edge since the extra factor $V^{1-\kappa'/d}$ introduces an extra factor $b^{d-\kappa'}$ in the recursion relation. One deduces the power law

$$V^{2-\kappa'/d}\,\overline{\psi^2(0)\psi^2(r)}\sim r^{-\kappa'} \tag{4.27}$$

at the mobility edge.

5. Two-Particle Correlation Functions; Electric Conductivity

We discuss two special cases of the two-particle correlation function

$$\begin{aligned}&G_2(r_1, r_1', E_1, r_2, r_2', E_2)\\ &=\overline{\langle r_1'|\delta(H-E_1)|r_1\rangle\langle r_2'|\delta(H-E_2)|r_2\rangle}\\ &=\overline{\sum_{jj'}\psi_j(r_1)\psi_j(r_1')\psi_{j'}(r_2)\psi_{j'}(r_2')\delta(E_j-E_1)\delta(E_{j'}-E_2)}.\end{aligned} \tag{5.1}$$

These special cases are

$$\begin{aligned}&G_\sigma(r, r', \omega, \tau)\\ &=G_2\left(r, r', E(\tau)+\frac{\omega}{2}, r, r', E(\tau)-\frac{\omega}{2}\right)\end{aligned} \tag{5.2a}$$

and

$$\begin{aligned}&G_n(r, r', \omega, \tau)\\ &=G_2\left(r, r, E(\tau)+\frac{\omega}{2}, r', r', E(\tau)-\frac{\omega}{2}\right).\end{aligned} \tag{5.2b}$$

The correlation G_σ serves to calculate the electric conductivity at low temperatures by means of the

Kubo-Greenwood-formula, whereas the correlation G_n describes the density correlation of two electrons at different energies.
Similarly to Section 4 we express G in terms of the eigenfunctions ψ

$$G_\sigma(r,r',\omega,\tau)=\rho^2 V^2 \overline{\psi_j(r)\psi_j(r')\psi_{j'}(r)\psi_{j'}(r')}. \tag{5.3}$$

Using the recurrence relation (3.10) one obtains in analogy to Section 4

$$G_\sigma(r^{(l-1)}, r'^{(l-1)}, \omega_{l-1}, \tau_{l-1}) = \phi_\sigma G_\sigma(r^{(l)}, r'^{(l)}, \omega_l, \tau_l) \tag{5.4}$$

with

$$\omega_l = b^d \omega_{l-1} \tag{5.5}$$

and

$$\phi_\sigma = n^2 \overline{U^{(\dots j_l \dots; i_{l+1} \dots)}_{j_l i_l} U^{(\dots j_{l-1}; i'_{l+1} \dots)}_{j_l i'_l}} \cdot \overline{U^{(\dots j'_{l-1}; i_{l+1} \dots)}_{j'_l i_l} U^{(\dots j'_{l-1}; i'_{l+1} \dots)}_{j'_l i'_l}}. \tag{5.6}$$

Similar expressions are obtained for G_n. To determine the renormalization factors ϕ one has not only to distinguish the three cases α, β, γ of Section 4 but also the associated energies

α') $\varepsilon^{(l)} = \varepsilon'^{(l)}$, the approximate energies (3.7) are degenerate,
β') $\varepsilon^{(l-1)} = \varepsilon'^{(l-1)}$, $\varepsilon^{(l)} \neq \varepsilon'^{(l)}$, this RG step removes the degeneracy of the energies,
γ') $\varepsilon^{(l-1)} \neq \varepsilon'^{(l-1)}$, the energies are not degenerate.

Depending on these 3×3 cases the matrix elements U_{ji} come from the same or from different matrices. The values of the renormalization factors ϕ_σ and ϕ_n are listed in Tables 2 and 3, respectively. The first line coincides apart from a factor n (due to the different power in V) with those of ϕ_D. The determination of ϕ in the first row and the third line is simple since the expectation values factorize and reduce to expressions evaluated in determining ϕ_G. For the case $\beta\beta'$ one obtains

$$\phi_\sigma = n^2 \overline{U_{ji} U_{ji'} U_{j'i} U_{j'i'}} \tag{5.7}$$

and

$$\phi_n = n^2 \overline{U_{ji}^2 U_{j'i'}^2} \tag{5.8}$$

with $i \neq i'$, $j \neq j'$ which in view of the orthonormality relation (4.12) can be expressed by α. The case $\gamma\beta'$ gives

$$\phi_\sigma = \phi_n = n^2 \overline{U_{ji}^2 U_{ji'}^2} \tag{5.9}$$

which yields the same result as $\beta\alpha'$.
The correlation function G_σ is evaluated by repeated application of Equation (5.4). If initially $|r-r'| \gg a_0$ and $0 < |\omega| \ll \varepsilon_0$ the first factor ϕ_σ is in the upper left corner of Table 2. As during the RG procedure

Table 2. Renormalization factors ϕ_σ for the recurrence relation of the two-particle function G_σ

	$r^{(l)} \neq r'^{(l)}$	$r^{(l)} = r'^{(l)}$ $r^{(l-1)} \neq r'^{(l-1)}$	$r^{(l-1)} = r'^{(l-1)}$
$\varepsilon^{(l)} = \varepsilon'^{(l)}$	1	$(n-\alpha)/(n-1)$	α
$\varepsilon^{(l)} \neq \varepsilon'^{(l)}$, $\varepsilon^{(l-1)} = \varepsilon'^{(l-1)}$	0	$-(n-\alpha)/(n-1)^2$	$(n-\alpha)/(n-1)$
$\varepsilon^{(l-1)} \neq \varepsilon'^{(l-1)}$	x^2	0	1

Table 3. Renormalization factors ϕ_n for the recurrence relation of the two-particle function G_n

	$r^{(l)} \neq r'^{(l)}$	$r^{(l)} = r'^{(l)}$ $r^{(l-1)} \neq r'^{(l-1)}$	$r^{(l-1)} = r'^{(l-1)}$
$\varepsilon^{(l)} = \varepsilon'^{(l)}$	1	$(n-\alpha)/(n-1)$	α
$\varepsilon^{(l)} \neq \varepsilon'^{(l)}$, $\varepsilon^{(l-1)} = \varepsilon'^{(l-1)}$	1	$1+(\alpha-1)/(n-1)^2$	$(n-\alpha)/(n-1)$
$\varepsilon^{(l-1)} \neq \varepsilon'^{(l-1)}$	1	1	1

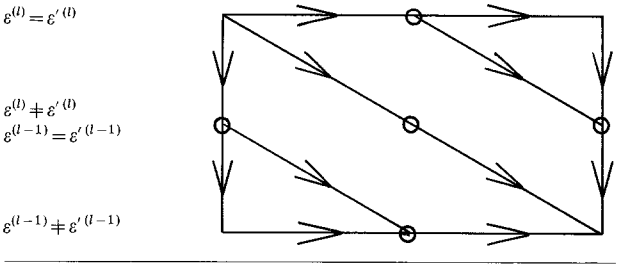

Fig. 1. The paths of this scheme indicate locations of factors ϕ_σ and ϕ_n in Tables 2 and 3 as they appear in successive steps of the recurrence relations for G_σ and G_n, respectively. The dots denote that the corresponding factors ϕ_σ and ϕ_n appear only once

$|r-r'|$ decreases and ω increases one approaches the factor ϕ_σ in the lower right corner along one of the paths shown in Figure 1. Let us determine k according to Equation (4.16) and m according to

$$\varepsilon^{(m)} \neq \varepsilon'^{(m)}, \qquad \varepsilon^{(m-1)} = \varepsilon'^{(m-1)}. \tag{5.10}$$

For $m > k$ one follows one of the two upper paths, for $m = k$ along the diagonal path and for $m < k$ along one of the two lower paths in Figure 1. Correspondingly one obtains close to the mobility edge (for $k \geqq 1$, $m \geqq 1$)

$$G_\sigma = \begin{cases} (n-\alpha)^2 \alpha^{m-k-1} (n-1)^{-2} \varepsilon_0^{-2} & m > k \\ -(n-\alpha)(n-1)^{-2} \varepsilon_0^{-2} & m = k \\ 0 & m < k \end{cases} \tag{5.11}$$

and similarly

$$G_n = \begin{cases} (n-\alpha)^2 \alpha^{m-k-1} (n-1)^{-2} \varepsilon_0^{-2} & m > k \\ (n^2-2n+\alpha)(n-1)^{-2} \varepsilon_0^{-2} & m = k \\ \varepsilon_0^{-2} & m < k. \end{cases} \tag{5.12}$$

The normalization has been determined by considering G_n in the limit $|r-r'|\to\infty$, $\omega\to\infty$. In this limit the correlation function factorizes $G_n=G_1^2=\varepsilon_0^{-2}$. The functions G_σ and G_n should agree in the limit $\omega\to 0$, that is for $m\gg k$, which yields the normalization of G_σ. The numbers k and m are roughly determined by the distance, Equation (4.18) and the energy difference

$$\omega\approx\varepsilon_0 b^{-md}. \tag{5.13}$$

Thus the correlations depend on the dimensionless quantity

$$A=\rho\omega|r-r'|^d=b^{(k-m)d} \tag{5.14}$$

where $a_0^d\varepsilon_0\rho=1$ has been used. For small A the functions G obey the power law

$$G_\sigma\approx G_n\sim A^{\kappa'/d}\sim\omega^{-\kappa'/d}r^{-\kappa'}. \tag{5.15}$$

For $A\approx 1$ the function G_σ becomes negative. This is expected since

$$\sum_r G_\sigma(r,r',\omega,\tau)=0 \quad \text{for } \omega\neq 0 \tag{5.16}$$

follows from the orthogonality of eigenstates at different energies. Thus the positive correlation (5.15) has to be compensated by a negative contribution. For $A\gg 1$ there is no correlation between the eigenstates. The function G_n approaches for $A\gg 1$ the value of uncorrelated states. For $A\approx 1$ this correlation is enhanced by a factor $1+(\alpha-1)/(n-1)^2$. One expects this since two electrons repel each other due to the Fermi principle. Since on the other hand

$$\sum_{r'}(G_n(r,r',\omega,\tau)-\varepsilon_0^{-2})=0 \tag{5.17}$$

there has to be some region where G_n exceeds its average value. Surprisingly, however, for large $m-k$, that is, for small distances G_n exceeds again ε_0^{-2}. There is some probability that two electrons share a part of one wave-function. In general α depends on τ. Therefore Equations (5.11) and (5.12) hold only if $\alpha(\tau_k)$ and $\alpha(\tau_m)$ are close to the fixed point value $\alpha^*=\alpha(0)$. We observe, however, that for $\alpha(\tau)\approx\alpha^*$ and

$$|r-r'|\gg a_0, \quad |\omega|\ll\varepsilon_0 \tag{5.18}$$

the homogeneity law

$$G_{n,\sigma}(r,r',\omega,\tau)=G_{n,\sigma}(rb^{-1},r'b^{-1},\omega b^d,\tau b^y) \tag{5.19}$$

holds since in this case $\phi_n=\phi_\sigma=1$.
At low temperatures thermal assisted hopping can be neglected and if the electron-electron interaction can be ignored then the Kubo-Greenwood formula ($\hbar=1$)

$$\sigma(\omega,\tau)=\frac{e^2\pi}{V}\sum_{r,r'}\int dE\,G_2(r,r',E,r,r',E+\omega)\cdot xx'\omega(\tilde f(E)-\tilde f(E+\omega)) \tag{5.20}$$

yields the real part of the electric conductivity σ (where $\tilde f$ is the Fermi function). Due to Equation (5.16) we may add $-(x^2+x'^2)/2$ to xx' which yields $-(x-x')^2/2$ instead of xx'. Now integrate over E and regard G_2 to be practically constant in the interval $E_f-\omega\ldots E_f$ and consider G_2 to be approximately translational and rotational invariant then

$$\sigma(\omega,\tau)=-\frac{e^2\pi}{2da_0^d}\omega^2\sum_r r^2 G_\sigma(0,r,\omega,\tau) \tag{5.21}$$

is obtained with τ given by $E_f=E(\tau)$. For small ω the function G_σ is different from zero in a large region $r\approx(\rho\omega)^{-1/d}$. The main contribution to the sum in Equation (5.21) comes from the negative correlation for $k=m$ in Equation (5.11). There are $n^m-n^{m-1}\approx(1-n^{-1})\varepsilon_0/\omega$ such terms. The other n^{m-1} terms yield a small contribution since r^2 is smaller by a factor b^{-2}. Thus at the mobility edge the power law

$$\sigma(\omega,0)\sim\omega^2 r^2 n^m\sim\omega^{1-2/d} \tag{5.22}$$

emerges. For general but small τ we use Equation (5.19)

$$\sigma(\omega,\tau)=-\frac{e^2\pi}{2da_0^d}\omega^2\sum_r r^2 G_\sigma(0,rb^{-1},\omega b^d,\tau b^y)$$
$$=-\frac{e^2\pi}{2da_0^d}\omega^2 b^{2+d}\sum_{r^{(1)}} r^{(1)2} G_\sigma(0,r^{(1)},\omega b^d,\tau b^y) \tag{5.23}$$

which yields the homogeneity law

$$\sigma(\omega,\tau)=b^{2-d}\sigma(\omega b^d,\tau b^y). \tag{5.24}$$

The factor b^{2+d} in Equation (5.23) comes from $r\approx br^{(1)}$ and from the fact that the sum over r contains b^d times the number of terms of the sum over $r^{(1)}$. Assuming a finite d.c. conductivity above the mobility edge one obtains from Equation (5.24)

$$\sigma(0,\tau)\sim\tau^{(d-2)/y}. \tag{5.25}$$

From the $\omega^2(\ln\omega)^{d+1}$ law [13] for the dependence of σ on ω below the mobility edge one obtains

$$\sigma(\omega,\tau)\sim|\tau|^{-(d+2)/y}\omega^2(\ln(c|\tau|^{d/y}\omega^{-1}))^{d+1} \tag{5.26}$$

with some constant c.
This method to calculate the correlation functions can also be used for the determination of the complete correlation function G_2 and other correlations. In general new constants besides x, c, c' and α will be necessary. A particular simple situation arises for $n=2$. In this special case all correlations

$$G=\overline{\prod_s\langle r_s|\delta(H-E_s)|r'_s\rangle} \tag{5.27}$$

vanish unless all arguments r and r' occur pairwise. Thus for $n=2$ G_2 vanishes except for the correlations G_σ and G_n. This special case exhibits gauge invariance

since the correlations are invariant against any transformation

$$|r\rangle \to \eta_r |r\rangle, \quad \eta_r = \pm 1. \tag{5.28}$$

The proof is given in Appendix B. This property raises the question whether the fixed point distribution is gauge-invariant which would imply that the probability distribution is invariant against any transformation

$$f_{rr'} \to \eta_r \eta_{r'} f_{rr'}. \tag{5.29}$$

Besides the homogeneity laws derived from the cell model one is interested in the critical exponents y, κ and κ' and the scaling functions $\tilde{C}, \tilde{D}$ and similar ones for the two-point correlations and the electric conductivity. Although the described RG procedure is expected to converge below the mobility edge the scaling functions from the cell model will hardly yield a good approximation. As soon as the extension of the cells is large in comparison to the correlation length ξ the probability distribution of U will not be invariant against all permutations of i in general, nor will succeeding transformations U be statistically independent as presupposed in Section 3. It will become important for the l-th RG step whether the distance between two wave-functions $\psi^{(l-1)}$ is small or large in comparison to the correlation length. The homogeneity laws, however, are still valid if the assumptions of the cell model are abandoned as will be argued in the next section.

6. General Arguments for Scaling

In this section general arguments for the scaling laws (1.7) are given on the basis of the RG ideas introduced in Section 2. We will not consider deviations from the fixed point distribution. (If $\tau \neq 0$, then during the first l steps the arguments will be still applicable provided τ_l is sufficiently small.) Performing l RG steps one obtains the recurrence relation

$$\psi_j(R) = \sum_t U^{(l,0)}(R_t^{(l)}, R) \psi_j^{(l)}(R_t^{(l)}) \tag{6.1}$$

where $R = (r, \varepsilon)$ and $R^{(l)} = (r^{(l)}, \varepsilon^{(l)})$ indicate position and energy initially and after l RG steps. $U^{(l,0)}$ is the orthogonal matrix produced during the RG procedures. Consider (as an example) the correlation function

$$G_\sigma = \overline{\sum_{jk} \psi_j(R)\psi_k(R)\psi_j(R')\psi_k(R')\delta(E_j - E_1)\delta(E_k - E_2)}. \tag{6.2}$$

Application of the RG makes r shrink to rb^{-l}. Since the single transformations U are short range and since the local space variable shrinks during the RG steps the matrix elements of $U^{(l,0)}$ will differ appreciably from zero only within some finite distance

$$|r^{(l)} - rb^{-l}| < r_c. \tag{6.3a}$$

Since on the other hand the single particle energies ε increase by a factor b^d during each RG step apart from some finite shifting only those levels $\tilde{R}_t$ will contribute appreciably which differ from E by less than some energy

$$|\varepsilon_t^{(l)} - E b^{dl}| < \varepsilon_l. \tag{6.3b}$$

The matrix elements $U^{(l,0)}$ will be correlated to the energies $\varepsilon^{(l)}$ and $f^{(l)}$ of sites in the region (6.3a) but not outside. The wave-function $\psi^{(l)}$ will be determined by the energies $\varepsilon^{(l)}$ and $f^{(l)}$ of levels falling in the energy interval (6.3b). Therefore the only quantities correlated both with $U^{(l,0)}$ and $\psi^{(l)}$ are the positions $r^{(l)}$, the energies $\varepsilon^{(l)}$, and the transfer energies $f^{(l)}$ of the levels $|R_t^{(l)}\rangle_l$ which fall in the region (6.3). We denote the set of all these quantities and the matrix elements $U^{(l,0)}$ of this region by Γ. Similarly we introduce the notation Γ' for the corresponding quantities around $r'b^{-l}$ and Eb^{dl}. We assume $|E_1 - E_2| b^{dl} \ll \varepsilon_c$ so that the regions (6.3b) practically coincide for the energies $E_1 b^{dl}$ and $E_2 b^{dl}$. If $|r - r'| b^{-l} \gg r_c$ then the configurations of Γ and Γ' are statistically independent. With the probability density $P_l(\Gamma)\mathrm{d}\Gamma$ one obtains

$$\begin{aligned} G_\sigma = \int \mathrm{d}\Gamma \mathrm{d}\Gamma' P_l(\Gamma) P_l(\Gamma') \sum_{t,u,t',u'} & U_\Gamma(R_t^{(l)}, R) U_\Gamma(R_u^{(l)}, R) \\ \cdot & U_{\Gamma'}(R_{t'}^{(l)}, R') U_{\Gamma'}(R_{u'}^{(l)}, R') \overline{\psi_j^{(l)}(R_t^{(l)}) \psi_k^{(l)}(R_u^{(l)})} \\ \cdot & \overline{\psi_j^{(l)}(R_{t'}^{(l)}) \psi_k^{(l)}(R_{u'}^{(l)})} \delta(E_j^{(l)} b^{-dl} - E_1) \delta(E_k^{(l)} b^{-dl} - E_2) \end{aligned} \tag{6.4}$$

where the average has to be taken over all systems of given configurations Γ and Γ'.

Let us introduce the normalization

$$a^2 = \sum_t U_\Gamma^2(R_t^{(l)}, R) \tag{6.5}$$

where the sum is restricted to the region (6.3).

Denote all configurations by γ which differ only in the norm a

$$U_\Gamma = a U_\gamma \tag{6.6a}$$

$$R_\Gamma^{(l)} = R_\gamma^{(l)}, \quad \varepsilon_\Gamma^{(l)} = \varepsilon_\gamma^{(l)}, \quad f_\Gamma^{(l)} = f_\gamma^{(l)}. \tag{6.6b}$$

The average in Equation (6.4) depends only on γ and γ' since only the quantities of Equation (6.6b) are involved but not those of (6.6a). Introducing the new probability density

$$P_l(\Gamma)\mathrm{d}\Gamma = \tilde{P}_l(\gamma, a)\mathrm{d}\gamma\,\mathrm{d}a \tag{6.7}$$

one obtains

$$\begin{aligned} G_\sigma = & b^{2dl} \int \mathrm{d}\gamma \mathrm{d}\gamma' \hat{P}_l^{(2)}(\gamma) \hat{P}_l^{(2)}(\gamma') \\ & \cdot F_\sigma(\gamma, \gamma', rb^{-l}, r'b^{-l}, E_1 b^{dl}, E_2 b^{dl}) \end{aligned} \tag{6.8}$$

with

$$\tilde{P}_l^{(s)}(\gamma) = \int \mathrm{d}a \tilde{P}_l(\gamma, a) |a|^s \tag{6.9}$$

and

$$F_\sigma = \sum U_\gamma(R_t^{(l)}, R)\, U_\gamma(R_u^{(l)}, R)\, U_{\gamma'}(R_t^{(l)}, R')\, U_{\gamma'}(R_{u'}^{(l)}, R') \cdot \overline{\psi_j^{(l)}(R_t^{(l)})\psi_k^{(l)}(R_{u'}^{(l)})}\,\overline{\psi_j^{(l)}(R_t^{(l)})\psi_k^{(l)}(R_{u'}^{(l)})} \cdot \overline{\delta(E_j^{(l)} - E_1 b^{dl})\delta(E_k^{(l)} - E_2 b^{dl})}. \tag{6.10}$$

The structure of Equation (6.8) is characteristic for *all* correlation functions G. They consist of integrals which contain factors $\hat{P}_l^{(s)}(\gamma)$ multiplied by some function F depending on the various arguments $\gamma, r b^{-l}$, and $E b^{dl}$. A function $\hat{P}_l^{(s)}(\gamma)$ is associated to each power $\psi^s(r)$ in the correlation function.

The behaviour of $\hat{P}_l^{(s)}(\gamma)$ as a function of l determines the scaling behaviour. Denote the probability for the transition from a configuration (γ', a') during a RG step to (γ, a) in the interval $\mathrm{d}\gamma$ and $a' q \leqq a \leqq a'(q+\mathrm{d}q)$ by $T(\gamma, \gamma', q)$ then one obtains

$$\tilde{P}_l(\gamma, a)\mathrm{d}a\mathrm{d}\gamma = \int T(\gamma, \gamma', q)\, \tilde{P}_{l-1}(\gamma', a')\mathrm{d}\gamma'\,\mathrm{d}a'\,\mathrm{d}q\,\mathrm{d}\gamma. \tag{6.11}$$

With the definition

$$\hat{T}_s(\gamma, \gamma') = \int T(\gamma, \gamma', q)|q|^s \mathrm{d}q \tag{6.12}$$

one has

$$\hat{P}_l^{(s)}(\gamma) = \int \mathrm{d}\gamma'\, \hat{T}_s(\gamma, \gamma')\, \hat{P}_{l-1}^{(s)}(\gamma'). \tag{6.13}$$

The largest eigenvalue λ_s of the kernel $\hat{T}_s$ dominates the behaviour of $\hat{P}_l^{(s)}(\gamma)$ for large l

$$\hat{P}_l^{(s)}(\gamma) \sim \lambda_s^l. \tag{6.14}$$

This power law yields Equations (1.7) with

$$b^{-\kappa} = b^d \lambda_1^2 \tag{6.15a}$$

$$1 = b^d \lambda_2 \tag{6.15b}$$

$$b^{\kappa'} = b^{2d} \lambda_4. \tag{6.15c}$$

The lower eigenvalues yield corrections to the leading scaling behaviour. Whereas Equations (6.15a) and (6.15c) define the exponents κ and κ', Equation (6.15b) is an equality for λ_2. This can be seen as follows: Similarly to Equation (6.8) one obtains

$$G_1(r, r, E) = b^{dl} \int d\gamma\, \hat{P}_l^{(2)}(\gamma) F_1(\gamma, r b^{-l}, E b^{dl}). \tag{6.16}$$

Since the system is homogeneous in space and energy F_1 does not depend on r and E which yields $G \sim b^{dl} \lambda_2^l$. On the other hand from Equation (4.15) which is valid independent of the cell model one obtains $G_1 = 1/\varepsilon_0$. Thus G_1 is independent of l which gives Equation (6.15b).

7. Inhomogeneous Ensemble

The inhomogeneous ensemble differs from the homogeneous ensemble in two important features, (a) the energy scale changes by a factor b^y, (1.4b); there is only one scale of energies (in contrast to the homogeneous model where ω scales like b^d, and τ like b^y), and (b) there is no relevant perturbation. The mobility edge is located at $E = 0$.

The scaling arguments of the last section still apply in the vicinity of the mobility edge provided the factors b^{dl} associated with the energies are replaced by b^{yl}. Thus for

$$|E| b^{yl} \ll \varepsilon_c \tag{7.1}$$

Equations (6.4) and (6.16) hold if d is replaced by y. Replacing l by $l+1$ in this modified Equation (6.16) one has

$$G_1(r b, r b, \tau b^{-y}) = b^{y(l+1)} \int \mathrm{d}\gamma\, \hat{P}_{l+1}^{(2)}(\gamma) F_1(\gamma, r b^{-l}, \tau b^{yl}) \tag{7.2}$$

which yields

$$G_1(r b, r b, \tau b^{-y}) = \lambda_2 b^y G_1(r, r, \tau) \tag{7.3}$$

by means of the asymptotic behaviour (6.14). Summation over r of Equation (7.3) gives for the density of states

$$\rho(\tau b^{-y}) = \lambda_2 b^y \rho(\tau). \tag{7.4}$$

We determine λ_2. The eigenvalues E are multiplied by the factor b^y during each RG step. Thus the number of eigenstates in the interval $\tau \ldots \tau + \mathrm{d}\tau$ obeys the homogeneity relation

$$V\rho(\tau)\mathrm{d}\tau = V^{(1)}\rho(\tau_1)\mathrm{d}\tau_1 = V b^{y-d}\rho(\tau b^y)\mathrm{d}\tau. \tag{7.5}$$

Comparison with Equation (7.4) shows $b^d \lambda_2 = 1$, Equation (6.15b). Equation (7.5) implies the power law

$$\rho(\tau) \sim |\tau|^{(d-y)/y}. \tag{7.6}$$

Using the definitions (6.15a) and (6.15c) one obtains the homogeneity relations

$$C(r, r', \tau) = b^{-\kappa+y-d} C(r b^{-1}, r' b^{-1}, \tau b^y) \tag{7.7a}$$

$$D(0, 0, \tau) = b^{\kappa'+y-2d} D(0, 0, \tau b^y) \tag{7.7b}$$

$$D(r, r', \tau) = b^{y-2d} D(r b^{-1}, r' b^{-1}, \tau b^y) \tag{7.7c}$$

$$G_{\sigma,n}(r, r', \tau) = b^{2(y-d)} G_{\sigma,n}(r b^{-1}, r' b^{-1}, \tau b^y). \tag{7.7d}$$

Recalling Equations (4.3) and (5.21) one easily verifies the power and homogeneity laws given in Section 1.

Appendix A

By means of Schwarz's inequality $\overline{a^2}\,\overline{b^2} \geqq \overline{ab}^2$ we derive the inequality (4.14c). We use the abbreviation $w = \sqrt{n}\,|U_{ji}|$. First we note $b^{-\kappa} = \bar{w}^2$, $1 = \overline{w^2}$, $b^{\kappa'} = \overline{w^4}$. Now $b^{-\kappa} = \bar{w}^2 \leqq \overline{w^2} = 1$, thus $0 \leqq \kappa$. Next $\overline{w^3}\,\bar{w} \geqq \overline{w^2}^2 = 1$ and $\overline{w^4} = \overline{w^4}\,\overline{w^2} \geqq \overline{w^3}^2$ which yields $b^{\kappa-\kappa'} = \overline{w^4}\,\bar{w}^2 \geqq 1$,

thus $\kappa \leqq \kappa'$. Finally

$$b^{\kappa'} = n^2 \overline{U_{ji}^4} = n \sum_{i=1}^{n} \overline{U_{ji}^4} \leqq n \sum_{ii'} \overline{U_{ji}^2 U_{ji'}^2} = n = b^d$$

which yields $\kappa' \leqq d$.

Appendix B

We prove the gauge invariance property for the cell model with $n=2$ stated in Section 5. The states $|r_s\rangle$ and $|r_s'\rangle$ are transformed by the sequence of RG steps into states $|j_s\rangle$ which occur pairwise. If in total an odd number of states $|r\rangle$ and $|r'\rangle$ equal some state $|\tilde{r}\rangle$ then there will be at least one RG step in which in one cell u_1 states $i_l=1$ and u_2 states $i_l=2$ transform into v_1 states $j_l=1$ and v_2 states $j_l=2$ with u_1 and u_2 odd, v_1 and v_2 even. Thus the factor

$$\overline{U_{11}^a U_{21}^b U_{12}^c U_{22}^d} \tag{B.1}$$

occurs with $a+b=u_1, c+d=u_2, a+c=v_1, b+d=v_2$. It follows that $b+c=u_1+v_1-2a$ and $a+d=u_1+v_2-2b$ are both odd. Since the expectation value (B.1) is invariant against the permutation $i \leftrightarrow i'$ and $j \leftrightarrow j'$ independently it equals the average of expression (B.1) and the three expressions obtained in replacing (a, b, c, d) by (b, a, d, c), (c, d, a, b), and (d, c, b, a). Since all 2×2 orthogonal matrices are of the form

$$\begin{pmatrix} \beta & \gamma \\ \mp\gamma & \pm\beta \end{pmatrix}$$

one easily verifies that the sum of these four terms vanishes if both $b+c$ and $a+d$ are odd.

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