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A Geometric Theory for L^2-Stability of the Inverse Problem in a One-Dimensional Elliptic Equation from an H^1-Observation*

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Abstract. This study provides a stability theory for the nonlinear least-squares formulation of estimating the diffusion coefficient in a two-point boundary-value problem from an error-corrupted observation of the state variable. It is based on analysing the projection of the observation on the nonconvex attainable set.

Key Words. Parameter estimation, Nonlinear least squares, Stability analysis, Elliptic boundary-value problems.

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1. Introduction

The purpose of this research is the study of the stability of estimating the diffusion coefficient in a two-point boundary-value problem from possibly error-corrupted data of the state-variable of the equation. The estimation problem is stated as a nonlinear least-squares problem in Hilbert space. A geometrically motivated abstract stability theory developed by the first author is used.

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The following notations and definitions are needed:

E is a normed space with norm $|\cdot|_E$.
$C \subset E$ is a given convex set.
F is a Hilbert space with norm $|\cdot|_F$.
$\varphi: C \to F$.

Definition 1. The problem

$$\text{(P)} \quad \text{Min } J(x) = |\varphi(x) - z|_F^2 \quad \text{over } C$$

is said to be quadractically well-posed (Q-well-posed) in an open neighborhood $\mathscr{V}$ of $\varphi(C)$ for the norm $|\cdot|_E$ if:

(i) (P) has a unique solution $\hat{x}$ for any $z \in \mathscr{V}$.
(ii) J has no local minimum for any $z \in \mathscr{V}$.
(iii) Any minimizing sequence converges to $\hat{x}$ in the norm of E.
(iv) The mapping $z \to \hat{x}$ is locally Lipschitz continuous from $(\mathscr{V}, |\cdot|_F)$ to $(C, |\cdot|_E)$.

While Q-stability is a desirable property for nonlinear least-squares problems—it allows, for example, the data z to be outside of $\varphi(C)$—it requires strong hypotheses which imply this property. It is the purpose of this paper to give an example in which Q-stability with an infinite-dimensional set C holds and to provide precise estimates for the geometrical quantities required.

The problem under investigation is

$$-(au_x)_x = f \tag{1.1}$$

with $a \in C$, where

$$C = \{a: [0, 1] \to \mathbb{R} | a \text{ is measurable}, 0 < a_m \le a(x) \le a_M, \text{ a.e. on } [0, 1]\}$$

and the data are assumed to correspond to u_x. The theory in [4], [5], and [2] requires us to connect all points of the parameter set C with a path. We shall see that the simple path of a segment between a_0 and a_1 given by $t \to (1 - t)a_0 + ta_1$ for $t \in [0, 1]$ is useless, since it leads to unbounded curvature in the image space. The correct parametrization is described in Section 3. Sections 4 and 5 consider the case of a rough source (f equal to a linear combination of delta functions) and of a smooth source ($f \in L^2$), respectively. It is necessary to provide additional information in the neighborhood of the singular points of the output u_x. Here we refer to a point $\bar{x}$ as singular if u_x does not exist at $\bar{x}$ (rough case) or if $u_x(\bar{x}) = 0$ (smooth case). The extra information which is required is that the coefficient a is constant in the neighborhood of singular points. An essential ingredient to obtain the desired stability results is the availability of a lower bound of the linearization of the parameter to output mapping. In this respect we rely on recent results from [6].

For another application of the geometrical theory of [4], [5], and [2] we refer to a paper by Symes [10] concerned with plane wave detection.

For related results on the identification of the diffusion coefficient in elliptic equations we refer to [1], and [7]–[9], for example. The main difference between these contributions and ours is given by the fact that the observation z in this paper need not be obtainable, i.e., z need not necessarily be an element of $\varphi(C)$. This situation occurs due to modeling and measurement errors.

The assumption on the availability of u_x as opposed to, e.g., u as data, together with appropriate input functions f, guarantees the existence of a continuous inverse to $a \to u_x(a)$ from L^2 to L^2, if a is restricted to lying in an appropriate subset of L^2 (see Sections 5.2 and 5.3). The case of data z for u in L^2 can only be treated using regularization techniques and will be studied elsewhere.

Extension of the results of this paper to the multidimensional case is not easy. The motivation for examining the one-dimensional case independently in this paper is that, due to an explicit formula for the solution of (1.1), we can derive precise stability estimates.

2. An L^2-Stability Estimate for H^1-Observations

We begin by a stability estimate which is crucial for the subsequent proofs, and which is a reformulation of Theorem 2.5 of [6].

Let us first remark that if u and a satisfy the $1-D$ elliptic equation for some given f,

$$-(au_x)_x = f, \qquad 0 \le x \le 1, \tag{2.1}$$

then u will still satisfy the same equation with a replaced by $a + k/u_x$, k small enough (provided this makes sense). Hence the determination of a from u using (2.1) only is underdetermined, the problem being caused by pairs of coefficients a whose difference is proportional to $1/u_x$. This should be reflected in the stability estimate to come.

So let $(a_j, u_j) \in L^\infty(0, 1) \times H^1(0, 1)$, $j = 0, 1$, satisfy (2.1). Calculating the difference of the two equations and integrating once yields

$$(a_1 - a_0)u_{0x} = a_1(u_{0x} - u_{1x}) + \text{an unknown constant.} \tag{2.2}$$

Lemma 1. *Let $d \in L^2(0, 1)$, $w \in L^\infty(0, 1)$, $h \in L^2(0, 1)$ satisfy*

$$\frac{d}{w} = h + \textit{an unknown constant.} \tag{2.3}$$

Then

$$\frac{|w|}{|w|_\infty} \sin\psi \left|\frac{d}{w}\right| \le |h|, \tag{2.4}$$

where

$$\begin{cases} \psi \in \left[0, \frac{\pi}{2}\right] \textit{ is the angle between directions } d \textit{ and } w, \\ \sin\psi = \sqrt{1 - \dfrac{\langle d, w\rangle^2}{|d|^2 |w|^2}}. \end{cases} \tag{2.5}$$

Proof. If we denote by $L^2(0, 1)/\mathbb{R}$ the space of class of functions defined up to a constant, we know that

$$\left|\frac{d}{w}\right|_{L^2/\mathbb{R}} = \inf_{\text{cst} \in \mathbb{R}} \left|\frac{d}{w} + \text{cst}\right| \le |h|$$

and

$$\left|\frac{d}{w}\right|^2_{L^2/\mathbb{R}} = \left|\frac{d}{w}\right|^2 - \left(\int_0^1 \frac{d}{w}\right)^2$$

so that

$$\left|\frac{d}{w}\right|^2_2 - \left(\int_0^1 \frac{d}{w}\right)^2 \leq |h|^2. \tag{2.6}$$

Let us now define y as the projection of w onto the subspace of $L^2(0, 1)$ orthogonal to d. Of course,

$$\langle d, y\rangle = 0, \tag{2.7}$$

$$|y| = \sin \psi |w|. \tag{2.8}$$

For any $k \in \mathbb{R}$, (2.6) can be rewritten in, view of (2.7), as

$$\left|\frac{d}{w}\right|^2 - \left(\int_0^1 d\left(\frac{1}{w} + ky\right)\right)^2 \leq |h|^2, \qquad k \in \mathbb{R}. \tag{2.9}$$

However,

$$\begin{aligned}\left(\int_0^1 d\left(\frac{1}{w} + ky\right)\right)^2 &\leq \left|\frac{d}{w}\right|^2 \left\{1 - 2k \int_0^1 wy + k^2 \int_0^1 v^2y^2\right\} \\ &\leq \left|\frac{d}{w}\right|^2 \left\{1 - 2k \int_0^1 wy + k^2 |w|^2_\infty |y|^2_2\right\}.\end{aligned}$$

Choosing for k the value that minimizes the second-order polynomial on the right-hand side yields

$$\left(\int_0^1 d\left(\frac{1}{w} + ky\right)\right)^2 \leq \left|\frac{d}{w}\right|^2 \left\{1 - \frac{|w|^2_2}{|w|^2_\infty} \sin^2 \psi\right\}.$$

Plugging the last inequality in (2.9) yields the expected result (2.4). □

We can now apply Lemma 1 to (2.2), provided we suppose that we know *a priori* lower and upper bounds to u_x:

$$0 < u_m \leq u_x(x) \leq u_M, \tag{2.10}$$

which yield the stability estimate

$$\frac{u_m}{u_M} \sin \psi |(a_1 - a_0)u_{0x}| \leq |a_1(u_{0x} - u_{1x})|, \tag{2.11}$$

where

$$\psi = \text{angle between the directions of } a_1 - a_0 \text{ and } \frac{1}{u_{0x}}. \tag{2.12}$$

As expected, this estimate vanishes when $a_1 - a_0$ becomes proportional to $1/u_{0x}$!

Besides Lemma 1 which we apply later to a reparametrization of the same problem, the useful findings of this section are that we can obtain an explicit L^2-Lipschitz stability estimate for a from an H^1-observation of u provided that:

- u_x can be bounded to stay away from zero as in (2.10).
- The angle between two admissible parameters and $1/u_x$ can also be bound to stay away from zero.

3. A Size × Curvature Condition for the Well-Posedness of Nonlinear Least-Squares Problems

In this section we describe a sufficient condition for the Q-well-posedness of our abstract nonlinear least-squares problem, based on the quasi-convexity and size × curvature conditions approach developed by Chavent in [4], [5], and [2].

Let

$$\begin{cases} C = \text{convex subset of some vector space} \quad & \text{(admissible set)}, \\ F = \text{Hilbert space} \quad & \text{(observation space)}, \\ z \in F \quad \text{(data)}, \\ \varphi: C \to F \quad & \text{(input} \to \text{output mapping)} \end{cases} \tag{3.1}$$

be given. We consider in this section *the nonlinear least-squares problem*

$$\text{find} \quad \hat{x} \in C \quad \text{which minimizes} \quad J(x) = |\varphi(x) - z|_F^2 \quad \text{over } C. \tag{3.2}$$

We only require from φ that it is regular along any segment of C, precisely:

$$\forall x_0, x_1 \in C, \qquad P: t \in [0, 1] \to P(t) = \varphi((1-t)x_0 + tx_1) \quad \text{is in } W^{2,\infty}(0, 1). \tag{3.3}$$

We call P a *path* in $\varphi(C)$, and throughout the paper we use the notation

$$\begin{aligned} V(t) &= P'(t) \in F \qquad \text{(velocity along the path)}, \\ A(t) &= P''(t) \in F \qquad \text{(acceleration along the path)}. \end{aligned} \tag{3.4}$$

Of course, $V(t)$ and $A(t)$ are implicitly related to the path P associated to (x_0, x_1) which will always be clear from the context.

Suppose now we have been able to find some Banach space E, with a norm $|\cdot|_E$, such that

$$C \subset E, \qquad C \text{ closed set in } E, \tag{3.5}$$

$$\begin{cases} \text{there exists} \quad 0 < \alpha_m \le \alpha_M \\ \text{such that} \quad \forall x_0, x_1 \in C \quad \text{and for a.e.} \quad t \in]0, 1[\\ \alpha_m |x_1 - x_0|_E \le |V(t)|_F \le \alpha_M |x_1 - x_0|_E, \end{cases} \tag{3.6}$$

$$\begin{cases} \text{there exists} \quad \Theta > 0 \quad \text{and} \quad R > 0 \\ \text{such that} \quad \forall x_0, x_1 \in C \quad \text{and for a.e.} \quad t \in]0, 1[\\ \dfrac{|A(t)|}{|V(t)|} \le \Theta, \qquad \dfrac{|A(t)|}{|V(t)|^2} \le \dfrac{1}{R}. \end{cases} \tag{3.7}$$

We comment first on hypothesis (3.6). In the finite-dimensional case α_m and α_M are lower and upper bounds to the singular values of $\varphi'(x)$ for all x of C. Hence (3.6) corresponds in some sense to the fact that the "linearized" least-squares problem is well-posed for any linearization point x of C. It allows us to obtain stability results on C for the $|\cdot|_E$ norm from stability results on $\varphi(C)$ for the arc length v along a path P (remember that $dv = |V(t)|\,dt$).

We now comment on hypothesis (3.7). The quantities Θ and R have a geometrical interpretation [5]: *Θ is an upper bound to the deflection* (i.e., angle of tangents) between any two points of any path P of the shape (3.3), and *R is a lower bound to the usual radius of curvature* along any path P of the shape (3.3).

We may then define an upper bound Δ to the length of any path P by

$$\Delta = \alpha_M \operatorname{diam} C. \tag{3.8}$$

As the geometrical intuition shows us that the deflection along a path P is necessarily smaller than Δ/R (see [5] for the proof), we suppose in what follows that the upper bound Θ to the deflection of paths given in (3.7) is at least as good as Δ/R, i.e., that

$$\Theta = \frac{\tau\Delta}{R}, \qquad 0 \le \tau \le 1, \tag{3.9}$$

where τ is called the "shape coefficient" of the estimation (3.6), (3.7), see [3].

Remark 1. Hypotheses (3.7) and (3.9) are satisfied if the following majorization holds for the acceleration $A(t)$:

$$\begin{cases} \text{there exists} \quad \beta > 0 \quad \text{such that} \\ \forall x_0, x_1 \in C \quad \text{and for a.e.} \quad t \in]0, 1[\\ |A(t)|_F \le \beta |x_0 - x_1|_E^2, \end{cases} \tag{3.10}$$

with Θ, R, and τ defined by

$$\begin{cases} \Theta = \left(\dfrac{\beta}{\alpha_m}\right) \operatorname{diam} C, \\ R = \dfrac{\alpha_m^2}{\beta}, \\ \tau = \dfrac{\alpha_m}{\alpha_M}. \end{cases} \tag{3.11}$$

If φ were twice differentiable on (C, E), then β would be an upper bound to $\|\varphi''(x)\|$ for $x \in C$. Notice also from (3.11) that the upper bound Θ to the deflection of $\varphi(C)$ can be made arbitrarily small by reducing the diameter of C (and hence of $\varphi(C)$ because of (3.8)!).

The numbers Θ, R, and τ give information on the shape of the set $\varphi(C)$, which is useful for the projection of z onto $\varphi(C)$, which is one of the steps involved in the solution of the nonliner least-squares problem (3.2). However, the relevant quantity for the Q-well-posedness of this projection is neither Θ nor R, but rather the smallest *global radius of curvature* between any two points of any path P defined in (3.3). We refer to [5] for the precise definition of this notion, and recall here only how to obtain a *a lower bound R_G to all these global radii of curvature*:

$$R_G = \begin{cases} R & \text{if } 0 \le \Theta \le \dfrac{\pi}{2}, \\ R(\sin\Theta + (\tau^{-1} - 1)\Theta\cos\Theta) & \text{if } \dfrac{\pi}{2} \le \Theta < \pi. \end{cases} \tag{3.12}$$

Of course, formula (3.12) has to be taken here as a definition. We have illustrated in Figure 3.1 the function $\Theta \to R_G$ for given R, and τ, and *define the maximum deflection Θ_M by*

$$\Theta_M = \text{unique solution in } \left]\frac{\pi}{2}, \pi\right[\text{ of the equation } \tan\Theta + (\tau^{-1} - 1)\Theta = 0,$$

which is an *upper bound to deflection values which ensure* $R_G > 0$. (3.13)

In Figure 3.2 we give numerical values of this maximum deflection Θ_M for values of the shape coefficient τ ranging in $[0, 1]$. It can be seen that Θ_M becomes close to $\pi/2$ very quickly when $1/\tau$ becomes larger than a few units. When Remark 1 applies, $1/\tau = \alpha_M/\alpha_m$ is an upper bound to the condition number of the linearized problems, so that we can expect a value of Θ_M (larger than but) close to $\pi/2$ in all applications whose condition number α_M/α_m is larger than, say, five.

We now state the size $\times$ curvature condition for the Q-well-posedness of the nonlinear least-squares problem:

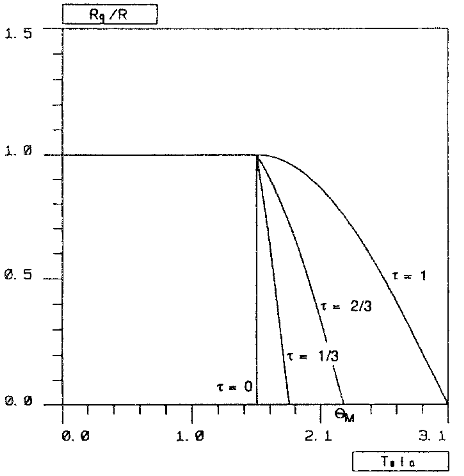

Figure 3.1. The lower bound R_G to the global radii of curvature as a function of the upper bound Θ to deflection, for various values of the shape coefficient τ.

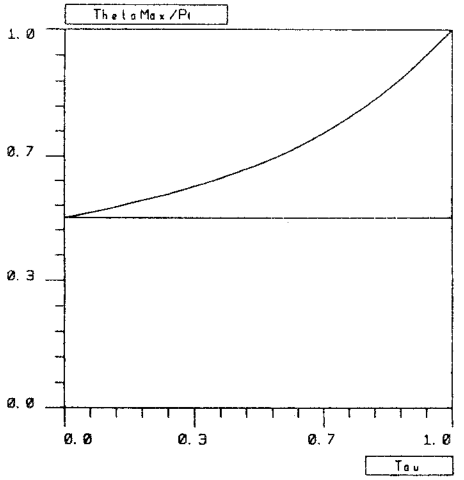

Figure 3.2 The maximum deflection Θ_M as a function of the shape coefficient τ.

Theorem 3.1 *Let C, F, and φ be given satisfying* (3.1), (3.3), (3.5), (3.6), (3.7), *and* (3.9) *and let R_G* and *Θ_M be defined by* (3.12) *and* (3.13).

If the **deflection size × curvature condition**

$$\Theta < \Theta_M \tag{3.14}$$

is satisfied, then the nonlinear least-squares problem (3.2) *is Q-well-posedness on the neighborhood*

$$\mathscr{V} = \{z \in F \mid d(z, \varphi(C)) < R_G\} \tag{3.15}$$

for the $|x_0 - x_1|_E$ distance on C, and the following stability estimate holds:

$$\alpha_m |\hat{x}_1 - \hat{x}_0|_E \le \int_0^1 |V(t)|_F \, dt \le \frac{1}{1 - \chi(R_G/R)} |z_1 - z_0|_F \tag{3.16}$$

as soon as z_0 and z_1 satisfy

$$|z_0 - z_1|_F + \max_{j=0,1} d(z_j, \varphi(C)) \le \chi R_G \qquad \textit{for some} \quad 0 < \chi < 1. \tag{3.17}$$

In (3.16), $V(t)$ is the velocity along the path P associated to $\hat{x}_0$ and $\hat{x}_1$.

Notice first (see Figure 3.1) that the size × curvature condition (3.14) will hold as soon as either $0 \le \Theta \le \pi/2$ (and then $R_G = R$) or $\pi/2 < \Theta < \Theta_M$ (and then $R_G < R$ becomes close to zero when Θ becomes close to Θ_M). However, as we have seen in Remark 1, Θ is usually proportional to the size of C. So there is some balancing between the size of the set C of admissible parameters and the upper bound R_G to the size of errors on the data which yield a Q-well-posed least-squares problem (3.2).

Notice also that condition (3.17) for the stability estimate (3.16) to hold means simply that z_0 and z_1 have to be close together and that they are sufficiently near $\varphi(C)$. For $\Theta < 1$ it also requires that the observations are bounded away from the boundary of $\mathscr{V}$. Of course, the Lipschitz constant of the stability estimate deteriorates when $\chi \to 1^-$ and it blows up if in addition $R_G = R$ ($\chi = 1$ allows that z_i approaches some center of curvature of $\varphi(C)$).

To conclude this section, let us anticipate how Theorem 3.1 could be applied to the estimation of the diffusion coefficient a in an elliptic equation: comparing (2.11) and (3.6) shows that there is a chance of applying this theorem with $E = L^2(0, 1)$, as (2.11) will easily give us the first inequality of (3.6) with $\alpha_m > 0$. It remains to check if the other inequalities in (3.6), (3.7), and (3.9) will hold with $E = L^2(0, 1)$. We discuss these matters in the next sections.

4. The Boundary Source Case: How To Learn Something from a Trivial Case

In this section we consider the one-dimensional elliptic equation

$$-(au_x)_x = 0, \qquad 0 < x < 1, \tag{4.1}$$

together with the boundary conditions

$$u(0) = 0, \qquad a(1)u_x(1) = g, \tag{4.2}$$

where

$$g \in \mathbb{R} \tag{4.3}$$

is a given boundary injection rate.

We remark first that the boundary condition at $x = 1$ suppresses the under-determination inherent to (4.1) itself which was pointed out in Section 2.1. Thus the estimation of a from a measurement of u has a chance of being better behaved.

We also remark that the (unique) solution to (4.1), (4.2) can be given by a very simple explicit formula:

$$u(x) = g \int_0^x \frac{dy}{a(y)}, \qquad 0 \leq x \leq 1. \tag{4.4}$$

We now consider the estimation of a in (4.1), (4.2) from an H^1-observation of u, i.e., from a a measurement z of u_x: Given

$$0 < a_m \leq a_M \tag{4.5}$$

we define

$$C = \{a\colon [0, 1] \to \mathbb{R} \,|\, a \text{ measurable, } a_m \leq a(x) \leq a_M \text{ for a.e. } x \in [0, 1]\}$$
$$\text{(set of admissible parameters),} \tag{4.6}$$

$$F = L^2(0, 1) \qquad \text{(observation space),} \tag{4.7}$$

$$\varphi\colon C \to F \quad \text{defined by} \quad \varphi(a) = u_x = \frac{g}{a}, \qquad \forall a \in C$$
$$\text{(parameter} \to \text{output mapping).} \tag{4.8}$$

Then to any observation

$$z \in F \tag{4.9}$$

we associate the error function

$$J(a) = \| \varphi(a) - z \|_F^2 = \int_0^1 |u_x - z|^2 \tag{4.10}$$

and estimate the corresponding a by solving the nonlinear least-squares problem

$$\text{find} \quad \hat{a} \in C \quad \text{which minimizes } J(a) \text{ over } C. \tag{4.11}$$

Of course, this problem is trivial because the mapping φ has a very simple analytical form, and we should be able to apply Theorem 3.1 without any difficulty.

So we estimate the coefficients α_m, α_M, and β defined in (3.6) and (3.9): Given $a_0, a_1 \in L^\infty$ and $t \in [0, 1]$, we have

$$|V(t)|_F = |g| \left\{ \int_0^1 \frac{(a_1 - a_0)^2}{((1-t)a_0 + ta_1)^4}\, dx \right\}^{1/2}, \tag{4.12}$$

$$|A(t)|_F = |g| \left\{ \int_0^1 \frac{(a_1 - a_0)^4}{((1-t)a_0 + ta_1)^6}\, dx \right\}^{1/2}. \tag{4.13}$$

As we expected at the end of Section 3, choosing $E = L^2(0, 1)$ yields the first estimate of (3.6) with $\alpha_m = |g|/a_M^2$. However, estimate (3.9) on $|A(t)|_F$ has no chance to hold, as $(a_1 - a_0)^2$ will never be in $L^2(0, 1)$ when a_0, a_1 are only in $L^2(0, 1)$! (Basically, the problem comes from the fact that $a \to 1/a$ is not twice differentiable on $L^2(0, 1)$.)

So Theorem 3.1 does not apply to problem (4.6)–(4.10), but, obviously, it was silly to take a as an unknown parameter when u is proportional to $1/a$! So we make the one-to-one change of the unknown parameter

$$b = \frac{1}{a} \tag{4.14}$$

and define our set of admissible parameters

$$D = \{b: [0, 1] \to \mathbb{R} \mid b \text{ measurable}, b_m \le b(x) \le b_M \text{ for a.e. } x \in [0, 1]\}, \tag{4.15}$$

where, of course,

$$b_m = \frac{1}{a_M} > 0, \qquad b_M = \frac{1}{a_m} < +\infty. \tag{4.16}$$

Then the analytical form of φ simplifies further to

$$\varphi(b) = gb, \tag{4.17}$$

so that φ is perfectly linear, and hence $\varphi(D)$ is convex (which shows that $\varphi(C)$ is convex too!). Then Theorem 3.1 immediately applies with

$$\alpha_m = |g| = \alpha_M, \qquad \beta = 0. \tag{4.18}$$

Hence we have proved:

Theorem 4.1. *Let b_m, b_M, and g be given by* (4.16), (4.15), *and* (4.13) *and let D and φ be defined by* (4.15) *and* (4.17). *Then the least-squares problem*

$$\text{find } \hat{b} \in D \text{ which minimizes } J(b) = |\varphi(b) - z|^2 \text{ over } D \tag{4.19}$$

for the estimation of b from the measurement of u_x in L^2 is Q-well-posedness on $\mathcal{V} = F = L^2(0, 1)$ for the $|b_1 - b_0|_{L^2}$ distance on D.

The interest of this result for what follows is that we cannot use a as a parameter if we want to use the technique of Section 3 to obtain an L^2 well-posedness result, and that $b = 1/a$ is a better candidate for that purpose.

5. The Dirac Source Case

5.1. Setting the Problem

Following the suggestion made at the end of Section 4, we take as the unknown parameter $b = 1/a$ throughout the remainder of this paper. We consider in this section the one-dimensional elliptic equation (2.1), but with a source term f made of a combination of Dirac functions:

$$-(b^{-1}u_x)_x = \sum_{i\in J} f_i\delta(x - x_j), \qquad 0 < x \leq 1, \tag{5.1}$$

which we complement this time with Dirichlet boundary conditions

$$u(0) = u(1) = 0, \tag{5.2}$$

where, of course,

$$\begin{cases} J \text{ is a finite set of indices,} \\ x_j \in]0, 1[\text{ denotes the location of the } j\text{th source,} \\ f_i \in \mathbb{R} \text{ denotes the amplitude of the } j\text{th source.} \end{cases} \tag{5.3}$$

Different from the boundary source case of Section 4, the boundary conditions (5.2) do not allow any explicit information on the unknown parameter b. Hence we expect some underdetermination for the determination of b if $1/u_x$ happens to be bounded as seen in Section 2. We take advantage of the existence of an explicit solution to (5.1), (5.2):

$$u(x) = -\int_0^x b(y)\{H(y) - \bar{H}_b\}\, dy, \tag{5.4}$$

where

$$H(x) = \int_0^x \sum_{j\in J} f_i\delta(y - x_j)\, dy \qquad \text{(primitive of the right-hand side)}, \tag{5.5}$$

$$\bar{H}_b = \frac{\int_0^1 b(y)H(y)\, dy}{\int_0^1 b(y)\, dy} \qquad \text{(b-weighted mean value of H).} \tag{5.6}$$

We first consider the same set of admissible parameters as in (4.15):

$$D = \{b\colon [0, 1] \to \mathbb{R} \mid b \text{ measurable, } b_m \leq b(x) \leq b_M \text{ for a.e. } x \in [0, 1]\}, \tag{5.7}$$

where

$$0 < b_m \leq b_M \tag{5.8}$$

are known lower and upper bounds to b, and suppose we are able to find some measure of the solution u in $H_0^1(0, 1)$, or, equivalently, as $(|u| + |u_x|^2)^{1/2}$ and $|u_x|$ are equivalent norms on $H_0^1(0, 1)$, of its derivative u_x in $L^2(0, 1)$. Hence we take as data space

$$F = L^2(0, 1) \tag{5.9}$$

and define the parameter $\to$ output mapping φ by

$$\varphi\colon b \in D \to \varphi(b) = u_x = -b\{H - \bar{H}_b\} \in F. \tag{5.10}$$

Now given

$$z \in F = L^2(0, 1) \tag{5.11}$$

we consider the problem of estimating b in D from the data z. As we have seen in Section 2, we need a lower bound to $|u_x|$ in order to apply the stability lemma (Lemma 1). This can be easily achieved in our case by supposing that the experimental device (i.e., the sources) satisfies, for some $H_m, H_M \in \mathbb{R}$

$$0 < H_m \leq |H(x) - \bar{H}_b| \leq H_M, \qquad \forall x \in [0, 1], \quad \forall b \in D. \tag{5.12}$$

(Notice that (5.12) will be automatically satisfied if, for example, a finite number of sources and sinks of the same amplitude are located in an alternate way from the left to the right.) We define

$$\mathcal{H} = \frac{H_m}{H_M} \times \frac{b_m}{b_M} \tag{5.13}$$

and note that $\mathcal{H}$ converges to a constant if $b_M/b_m \to 1$ ("homogeneous case") and $\mathcal{H} \to 0$ when $b_M/b_m \to +\infty$ ("heterogeneous case"). In order to obtain a stability result for b in a weighted L^2-norm, we also define

$$h(x) = \inf_{b \in D} |H(x) - \bar{H}_b|, \qquad \forall x \in [0, 1], \tag{5.14}$$

which satisfies

$$\begin{cases} h(x) \geq H_m > 0, & \forall x \in [0, 1], \\ |h|_{L^2} \geq H_m > 0. \end{cases} \tag{5.15}$$

Hypothesis (5.12) immediately yields, using (5.4) and (5.7), the sought for lower bound to $|u_x|$,

$$|u_x(x)| \geq b_m H_m \qquad \text{for a.e.} \quad x \in [0, 1], \quad \forall b \in D, \tag{5.16}$$

which is required to apply Lemma 1, but at the same time (5.16) shows that $1/u_x$ is bounded, which leads to difficulties as it makes it possible for two different elements b of D to yield exactly the same solution u of (5.1), (5.2)!

5.2. *Using the Stability Estimate*

We now obtain, for $b = 1/a$, a stability estimate similar to the one we had in (2.11) for a. For any two $b_0, b_1 \in D$ we obtain (compare with (2.2))

$$\frac{b_0 - b_1}{b_1} \{H - \bar{H}_{b_0}\} = \frac{u_{0x} - u_{1x}}{b_1} + \text{an unknown constant}. \tag{5.17}$$

Then we have

Lemma 2. *Let $b_0, b_1 \in D$ and u_0, u_1 be the corresponding solutions to the Dirichlet problem* (5.1), (5.2), *and suppose that hypotheses* (5.8) *and* (5.12) *hold. Then*

$$\mathcal{H} \sin \psi |(b_1 - b_0)(H - \bar{H}_{b_0})| \leq |u_{1x} - u_{0x}|, \tag{5.18}$$

where $\mathcal{H}$ is defined in (5.13), *and*

$$\psi \in \left[0, \frac{\pi}{2}\right] \quad \text{is the angle between the directions } \frac{b_0 - b_1}{b_1} \text{ and } \{H - \bar{H}_{b_0}\}^{-1}. \tag{5.19}$$

Proof. Applying Lemma 1 to (5.17) with $d = (b_0 - b_1)/b_1$ and $w = \{H - \bar{H}_{b0}\}^{-1}$, noticing that

$$|d| \geq H_M^{-1}, \qquad |w|_\infty \leq H_m^{-1},$$

and defining ψ as in (5.19) yields

$$\frac{H_m}{H_M} \sin \psi \left| \frac{b_0 - b_1}{b_1} \{H - \bar{H}_{b_0}\} \right| \leq \left| \frac{u_{0x} - u_{1x}}{b_1} \right|,$$

which in turn yields (5.18) using (5.8) and (5.13). □

As we noticed in Section 4.1, estimate (5.18) vanishes if $(b_0 - b_1)/b_1$ happens to be proportional to the piecewise constant function $\{H - \bar{H}_b\}^{-1}$: the problem of estimating b in D from a measurement of u_x is underdetermined, or in other terms the parameter → output mapping φ is not injective. We take care of this in the next section.

5.3. *Eliminating the Underdetermination*

There are two ways for handling the noninjectivity of φ:

- either it is decided to live with it, so that the search for b is replaced by the search for (connected components of) equivalence classes of b's [5]
- or some additional information is added in order to suppress the underdetermination as, for example, in the regularization technique.

We follow the second approach here, but rather than adding a general-purpose regularizing term, we add the minimum amount of information that it is needed to suppress the underdetermination. The idea is to prevent $(b_0 - b_1)/b_1$ and $\{H - \bar{H}_{b_0}\}^{-1}$ becoming proportional, or better, in view of Lemma 2, to prevent the angle between the corresponding directions becoming smaller than some $\psi_m > 0$. This is done by noticing that $\{H - \bar{H}_{b_0}\}^{-1}$ necessarily has a discontinuity at each point source $x_j, j \in J$, and by requiring that $(b_0 - b_1)/b_1$ is constant around some (at least one!) of the source points x_j. So we define

$\tilde{J} \subset J$ nonempty subset of indexes of source points at which additional information on b is known, (5.20)

$\eta = (\eta_j > 0, j \in \tilde{J})$ vector of radii of balls on which information on b is known, (5.21)

which of course are supposed to satisfy

$$I_j =]x_j - \eta_j, x_j + \eta_j[\subset]0, 1[, \qquad \forall j \in \tilde{J}, \tag{5.22}$$

$$I_{j_1} \cap I_{j_2} = \varnothing, \qquad \forall j_1, j_2 \in \tilde{J}, \quad j_1 \neq j_2. \tag{5.23}$$

We may then define *our new admissible parameter set* by

$$D_\eta = \{b \in D \mid \forall j \in \tilde{J}, b(x) = b_j = \text{an unknown constant on } I_j\} \tag{5.24}$$

for which we have

Lemma 3. *Let hypotheses* (5.8) *and* (5.12) *of Section* 5.1 *hold, and let* D_η *be defined by* (5.20)–(5.24). *First we have*

$$0 \leq 1 - \frac{1}{4} \sum_{j \in \tilde{J}} \eta_j \Delta_j^2 < 1, \tag{5.25}$$

where

$$\Delta_j = \frac{|f_i| H_m}{H_M^2}, \qquad \forall j \in \tilde{J}, \tag{5.26}$$

which allows us to define the **underdetermination angle** ψ_m *by*

$$\begin{cases} 0 < \psi_m \leq \dfrac{\pi}{2}, \\ \cos \psi_m = 1 - \frac{1}{4} \sum\limits_{j \in \tilde{J}} \eta_i \Delta_j^2. \end{cases} \tag{5.27}$$

Moreover, for any $b_0, b_1 \in D_\eta$, *the angle* ψ *between the directions of* $(b_0 - b_1)/b_1$ *and* $\{H - \bar{H}_{b_0}\}^{-1}$ *satisfies*

$$\psi \geq \psi_m > 0. \tag{5.28}$$

Proof. Let $b_0, b_1 \in D_\eta$ be given. The angle ψ between the direction of $(b_0 - b_1)/b_1$ and $\{H - \bar{H}_{b_0}\}^{-1}$ is the angle between the unit vectors c and v defined by

$$c = \frac{(b_0 - b_1)/b_1}{|(b_0 - b_1)/b_1|}, \qquad v = \pm \frac{\{H - \bar{H}_{b_0}\}^1}{|\{H - \bar{H}_{b_0}\}^{-1}|}, \tag{5.29}$$

where the $\pm$ sign has been chosen such that

$$\langle c, v \rangle \geq 0. \tag{5.30}$$

So we have, by definition of c and v,

$$|c| = |v| = 1, \tag{5.31}$$

$$\cos \psi = \langle c, v \rangle \geq 0. \tag{5.32}$$

The latter equation rewrites, using the median theorem, as

$$\cos \psi = 1 - \tfrac{1}{2}|c - v|^2 \geq 0. \tag{5.33}$$

In order to find a lower bound to ψ, we have to find a lower bound to $|c - v|^2$. However, c and v have a very simple shape over each I_j interval surrounding a source point x_j where $j \in \tilde{J}$ (see Figure 5.1):

$$c(x) = c_j = \text{constant}, \qquad \forall x \in I_j \tag{5.34}$$

(by definition of D_η),

$$v(x) = \begin{cases} v_j^- = \text{constant}, & \forall x \in I_j, \quad x < x_j, \\ v_j^+ = \text{constant}, & \forall x \in I_j, \quad x > x_j \end{cases} \tag{5.35}$$

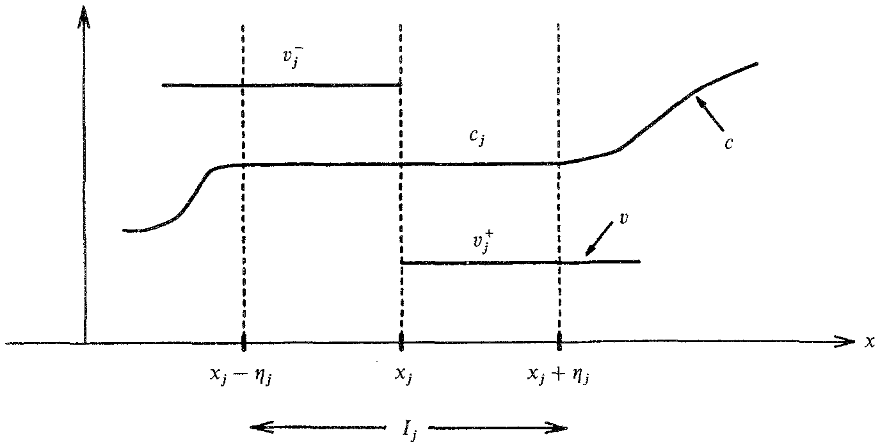

Figure 5.1. Behavior of c and v on the I_j interval for $j \in \tilde{J}$.

(by definition of the H function),
and a simple calculation shows that

$$|v_j^+ - v_j^-| \geq \Delta_j, \qquad \forall j \in J, \tag{5.36}$$

where Δ_j is defined in (5.26). Hence, from hypotheses (5.22) and (5.23) we obtain

$$\begin{aligned} |c - v|^2 &\geq \sum_{j \in \tilde{J}} \int_{I_j} |c(x) - v(x)|^2 \, dx \\ &= \sum_{j \in \tilde{J}} \eta[(c_j - v_j^-)^2 + (c_j - v_j^+)^2]. \end{aligned}$$

The second-order polynomial in c_j inside the bracket is minimum for

$$c_j = (v_j^- + v_j^+)/2.$$

Hence, using (5.36), we obtain

$$|c - v|^2 \geq \tfrac{1}{2} \sum_{j \in \tilde{J}} \eta_j \Delta_j^2. \tag{5.37}$$

However, because of (5.30) and (5.31) we have

$$|c - v|^2 = |c|^2 + |v|^2 - 2\langle c, v \rangle \leq 2$$

which, together with (5.37) proves (5.25), and allows us to define ψ_m by (5.27). From (5.33) and (5.37) we obtain

$$\cos \psi \leq 1 - \tfrac{1}{4} \sum_{j \in \tilde{J}} \eta_j \Delta_j^2,$$

i.e., using (5.27)

$$\cos \psi \leq \cos \psi_m,$$

which proves (5.28). □

At this point we already have a precise *stability estimate for the estimation of b in D_η in the zero-residual case* by combining Lemmas 2 and 3:

$$\mathscr{H} \sin \psi_m |(b_1 - b_0)(H - \bar{H}_{b_0})| \leq |\varphi(b_1) - \varphi(b_0)|, \qquad \forall b_0, b_1 \in D_\eta. \tag{5.38}$$

We combine in the two next sections this estimate with the geometrical approach of Section 3 in order to obtain a stability result for the nonzero residual case.

5.4. *Estimation of the Geometrical Quantities Associated to D_η and φ*

In this section we check the prerequisites for application of the geometric theory of Section 3 to the problem

$$\text{find} \quad \hat{b} \in D_\eta \quad \text{which minimizes} \quad J(b) = |\varphi(b) - z|_F^2 \quad \text{over } D_\eta, \tag{5.39}$$

where D_η is defined in (5.24), φ in (5.10), and z in (5.11). Given b_0, b_1 in D_η, $b_0 \neq b_1$, we define a path P in $\varphi(D_\eta)$ by

$$\begin{cases} P(t) = \varphi(b_t), & \forall t \in [0, 1], \\ b_t = (1 - t)b_0 + tb_1, & \forall t \in [0, 1]. \end{cases} \tag{5.40}$$

We notice first that P is infintely differentiable from $[0, 1]$ to F, as φ is well known to be infintely differentiable from D equipped with the $L^\infty(0, 1)$ norm to $F = L^2(0, 1)$. Hence,

$$V(t) = P'(t), \qquad A(t) = P''(t) \tag{5.41}$$

exist for any $t \in [0, 1]$. We choose a norm $|\cdot|_E$ on D_η such that E is a Banach space, D_η is a closed (convex) subset of E, and estimations (3.6)–(3.8) on $V(t)$ and $A(t)$ hold. In view of the L^2-stability estimate (5.38) obtained in Section 5.3, a proper choice for E is

$$E = L^2(0, 1) \tag{5.42}$$

for which D_η is clearly a closed (convex) subset. We are left with the estimation of the constants α_m, α_M of (3.6) and Θ and R of (3.7) which have to satisfy (3.9):

Estimation of α_m. We approximate $V(t)$ by the finite difference

$$(P(t + dt) - P(t))/dt.$$

For any $t \in [0, 1]$ and $dt \in \mathbb{R}$ such that $t + dt \in [0, 1]$ we have

$$P(t + dt) - P(t) = \varphi(b_{t+dt}) - \varphi(b_t)$$

and, using (5.38)

$$\begin{aligned} |P(t + dt) - P(t)| &\geq \mathscr{H} \sin \psi_m |(b_{t+dt} - b_t)(H - \bar{H}_{b_t})| \\ &= \mathscr{H} \sin \psi_m \, dt |(b_1 - b_0)(H - \bar{H}_{b_t})|. \end{aligned}$$

Dividing by dt and passing to the limit yields

$$|V(t)| \geq \mathscr{H} \sin \psi_m |(b_1 - b_0)(H - \bar{H}_{b_t})| \tag{5.43}$$

and, using (5.15)

$$\alpha_m = H_m \mathscr{H} \sin \psi_m. \tag{5.44}$$

Estimation of α_M. In order to calculate $V(t)$ from the closed formula (5.10) for $P(t) = \varphi(b_t)$, we begin by calculating the Gateaux derivative of $\bar{H}_b$ at $b \in D$ in the direction $c \in L^\infty(0, 1)$. For k sufficiently small we have

$$\bar{H}_{b+kc} = \frac{\int_0^1 (b + kc)H}{\int_0^1 (b + kc)} = \frac{\int_0^1 (b + kc)(u_x/b + \bar{H}_b)}{\int_0^1 (b + kc)} = \frac{\int_0^1 u_x + k \int_0^1 cu_x/b}{\int_0^1 (b + kc)} + \bar{H}_b.$$

However, $\int_0^1 u_x = 0$ and $u_x = b(H - \bar{H}_b)$. Hence,

$$\bar{H}_{b+kc} = \bar{H}_b + k \frac{\int_0^1 c(H - \bar{H}_b)}{\int_0^1 (b + kc)},$$

which shows that

$$\frac{d}{dk} \bar{H}_{b+kc}|_{k=0} = \frac{\int_0^1 c(H - \bar{H}_b)}{\int_0^1 b}. \tag{5.45}$$

This implies that

$$\frac{d\bar{H}_{b_t}}{dt} = \frac{\int_0^1 (b_1 - b_0)(H - \bar{H}_{b_t})}{\int_0^1 b_t}. \tag{5.46}$$

Using the closed form formula (5.10) yields

$$V(t) = -(b_1 - b_0)(H - \bar{H}_{b_t}) + \frac{b_t}{\int_0^1 b_t} \int_0^1 (b_1 - b_0)(H - \bar{H}_{b_t}), \tag{5.47}$$

which we rewrite as

$$V(t) = -(b_1 - b_0)(H - \bar{H}_{b_t}) + \int_0^1 (b_1 - b_0)(H - \bar{H}_{b_t})$$
$$-\left(1 - \frac{b_t}{\int_0^1 b_t}\right) \int_0^1 (b_1 - b_0)(H - \bar{H}_{b_t}).$$

For $v \in L^2(0, 1)$ we define $|v|_{L^2/\mathbb{R}} = |v - \int_0^1 v|_{L^2}$ and we remember that

$$|v|_{L^2/\mathbb{R}} \leq |v|_{L^2} \qquad \text{for all} \quad v \in L^2(0, 1). \tag{5.48}$$

Using $|v|_{L^1(0,1)} \leq |v|_{L^2(0,1)}$ we find

$$|V(t)|_{L^2} \leq |(b_1 - b_0)(H - \bar{H}_{b_t})|_{L^2} + \left|1 - \frac{b_t}{\int_0^1 b_t}\right|_{L^2} |(b_1 - b_0)(H - \bar{H}_{b_t})|.$$

A simple calculation shows that

$$\left|1 - \frac{b_t(x)}{\int_0^1 b_t}\right| \leq \frac{b_M}{b_m} - 1, \qquad \forall x \in [0, 1],$$

so that finally

$$|V(t)| \le \frac{b_M}{b_m} |(b_1 - b_0)(H - \bar{H}_{b_t})| \le \frac{b_M}{b_m} H_M(b_1 - b_0) \tag{5.49}$$

and hence

$$\alpha_M = H_M \frac{b_M}{b_m}. \tag{5.50}$$

Estimation of Θ *and* R. We differentiate formula (5.47) with respect to t in order to obtain $A(t)$. Letting $c = b_1 - b_0$ we easily obtain, using (5.46),

$$A(t) = 2 \frac{\int_0^1 c(H - \bar{H}_{b_t})}{\int_0^1 b_t} \left\{ c - \frac{b_t}{\int_0^1 b_t} \int_0^1 c \right\}. \tag{5.51}$$

Estimating the L^2-norm of the term in the parentheses by the same technique we have just used for $V(t)$ yields

$$|A(t)| \le 2 \frac{|c(H - \bar{H}_{b_t})|}{b_m} \times \frac{b_M}{b_m} |c|$$

which, together with the lower bound (5.43) on $|V(t)|$, yields, as $|c| \le b_M - b_m$,

$$\frac{|A(t)|}{|V(t)|^2} \le \frac{2}{b_m H_M \mathscr{H}^3 \sin^2 \psi_m},$$

$$\frac{|A(t)|}{|V(t)|} \le \frac{2(b_M/b_m - 1)(b_M/b_m)}{\mathscr{H} \sin \psi_m},$$

so that we can choose, using (3.7),

$$R = \tfrac{1}{2} b_m H_M \mathscr{H}^3 \sin^2 \psi_m, \tag{5.52}$$

$$\Theta_1 = \frac{2(b_M/b_m - 1)(b_M/b_m)}{\mathscr{H} \sin \psi_m}. \tag{5.53}$$

Refining the estimation. The output set $\varphi(C)$ tends to become convex when $|\eta|_\infty \to \frac{1}{2}$, i.e., when the largest interval on which b is known to be constant tends to fill the whole space domain [0, 1]. Let us first see whether this fact is reflected in our estimations of R and Θ: when $|\eta|_\infty \to \frac{1}{2}$, R given by (5.52) remains bounded, and we have not been able to improve upon that using the above technique (i.e., to find a better estimation such that $R \to \infty$ when $|\eta|_\infty \to \frac{1}{2}$). Also Θ, given by (5.33), does not vanish when $|\eta|_\infty \to \frac{1}{2}$. We next give an alternative estimate of the deflection which has this property. From (5.51) we find, using (5.48),

$$|A(t)| \le 2 \frac{|c(H - \bar{H}_{b_t})|}{b_m} \left\{ |c|_{L^2/\mathbb{R}} + \frac{|b_t|_{L^2/\mathbb{R}}}{b_m} |c|_{L^1} \right\}. \tag{5.54}$$

Let us denote by $\hat{j}$ the index of the source with the largest interval I_j, so that

$$|\eta|_\infty = \max_{j \in \bar{J}} \eta_j = \eta_{\hat{j}}. \tag{5.55}$$

But c takes a constant value $c_{\hat{j}}$ on the interval $I_{\hat{j}}$, so that $c - \hat{c}_j \equiv 0$ on $I_{\hat{j}}$, and $|c - \hat{c}_j| \leq 2(b_M - b_m)$ outsite of $I_{\hat{j}}$. Hence,

$$|c|_{L^2/\mathbb{R}} \leq |c - c_{\hat{j}}|_{L^2} \leq 2(1 - 2|\eta|_\infty)^{1/2}(b_M - b_m).$$

Similarly,

$$|b_t|_{L^2/\mathbb{R}} \leq |b_t - b_{t\hat{j}}|_{L^2} \leq (1 - 2|\eta|_\infty)^{1/2}(b_M - b_m)$$

which yields, as $|c|_{L^1} \leq b_M - b_m$,

$$|A(t)| \leq 2|c(H - \bar{H}_{bt})|(1 - 2|\eta|_\infty)^{1/2}\left(\frac{b_M}{b_m} - 1\right)\left(\frac{b_M}{b_m} + 1\right),$$

so that

$$\frac{|A(t)|}{|V(t)|} \leq \frac{2(1 - 2|\eta|_\infty)^{1/2}(1 + b_m/b_M)(b_M/b_m - 1)(b_M/b_m)}{\mathscr{H} \sin\psi_m} \tag{5.56}$$

which, in view of (3.7), gives the following upper bound for the deflection:

$$\Theta_2 = \frac{2(1 - 2|\eta|_\infty)^{1/2}(1 + b_m/b_M)(b_M/b_m - 1)(b_M/b_m)}{\mathscr{H} \sin\psi_m}. \tag{5.57}$$

When $|\eta|_\infty$ is small, estimation (5.57) is less precise than (5.53), so we take, as a final estimate for the deflection,

$$\Theta = \operatorname{Min}\left\{1, (1 - 2|\eta|_\infty)^{1/2}\left(1 + \frac{b_m}{b_M}\right)\right\} \frac{2(b_M/b_m - 1)(b_M/b_m)}{\mathscr{H} \sin\psi_m}. \tag{5.58}$$

The corresponding shape coefficient τ is then given by (see (3.8) and (3.9))

$$\tau = \frac{\Theta}{\Delta/R} = \frac{\Theta \times R}{\alpha_M \operatorname{diam} c}, \tag{5.59}$$

i.e.,

$$\tau = \operatorname{Min}\left\{1, (1 - 2|\eta|_\infty)^{1/2}\left(1 + \frac{b_m}{b_M}\right)\right\}\mathscr{H}^2 \sin\psi_m. \tag{5.60}$$

Knowledge of the shape coefficient τ allows the determination of the maximum deflection Θ_M by (3.13) and of the lower bound R_G to global radii of curvature by (3.12).

5.5. *The Final Stability Result*

Having estimated in the previous section all geometrical quantities associated in (3.6), (3.7), and (3.9) to $\varphi(D_\eta)$ we can now apply Theorem 3.1 to obtain the well-posedness of the least-squares problem (5.39).

Theorem 5.1. *Suppose that the lower and upper bounds b_m and b_M on b, the source locations and amplitudes x_j, f_j, $j \in J$, and the radii η_j, $j \in \tilde{J}$, of the balls surrounding the sources over which b is known to take constant values satisfy the following* *conditions:*

$$\begin{cases} 0 < b_m \le b_M, \\ I_j =]x_j - \eta_j, x_j + \eta_j[\subset]0,1[, \qquad \forall j \in \tilde{J}, \\ I_j \cap I_{j'} = \varnothing, \qquad \forall j, j' \in \tilde{J}, \quad j \ne j', \end{cases} \tag{5.61}$$

$$H_m > 0 \qquad \textit{(proper arrangement of sources)}, \tag{5.62}$$

where H_m *is defined by* (5.5) *and* (5.12),

$$\Theta < \Theta_M \qquad \textit{(deflection size} \times \textit{curvature condition)}, \tag{5.63}$$

where Θ *is defined by* (5.58) *and* Θ_M *by* (3.13) *and* (5.60). *Then, if* R_G *is defined by* (3.12), (5.52), *and* (5.60), *for any data* z *satisfying*

$$z \in \mathscr{V} = \{z \in L^2(0,1) \mid d(z, \varphi(D_\eta)) < R_G\}, \tag{5.64}$$

the **least-squares problem (5.39) for the estimation of b in D_η from the measurement of z of u_x is Q-well-posedness for the h-weighted L^2-norm on b,** *and the following stability estimate holds:*

$$\mathscr{H} \sin \psi_m |h(\hat{b}_0 - \hat{b}_1)|_{L^2} \le \frac{1}{1 - \chi R_G/R} |z_0 - z_1|_{L^2} \tag{5.65}$$

as soon as

$$|z_0 - z_1|_{L^2} + \operatorname*{Max}_{j=0,1} d(z_j, \varphi(D_\eta)) \le \chi R_G, \qquad 0 < \chi < 1. \tag{5.66}$$

Notice first from (5.58) that the size $\times$ curvature condition (5.63) will be satisfied as soon as b_M/b_m is close enough to 1 or $|\eta|_\infty$ is close enough to $\frac{1}{2}$! Hence, for each value of $|\eta|_\infty$, there exists an upper bound to the ratio b_M/b_m for which the inverse problem is well-posed, this upper bound being less and less restrictive when $|\eta|_\infty$ approaches $\frac{1}{2}$, i.e., when one of the balls over which the parameter is known to be constant tends to fill up the space domain.

Notice also that we obtain *stability of b for a weighted L^2-norm*: the stability of b is better at locations x where $h(x)$ is large, i.e., where $|u_x(x)|$ is large, which corresponds to the physical intuition.

Notice also from formula (3.12) defining R_G that the size of the neighborhood $\mathscr{V}$ on which stability holds will be R independent of the size D_η, provided that b_M/b_m is small enough so that Θ given by (5.58) is smaller than $\pi/2$. Allowing the size of D_η to grow beyond this limit will be paid for by a reduction in the size of $\mathscr{V}$ to $R_G < R$, with R_G approaching zero when b_M/b_m approaches its upper limit corresponding to $\Theta = \Theta_M$ given by (3.13).

To conclude this section we give the numerical values of all constants appearing in the stability theorem (Theorem 5.1) in the simple case where the right-hand side of the elliptic equations contains only one Dirac source of amplitude one located at the center of the interval. Hence we estimate the coefficient b in

$$-(b^{-1}u_x)_x = \delta(x - \tfrac{1}{2}), \quad 0 < x < 1, \tag{5.67}$$

$$u(0) = u(1) = 0, \tag{5.68}$$

from

$$z \in L^2(0, 1) = \text{measurement of } u_x \tag{5.69}$$

using the additional information that

$$b \in D_\eta = \{b \mid 0 < b_m \leq b(x) \leq b_M, b = \text{constant over }]\tfrac{1}{2} - \eta, \tfrac{1}{2} + \eta[\,\}. \tag{5.70}$$

The problem is hence completely specified as soon as we have chosen

$$\eta \in]0, \tfrac{1}{2}] \quad \text{(radius of the ball over which the parameter } b \text{ is known to be constant)}, \tag{5.71}$$

$$\zeta = \frac{b_M}{b_m} \in [1, +\infty[\quad \text{(upper to lower bound ratio for } b\text{)}, \tag{5.72}$$

$$b_m > 0 \quad \text{(lower bound to } b\text{)}. \tag{5.73}$$

We then immediately find that

$$\begin{aligned} h(x) &= \text{constant} = H_m(1 + \zeta)^{-1}, \\ H_M &= \zeta(1 + \zeta)^{-1}, \\ \mathscr{H} &= \zeta^{-2}, \end{aligned} \tag{5.74}$$

so that the stability estimate (5.65) rewrites as

$$|\hat{b}_0 - \hat{b}_1|_{L^2} \leq \frac{\zeta^2(1 + \zeta)}{\sin \psi_m} \times \frac{1}{1 - \chi(R_G/R)} |z_0 - z_1|_{L^2}. \tag{5.75}$$

For each value of the "regularization" parameter η, the size $\times$ curvature condition (5.63) imposes an upper limit ζ_M to the b_M/b_m ratio to ensure the well-posedness of the inverse problem. This upper limit ζ_M is shown in Figure 5.2. It becomes unbounded when η approaches 0.5 (for $\eta = 0.5$ the output set $\varphi(D)_\eta$) is convex!).

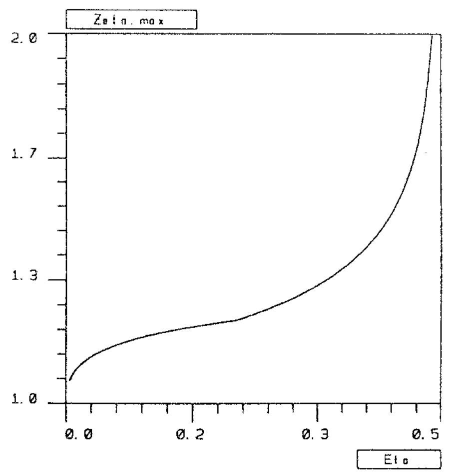

Figure 5.2. Upper limit ζ_M to b_M/b_m ensuring well-posedness of the inverse problem following Theorem 5.1 for example (5.67)–(5.70).

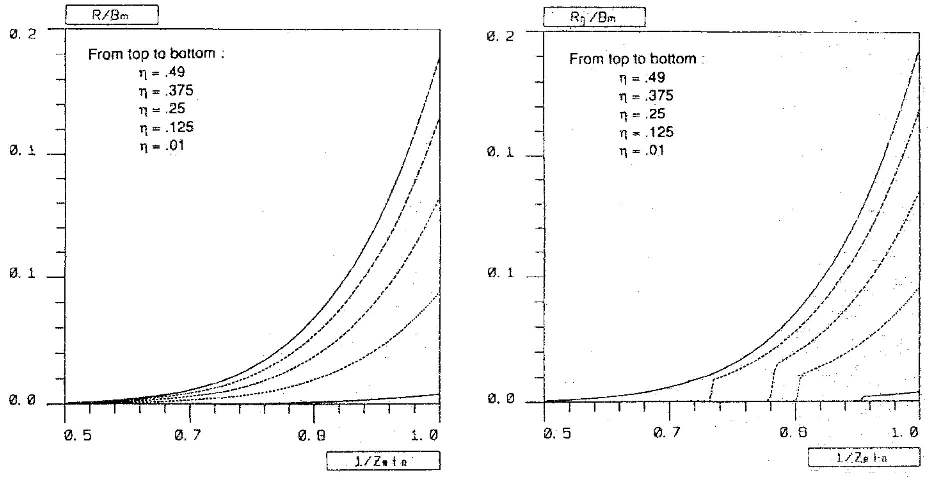

Figure 5.3. Values of R/b_m and R_G/b_m as functions of $1/\zeta = b_m/b_M$ and η.

In Figure 5.3 we show the radius of curvature R and the global radius of curvature R_G as functions of η and ζ^{-1}. Note that the interval over which R_G is strictly smaller than R but still positive is quite small (compare (3.12), (3.13), and Figure 3.1). Recall that positive values of R_G give the size of the cylindrical neighborhood of $\varphi(D_\eta)$ with respect to which the inverse problem is Q-well-posed, provided that $\zeta < \zeta_M$. Figure 5.4 gives the graphs for the deflection Θ in multiples of π and the shape coefficient τ. Notice that τ has very roughly the value 0.2 for values η and ζ which give a deflection Θ close to $\pi/2$. As it can be seen in Figure 3.2, this value of τ corresponds to a maximum deflection Θ_M only a little larger than $\pi/2$. Hence, for such values of τ the set $\{(\eta, \zeta) | (\pi/2) < \Theta(\eta, \zeta) \leq \Theta_M\}$ is small,

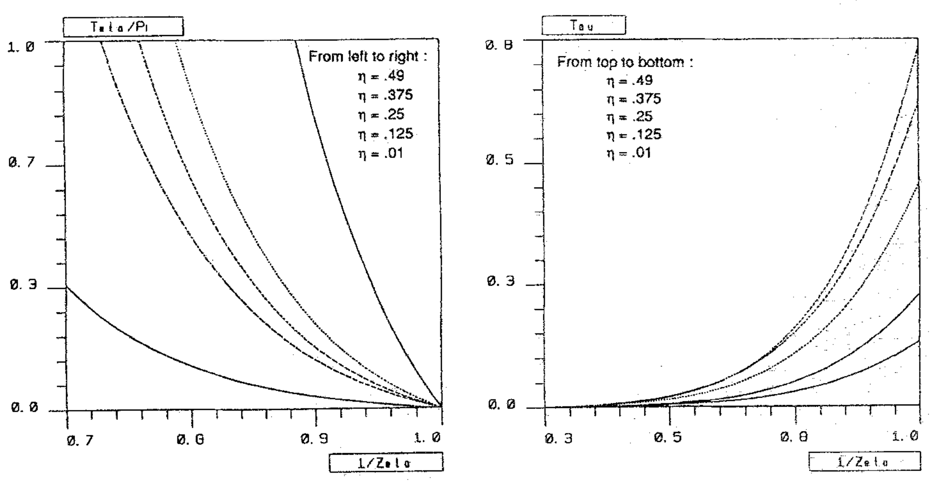

Figure 5.4. The upper bound Θ to the deflection of paths of $\varphi(D_\eta)$ and the shape of coefficient τ of the estimation R, Θ and Δ as functions of $1/\zeta = b_m/b_M$ and η.

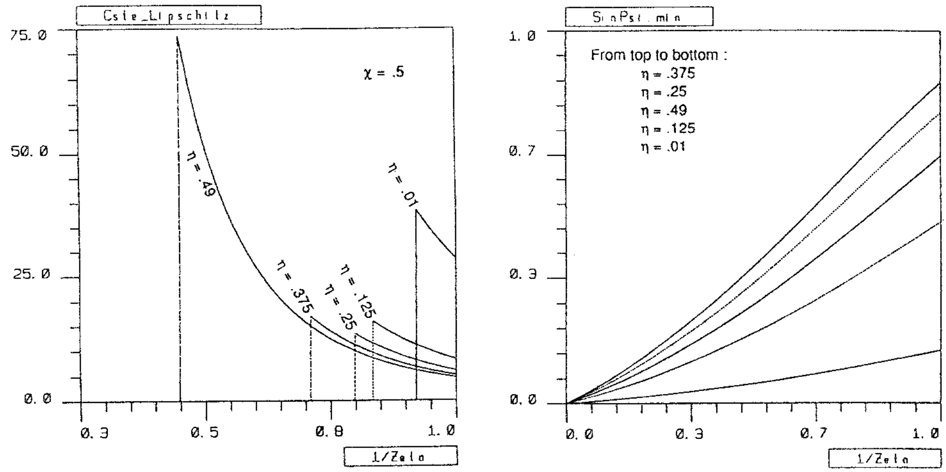

Figure 5.5. (a) The Lipschitz constant of the inverse problem for data in the first half of the security strip around $\varphi(D_\eta)$ (i.e., $\chi = 0.5$) as functions of $1/\zeta = b_m/b_M$ and η. (b) The lower bound $\sin \psi_m$ associated to the set D_η as functions of $1/\zeta = b_m/b_M$ and η.

and there is only little gain in allowing Θ to pass beyond $\pi/2$. We show in Figure 5.5(a) the Lipschitz constant of (5.75). This figure corresonds to the choice $\chi = 0.5$, so that the data z_0 and z_1 are located no further from $\varphi(D_\eta)$ than "in the middle" of the security strip around $\varphi(D_\eta)$ defined by R_G. For $\chi = 0.1$ the graphs look similar to those of Figure 5.5 but are scaled with the factor $\frac{1}{2}$. Finally, in Figure 5.5(b) we give the graph for $\sin \psi_m$ associated with the set D_η for various values of η and ζ^{-1}.

6. The Distributed Source Case

In this section we consider the estimation of b in

$$\begin{cases} -(b^{-1}u_x)_x = f, \\ u(0) = u(1) = 0 \end{cases} \tag{6.1}$$

from observation of u_x, and with $f \in L^2(0, 1)$. We put $H(x) = \int_0^x f(s)\,ds$ and recall the notation of $\bar{H}_b$, D, b_m, and b_M of the previous section. Due to the increased assumption in the regularity of f in this section there always exists at least one zero of

$$b^{-1}u_x = -H + \bar{H}_b, \quad b \in D.$$

If the coefficients b are restricted to be constant in the neighborhood of zeros of $H - \bar{H}_b$, then it will be possible to establish stability in the sense of Section3.

For $H_m > 0$ we define

$$\Omega_m = \bigcap_{b \in D} \{x \in [0, 1]: |H(x) - \bar{H}_b| \geq H_m\} \tag{6.2}$$

and for $b \in D$ we put

$$\Omega_m^+ = \{x \in \Omega_m: H(x) - \bar{H}_b \geq H_m\}$$

and

$$\Omega_m^- = \{x \in \Omega_m: H(x) - \bar{H}_b \leq - H_m\}.$$

It is assumed that Ω_m is not empty. Since D is a connected subset of L^∞ and since $b \to H(x) - \bar{H}_b$ is continuous from $D \subset L^\infty$ to $\mathbb{R}$ for every $x \in [0, 1]$ it follows that $\{H(x) - H_{\bar{b}} | b \in D\}$ is a connected subset of $\mathbb{R}$ for every $x \in [0, 1]$. Consequently, the definition of Ω_m^+ and Ω_m^- is independent of the representative $b \in D$ and $\Omega_m^+ \cup \Omega_m^- = \Omega_m$. Clearly, Ω_m is a closed set and hence its complement is open. It can therefore be represented as a countable union of nonintersecting open intervals. For simplicity we assume that there are only finitely many such intervals $\{S_j\}_{j=1}^N$, that they are indexed from left to right in the domain (0, 1), and that S_1 and S_N do not contain 0 or 1 in their closure. Between any pair of the endpoints 0 and 1 and of the intervals $\{S_j\}_{j=1}^N$ there are subsets of Ω_m^+ and Ω_m^-. Henceforth we assume that these subsets belong alternately to Ω_m^+ and Ω_m^-. Furthermore, let $\eta = (\eta_1, \ldots, \eta_N) \in \mathbb{R}^N$, $\eta_j > 0$, be a vector characterizing neighborhoods $I_j =]l_j - \eta_k, r_j + \eta_j[$ of $S_j =]l_j, r_j[$. These neighborhoods are assumed to be pairwise disjoint. The notation is illustrated in Figure 6.1.

In analogy to the case of point sources in Section 5 we may refer to the intervals S_j as "sources." We also note that the complement of Ω_m consists of one open interval in the case that $f \geq 0$, $f \neq 0$, and provided that $H_m > 0$ is small enough. It may also be useful to consider the following specific example.

Remark 2. To illustrate further the set Ω_m and its dependence on H_m, let us consider the specific case when $f = k \in \mathbb{R}$. In this case

$$H(x) = kx \quad \text{and} \quad \bar{H} = k \frac{\int_0^1 sb(s)\, ds}{\int_0^1 b(s)\, ds}.$$

Moreover, define

$$\underline{x} = \min_{b \in D} \frac{\int_0^1 sb\, ds}{\int_0^1 b\, ds} \quad \text{and} \quad \bar{x} = \max_{b \in D} \frac{\int_0^1 sb\, ds}{\int_0^1 b\, ds}.$$

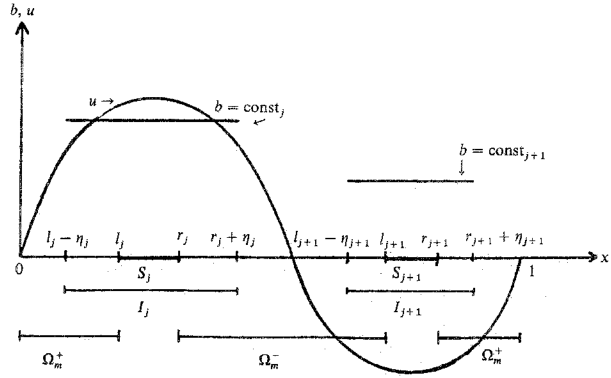

Figure 6.1. Notations for the distributed source case.

Then, for $H_m > 0$ sufficiently small,

$$\Omega_m = \Omega_m^- \cup \Omega_m^+ = \left[0, \underline{x} - \frac{H_m}{k}\right] \cup \left[\bar{x} + \frac{H_m}{k}, 1\right].$$

For the proof of Theorem 6.1 the following modification of Lemma 1 is required. It can be verified with techniques analogous to those of Lemma 1. We use $|\Omega|$ to denote the measure of a set $\Omega \subset \mathbb{R}$.

Lemma 4. *Let $\Omega \subset \mathbb{R}$ be a measurable set and let $d \in L^2(\Omega)$, $w \in L^\infty(\Omega)$, and $h \in L^2(\Omega)$ satisfy*

$$\frac{d}{w} = h + \textit{an unknown constant.}$$

Then we have

$$\frac{1}{|\Omega|^{1/2}} \frac{|w|_{L^2(\Omega)}}{|w|_{L^\infty(\Omega)}} \left|\frac{d}{w}\right|_{L^2(\Omega)} \sin\psi \le |h|_{L^2(\Omega)},$$

where $\psi \in [0, \pi/2]$ is the angle between the directions d and w and

$$\sin\psi = \sqrt{1 - \frac{\langle d, w\rangle_{L^2(\Omega)}}{|d|^2_{L^2(\Omega)}|w|^2_{L^2(\Omega)}}}.$$

The class of admissible coefficients is given by

$$D_\eta = \{b \in D \mid b(x) = b_j \in \mathbb{R} \text{ on } I_j, j = 1, \ldots, N\},$$

and the parameter to output mapping

$$\varphi\colon L^2(\Omega_m) \to L^2(\Omega_m)$$

is given by

$$\varphi(b) = u_x.$$

We note the following relationship between the $L^2(0, 1)$- and the $L^2(\Omega_m)$-norms for elements $b \in D_\eta$:

$$|b|_{L^2(\Omega^m)} \le |b|_{L^2(0,1)} \le \left(\frac{2\eta_{\max} + (r_{j\max} - l_{j\max})}{2\eta_{\min}}\right)^{1/2} |b|_{L^2(\Omega_m)}, \tag{6.3}$$

where

$$\eta_{\min} = \min\{\eta_i \mid i = 1, \ldots, N\}, \qquad \eta_{\max} = \max\{\eta_i \mid i = 1, \ldots, N\},$$

and j max is the index of the largest interval S_j. We further define H_M such that

$$|H(x) - \bar{H}_b| \le H_M, \qquad \forall x \in \Omega_m \quad \text{and} \quad b \in D_\eta, \tag{6.4}$$

and we put $\mathscr{H} = (H_m b_m)(H_M b_M)^{-1}$ and

$$J = \bigcup_{j=1}^{N} \left( ]l_j - \eta_j, l_j[\cup]r_j, r_j + \eta_j[\right).$$

The following stability estimate can be obtained in the zero residual case.

Theorem 6.1 *Let the assumptions made on H_m, Ω_m, and S_j hold and assume that*

$$r := \frac{H_m^4}{H_M^4} \sum_{j=1}^{N} \eta_i - \frac{H_M^2 |f|_{L^2(J)}^2 \eta_{\max}^2}{9H_m^4} > 0. \tag{6.5}$$

Then $r|\Omega_m|^{-1} \in]0, \frac{1}{2}]$, which allows us to define ψ_m as the unique solution in $]0, \pi/2]$ of $\cos \psi_m = 1 - r|\Omega_m|^{-1}/2$. Moreover, the estimate

$$\mathscr{H} \sin \psi_m |(b_1 - b_0)(H - \bar{H}_{b_0})|_{L^2(\Omega_m)} \leq |\varphi(b_0 - \varphi(b_1)|_{L^2(\Omega_m)} \tag{6.6}$$

holds for every b_0 and $b_1 \in D_\eta$, with $\sin \psi_m > 0$. In view of (6.2) and (6.3), Theorem 6.1 implies

$$|b_0 - b_1|_{L^2(0,1)} \leq K |\varphi(b_0) - \varphi(b_1)|_{L^2(\Omega_m)}$$

for some constant K which is independent of b_0 and b_1 in D_η.

Proof of Theorem 6.1. Let b_0 be in D_η and recall that

$$\frac{b_0 - b_1}{b_1} \{H - \bar{H}_{b_0}\} = \frac{u_{0x} - u_{1x}}{b_1} + \text{an unknown constant.}$$

Applying Lemma 4 with $d = (b_0 - b_1)/b_1$, $w = 1/(H - H_{b_0})$, and $\Omega = \Omega_m$ gives

$$\frac{1}{|\Omega_m|^{1/2}} \frac{|(H - \bar{H}_{b_0})^{-1}|_{L^2(\Omega_m)}}{|(H - \bar{H}_{b_0})^{-1}|_{L^\infty(\Omega_m)}} \left| \frac{b_0 - b_1}{b_1} (H - \bar{H}_{b_0}) \right|_{L^2(\Omega_m)} \sin \psi$$
$$\leq \left| \frac{u_{0x} - u_{1x}}{b_1} \right|_{L^2(\Omega_m)}, \tag{6.7}$$

where ψ is the angle between the directions given by d and w. From (6.7), (6.3), and (6.4) we conclude that

$$\frac{H_m}{H_M b_M} |(b_0 - b_1)(H - \bar{H}_{b_0})|_{L^2(\Omega_m)} \sin \psi \leq \frac{1}{b_m} |u_{0x} - u_{1x}|_{L^2(\Omega_m)}$$

and consequently

$$\mathscr{H} |(b_0 - b_1)(H - \bar{H}_{b_0})|_{L^2(\Omega_m)} \sin \psi \leq |u_{0x} - u_{1x}|_{L^2(\Omega_m)}. \tag{6.8}$$

Next define $c = d/|d|_{L^2(\Omega_m)}$ and $v = \pm w/|w|_{L^2(\Omega_m)}$, with the sign chosen such that $\langle c, v \rangle_{L^2(\Omega_m)} \geq 0$. As in the proof of Lemma 3 we have

$$\cos \psi = \langle c, v \rangle_{L^2(\Omega_m)} = 1 - \tfrac{1}{2} |c - v|_{L^2(\Omega_m)}^2 \geq 0. \tag{6.9}$$

Below we establish that

$$|c - v|_{L^2(\Omega_m)}^2 \geq \frac{1}{|\Omega_m|} \left(\frac{H_m^4}{H_M^4} \sum_{j=1}^{N} \eta_j - \frac{H_M^2 |f|_{L^2(J)}^2 \eta_{\max}^2}{9H_m^4} \right) = \frac{r}{|\Omega_m|}. \tag{6.10}$$

Combining (6.9) and (6.10) implies

$$0 \le \cos\psi \le 1 - \frac{r}{2|\Omega_m|} = \cos\psi_m < 1,$$

and therefore

$$\sin\psi \ge \sin\psi_m > 0.$$

This estimate together with (6.8) implies the desired result. It remains to verify (6.10). Obviously we have

$$|c - v|_{L^2(\Omega_m)} \ge \sum_{j=1}^{N} |c - v|_{L^2(U_j)}, \tag{6.11}$$

where $U_j = I_j^L \cup I_j^R =]l_j - \eta_j, l_j[\cup]r_j, u_x, r_j + \eta_j[$, and

$$|c - v|^2_{L^2(U_j)} = \int_{I_j^L} |c - v|^2\, dx + \int_{I_j^R} |c - v|^2\, dx. \tag{6.12}$$

Observe that c equals an unknown constant on U_j for $j = 1, \ldots, N$. A simple calculation shows that the expression on the right-hand side of (6.12) is minimum when c is equal to the following constant:

$$\hat{c} = \frac{1}{2\eta_i}\int_{I_j^L} v\, dx + \frac{1}{2\eta_i}\int_{I_j^R} v\, dx \qquad \text{on } U_j.$$

Therefore we find

$$|c - v|^2_{L^2(U_j)} \ge \int_{I_j^L}\left|\frac{1}{2\eta_j}\int_{I_j^L} v\, ds + \frac{1}{2\eta_j}\int_{I_j^R} v\, ds - v(x)\right|^2 dx$$
$$+ \int_{I_j^R}\left|\frac{1}{2\eta_j}\int_{I_j^L} v\, ds + \frac{1}{2\eta_j}\int_{I_j^R} v\, ds - v(x)\right|^2 dx. \tag{6.13}$$

For the first term on the right-hand side of (6.13) we have

$$\int_{I_j^L}\left|\frac{1}{2\eta_j}\int_{I_j^L} v\, ds + \frac{1}{2\eta_j}\int_{I_j^R} v\, ds - v(x)\right|^2 dx$$
$$= \int_{I_j^L}\left|\frac{1}{2\eta_j}\int_{I_j^L} [v(\xi) - v(x)]\, d(\xi) + \frac{1}{2\eta_j}\int_{I_j^R} (v(\xi) - v(x))\, d\xi\right|^2 dx$$
$$\ge \frac{1}{8\eta_j^2}\int_{I_j^L}\left(\int_{I_j^R} |v(\xi) - v(x)|\, d\xi\right)^2 dx - \frac{1}{4\eta_j^2}\int_{I_j^L}\left(\int_{I_j^L} |v(\xi) - v(x)|\, d\xi\right)^2, \tag{6.14}$$

where we used the fact that $(a + b)^2 \ge \frac{1}{2}a^2 - b^2$ for any $a, b \in \mathbb{R}$. (In the case of point sources the analogue to the last term is zero.) In the following estimates we use

$$\frac{|\Omega_m|^{1/2}}{H_M} \le \left|\frac{1}{H - \bar{H}_{b_0}}\right|_{L^2(\Omega_m)} \le \frac{|\Omega_m|^{1/2}}{H_m} \tag{6.15}$$

and

$$\frac{1}{H_M} \le (H(x) - \bar{H}_{b_0})^{-1}, \qquad \forall x \in \Omega_m. \tag{6.16}$$

From (6.14) we find

$$\begin{aligned}
&\int_{I_j^L} \left| \frac{1}{2\eta_j} \int_{I_j^L} v\, d\zeta + \frac{1}{2\eta} \int_{I_j^R} v\, d\zeta - v(x) \right|^2 dx \\
&\ge \frac{1}{8\eta_j^2} \frac{H_m^2}{|\Omega_m|} \int_{I_j^L} \left(\int_{I_j^R} \left| \frac{1}{H(\zeta) - \bar{H}_{b_0}} - \frac{1}{H(x) - \bar{H}_{b_0}} \right| d\zeta \right)^2 dx \\
&\quad - \frac{1}{4\eta_j^2} \frac{H_M^2}{|\Omega_m|} \int_{I_j^L} \left(\int_{I_j^L} \left| \int_\xi \left(\frac{1}{H(\sigma) - \bar{H}_{b_0}} \right)' d\sigma \right| d\xi \right)^2 dx \\
&\ge \frac{1}{8\eta_j^2} \frac{H_m^2}{|\Omega_m|} \frac{1}{H_M^4} \int_{I_j^L} \left(\int_{I_j^R} |(H(\xi) - \bar{H}_{b_0}) - (H(x) - \bar{H}_{b_0})|\, d\xi \right) dx \\
&\quad - \frac{H_M^2}{4\eta_j^2 |\Omega_m| H_m^4} \int_{I_j^L} \left(\int_{I_j^L} \int_\xi^x |f(\sigma)|\, d\sigma\, d\xi \right)^2 dx \\
&\ge \frac{H_m^2}{8\eta_j^2 |\Omega_m| H_M^4} 4H_m^2 \eta_j^3 - \frac{H_M |f|^2_{L^2(I_j^L)}}{4\eta_j^2 |\Omega_m| H_m^4} \int_{I_j^L} \tfrac{4}{9}\eta_j^3\, dx \\
&= \frac{\eta_j}{|\Omega_m|} \left[\frac{H_m^4}{2H_M^4} - \frac{H_M^2 |f|^2_{L^2(I_j^L)} \eta_i}{9H_m^4} \right].
\end{aligned}$$

The last term in (6.13) is estimated in an analogous manner. We obtain

$$|c - v|_{L^2(U_j)} \ge \frac{\eta_i}{|\Omega_m|} \left[\frac{H_m^4}{H_M^4} - \frac{H_M^2 |f|^2_{L^2(U_j)} \eta_j}{9H_m^4} \right], \qquad j = 1, \ldots, N.$$

Using this estimate in (6.11) we obtain

$$\begin{aligned}
|c - v|^2_{L^2(\Omega_m)} &\ge \frac{1}{|\Omega_m|} \sum_{j=1}^N \eta_j \left[\frac{H_m^4}{H_M^4} - \frac{H_M^2 |f|^2_{L^2(U_j)} \eta_j}{9H_m^4} \right] \\
&\ge \frac{1}{|\Omega_m|} \left[\frac{H_m^4}{H_M^4} \sum_{j=1}^N \eta_j - \frac{H_M^2 |f|^2_{L^2(J)} \eta_{\max}^2}{9H_m^4} \right],
\end{aligned}$$

which is the desired estimate (6.10). □

With the estimates of Theorem 6.1 it is simple to argue Q-stability of the least-squares problem

$$\min_{b \in D_\eta} |\varphi(b) - z|^2_{L^2(\Omega_m)}, \tag{6.17}$$

where $z \in L^2(\Omega_m)$ and $\varphi(b) = u_x(b)$, with $u(b)$ the solution of (6.1). In fact, the estimates for the geometric quantities α_m, α_M, θ_1, and R are obtained in the same manner as in Section 5. The only change occurs in the estimate of Θ_2, where in formulas (5.56)–(5.60) the term $(1-2|\eta|_\infty)^{1/2}$ can be replaced by $(|S_{j\max}|-2|\eta|_{\max})^{1/2}$. Thus, for problem (6.17) the deflection Θ and the shape coefficient τ are given by

$$\Theta = \operatorname{Min}\left\{1, (|S_{j\max}|-2|\eta_{\max}|)^{1/2}\left(1+\frac{b_m}{b_M}\right)\right\}\frac{2(b_M/b_m-1)}{\mathscr{H}\sin\psi_m}$$

and

$$\tau = \operatorname{Min}\left\{1, (|S_{j\max}|-2|\eta_{\max}|)^{1/2}\left(1+\frac{b_m}{b_M}\right)\right\}\mathscr{H}^2\sin\psi_m,$$

respectively, while Θ_M, R, and R_G are as given in Section 5. Furthermore, we put

$$h(x) = \operatorname*{Inf}_{b\in D_\eta}|H(x)-\bar{H}_b| \qquad \text{for} \quad x\in\Omega_m,$$

and we note that $h(x)\geq H_m$ for $x\in\Omega_m$. We obtain the following:

Theorem 6.2. *Under the assumptions of Theorem* 6.1 *the least-squares problem* (6.17) *with*

$$\mathscr{V} = \{z\in L^2(\Omega_m)\,|\,d(z,\varphi(D_\eta)) < R_G\}$$

is Q-well-posed with the h-weighted L^2-norm on b, and the following stability estimate holds:

$$\mathscr{H}\sin\psi_m|h(\hat{b}_0-\hat{b}_1)|_{L^2(\Omega_m)} \leq \frac{1}{1-\chi R_G/R}|z_0-z_1|_{L^2(\Omega_m)}$$

as soon as

$$|z_0-z_1|_{L^2(\Omega_m)} + \operatorname*{Max}_{j=0,1} d(z_j,\varphi(D_\eta)) \leq \chi R_G, \qquad \textit{where} \quad 0<\chi<1.$$

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