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Simple groups all of whose second maximal subgroups are (A)-groups

By

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1. Introduction. Throughout this paper the term group always means a group of finite order. A subgroup H of G is called pronormal in G if for every element x of G, H is conjugate to H^x in $\langle H, H^x \rangle$. We say that G is an (A)-group if every subgroup of G of prime order is pronormal in G and either the Sylow 2-subgroups of G are abelian or every cyclic subgroup of G of order 4 is pronormal in G. A finite group G is called a PN-group if every subgroup of prime order is normal in G. In [1], Buckley proved that a PN-group of odd order is supersolvable. The structure of a non PN-group, each of whose proper subgroups is a PN-group, has been analyzed by Sastry [2].

The purpose of this paper is to investigate the structure of a finite group, each of whose maximal subgroups is an (A)-group. We also classify a finite simple group, each of whose second maximal subgroups is an (A)-group.

The following result will be repeatedly used: A subgroup H of G is normal in G iff it is both subnormal and pronormal in G [3, Exercise 6, p. 14].

The notation used in this paper is standard. In addition, $W(|G|)$ denotes the number of the distinct prime divisors of the order of G.

2. Preliminaries. In this section, we collect some of the results that are needed in this paper.

(2.1) Suppose that p is the largest prime divisor of the order of G, every subgroup of G of prime order $q \neq p$ is pronormal in G and either (i) the Sylow 2-subgroups of G are abelian, or (ii) every cyclic subgroup of G of order 4 is pronormal in G. Then G possesses an ordered Sylow tower and $G/O_p(G)$ is supersolvable. In particular, G is solvable.

Proof. This is [4, Theorem 4.2].

(2.2) If G/H is supersolvable, $(|H|, 2) = 1$, and every subgroup of H of prime order is pronormal in G, then G is supersolvable.

Proof. This is [5, Theorem B].

(2.3) If G is a minimal non-supersolvable group (non-supersolvable group all of whose proper subgroups are supersolvable), then:

(i) G possesses an ordered Sylow tower or G is a minimal non-nilpotent group (non-nilpotent group all of whose proper subgroups are nilpotent).
(ii) G possesses a unique normal Sylow p-subgroup P for some prime p.
(iii) P has exponent p if $p > 2$ and exponent at most 4 if $p = 2$.
(iv) $P/\phi(P)$ is a minimal normal subgroup of $G/\phi(P)$.
(v) P' has exponent p.

Proof. This is (Doerk [6]; see also [7, Aufgaben 16, p. 721]).

(2.4) If G is a non-nilpotent dihedral group of order $2n$ or $4n$, where n is odd, then G is an (A)-group.

Proof. We consider two cases:

Case 1. $|G| = 2n$, n is odd. Then $G = SH$, where $S \in \mathrm{Syl}_2(G)$ and H is a normal cyclic subgroup of G of order n. For any $x \in G$, S is conjugate to S^x in $\langle S, S^x \rangle$ by Sylow's theorem. Hence S is pronormal in G. Clearly, all subgroups of G of odd order are normal in G. Therefore, G is an (A)-group.

Case 2. $|G| = 4n$, n is odd. Let $S \in \mathrm{Syl}_2(G)$ and let K be a subgroup of G such that $K < S$. If $K^x < S$, $x \in G$, then K and K^x are conjugate in $N_G(S)$ by Burnside's theorem [3, Theorem 1.1, p. 240]. Since G is non-nilpotent, it follows that $N_G(S) < G$. Hence $K = K^x$. Thus every Sylow 2-subgroup of G contains exactly one conjugate of K. Now [3, Exercise 4 (ii), p. 13] implies that K is pronormal in G. Clearly, all subgroups of G of odd order are normal in G. Therefore, G is an (A)-group.

(2.5) Set $G = \mathrm{PSL}(2, r)$ with $r = p^f$, p a prime and $r > 3$. Then we have:

(i) The groups $\mathrm{PSL}(2, r)$ are simple;
(ii) $\mathrm{PSL}(2, 4) \cong \mathrm{PSL}(2, 5) \cong \mathrm{Alt}(5)$;
(iii) All second maximal subgroups of Alt(5) are (A)-groups.

Proof. For (i) and (ii); see [3, Theorem 1.2, p. 419]. It is known that all the second maximal subgroups of Alt(5) are abelian. Now it follows easily that all the second maximal subgroups of Alt(5) are (A)-groups. Thus (iii) holds.

(2.6) Set $G = \mathrm{PSL}(2, r)$, with $r = p^f$, p a prime and $r > 3$. Then G is a Zassenhaus group of degree $(r + 1)$ and the subgroup N fixing a letter is a Frobenius group with elementary abelian kernel K of order r and a cyclic complement H of order $(r - 1)/d$, where $d = (r - 1, 2)$. Further, $N_G(K) = N$ is a maximal subgroup of G and H acts irreducibly on K.

Proof. See [3, Theorem 8.2, p. 41], [3, Theorem 7.3 (iii), p. 35] and [3, Lemma 1.1, p. 418].

(2.7) Suppose that G is one of the following groups:

(1) $\mathrm{PSL}(2, p)$, where p is a prime with $p > 5$, $p^2 - 1 \not\equiv 0(5)$ and $p^2 - 1 \not\equiv 0(16)$;
(2) $\mathrm{PSL}(2, 2^q)$, where q is an odd prime and $2^q - 1 =$ prime;
(3) $\mathrm{PSL}(2, 3^q)$, where q is an odd prime and $(3^q - 1)/2 =$ prime.

Then every second maximal subgroup of G is an (A)-group.

Proof. A maximal subgroup of G is one of the following groups:

(i) A dihedral group of order $2(r-1)/d$, where $d=(r-1,2)$ and $r=p$ or $r=2^q$ or $r=3^q$. See [7 § 8 Die Untergruppen von PSL(2, r), pp. 191–213].

(ii) A Frobenius group N with elementary abelian kernel K of order r and a cyclic complement H of order $(r-1)/d$, where $d=(r-1,2)$ and $r=p$ or $r=2^q$ or $r=3^q$. See remark (2.6).

(iii) Alt(4).

Remark (2.4) implies that groups of type (i) are (A)-groups. Remark (2.6) implies that N is a minimal non-abelian group. Hence all maximal subgroups of N are (A)-groups. Since Alt(4) is a minimal non-abelian group, it follows that all maximal subgroups of Alt(4) are (A)-groups.

3. Groups all of whose maximal subgroups are (A)-groups. In this section, we consider the structure of groups all of whose maximal subgroups are (A)-groups. We prove the following theorem:

Theorem 3.1. *If every maximal subgroup of G is an* (A)*-group, then one of the following statements is true*:

(1) *G is supersolvable.*

(2) *$G=PQ$, where $P \lhd G$, $P \in \mathrm{Syl}_p(G)$, P is elementary abelian, $Q \in \mathrm{Syl}_q(G)$ is non-normal and cyclic, and $p>q$.*

(3) *$G=PQ$, where $P \lhd G$, $P \in \mathrm{Syl}_p(G)$, P is elementary abelian, $Q \in \mathrm{Syl}_q(G)$ is non-normal and cyclic, $p<q$, and G is a minimal non-abelian group* (*non-abelian group all of whose proper subgroups are abelian*).

(4) *$G=PQ$, where $P \lhd G$, $P \in \mathrm{Syl}_2(G)$, P is quaternion of order* 8, *$Q \in \mathrm{Syl}_3(G)$ is non-normal and cyclic, and G is a minimal non-nilpotent group.*

Proof. Suppose that G is not supersolvable. Let M be an arbitrary maximal subgroup of G. Then M is an (A)-group by hypothesis. By (2.1), $M/O_p(M)$ is supersolvable, where p is the largest prime divisor of the order of M. Applying (2.2), we conclude that M is supersolvable. Thus G is a minimal non-supersolvable group. By Huppert's theorem, G is solvable [7, Satz 9.6, p. 718]. We argue that $W(|G|)=2$. If not, $W(|G|) \geqq 3$. Then it follows from (2.3 (i)) that G possesses an ordered Sylow tower, and so $P \lhd G$, where $P \in \mathrm{Syl}_p(G)$ and p is the largest prime divisor of the order of G. By hypothesis, P is an (A)-group. Then every subgroup H of P of order p is pronormal in P. On the other hand, every subgroup of P is subnormal. Now we conclude that every subgroup H of P of order p is normal in P and so $\Omega_1(P)=\langle x \in P \mid x^p=1\rangle \leqq Z(P)$. By (2.3 (ii)), P is the unique normal Sylow subgroup of G. Then by (2.3 (iii)) P has exponent p. Hence $\Omega_1(P)=P=Z(P)$. It follows now from (2.3 (iv)) that P is a minimal normal subgroup of G. Since G is solvable and $W(|G|) \geqq 3$, it follows from [7, Haupsatz 1.8, p. 662] that there exist two Hall subgroups K_1 and K_2 of G such that $P \leqq K_1 \cap K_2$ and $(|G:K_1|, |G:K_2|)=1$. Let H be a subgroup of K_i $(i=1,2)$ of order p. By hypothesis, K_i $(i=1,2)$ are (A)-groups. Then H is pronormal in K_i $(i=1,2)$. Clearly, H is subnormal in K_i $(i=1,2)$. Now we conclude that $H \lhd K_i$ $(i=1,2)$ and so $H \lhd G$. Since P is a minimal normal subgroup of G, it follows that $H = P$. By [7, Hauptsatz 1.8, p. 662], $G = HK$, where K is a p'-Hall subgroup of G. Clearly, $G/H \cong K$ is supersolvable. Now it follows easily that G is supersolvable, a contradiction. Thus $W(|G|) = 2$.

If G has an ordered Sylow tower, then there exists a normal Sylow p-subgroup P of G, where p is the largest prime divisor of the order of G. Hence, as in the preceding paragraph, P is elementary abelian. Let $Q \in \mathrm{Syl}_q(G)$, where $p > q$. We argue that Q is cyclic. If not, Q has at least two distinct maximal subgroups Q_i $(i = 1, 2)$. By hypothesis, PQ_i $(i = 1, 2)$ are (A)-groups. Let H be a subgroup of P of order p. Then H is both subnormal and pronormal in PQ_i $(i = 1, 2)$ and so $H \lhd PQ_i$ $(i = 1, 2)$. Hence $H \lhd G$. Now (2.3 (iv)) implies that $H = P$ and hence G is supersolvable, a contradiction. Thus Q is non-normal cyclic. This proves (2).

If G has not an ordered Sylow tower, then, by (2.3 (i)), G is a minimal non-nilpotent group. By (2.3 (ii)), G possesses a unique normal Sylow p-subgroup P. Obviously, $p < q$, where q is a prime divisor of the order of G and $W(|G|) = 2$. By (2.3 (iii)), P is of exponent p or P is of exponent 4.

Case 1. P is of exponent p. If $p = 2$, then it follows easily that P is elementary abelian 2-group and so G is a minimal non-abelian group. If $p \neq 2$, then, as in the preceding first paragraph, P is elementary abelian p-group and so G is a minimal non-abelian group. This proves (3).

Case 2. P is of exponent 4. By hypothesis, P is an (A)-group. Then every subgroup of P of order 2 or every cyclic subgroup of P of order 4 is pronormal in P. On the other hand, every subgroup of P is subnormal. Now we conclude that every subgroup of P of order 2 or every cyclic subgroup of P of order 4 is normal in P. Hence every subgroup of P is normal in P (recall that P is of exponent 4). Since P is non-abelian, it follows that P is Hamiltonian group (P is non-abelian group in which every subgroup is normal). From [7, Satz 7.12, p. 308], we conclude that $P = P_1 \times P_2$, where P_1 is quaternion group of order 8 and P_2 is an elementary abelian 2-group. Since P is Hamiltonian group, it follows that $\Omega_1(P) \leqq Z(P)$. Since $P' \neq 1$ and G is a minimal non-nilpotent group, it follows from Re'dei [8], that $P' = Z(P)$. By (2.3 (v)), $P' \leqq \Omega_1(P)$. Hence $\Omega_1(P) = P'$. On the other hand, $\Omega_1(P) = \Omega_1(P_1) \times P_2 = P_1' \times P_2$ and $P_1' = P'$. Now it follows that $P_2 = 1$ and so $P = P_1$. Since G is a minimal non-nilpotent group, it follows that $Z(G) = P' \times Q_1$, where Q_1 is a maximal subgroup of Q and $Q \in \mathrm{Syl}_q(G)$, and hence $G/Z(G) = \bar{G}$ is a minimal non-abelian group and $|\bar{P}| = 4$, where $\bar{P} \lhd \bar{G}$ and $\bar{P} \in \mathrm{Syl}_2(\bar{G})$. Clearly, $\bar{P}$ is elementary abelian. Since $\bar{G}/\bar{P} = \bar{G}/C_{\bar{G}}(\bar{P}) \cong \mathrm{Aut}(\bar{P})$, and $|\mathrm{Aut}(\bar{P})| = 6$, it follows that $q = 3$. This proves (4).

As an immediate corollary we have:

Corollary 3.2. *If every maximal subgroup of G is an* (A)*-group, then G' is nilpotent. Further, G is supersolvable when $W(|G|) \geqq 3$.*

Proof. Theorem 3.1 implies that either (i) G is supersolvable or (ii) $G = PQ$, where $P \lhd G$, $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ and Q is non-normal and cyclic. If G is as in (i), then G' is nilpotent [7, Satz 9.1 b), p. 716]. If G is as in (ii), then $G/P \cong Q$ and hence $P \geqq G'$.

4. Simple groups all of whose second maximal subgroups are (A)-groups. In this section, we prove the following theorem:

Theorem 4.1. *Let G be a non-abelian simple group with the property that all its second maximal subgroups are* (A)*-groups. Then G is one of the following groups*:

(a) PSL(2, p), *where p is a prime with* $P > 3$, $p^2 - 1 \not\equiv 0(5)$ *and* $p^2 - 1 \not\equiv 0(16)$;
(b) PSL(2, 2^q), *where q is a prime and* $2^q - 1 =$ *prime*;
(c) PSL(2, 3^q), *where q is an odd prime and* $(3^q - 1)/2 =$ *prime.*

P r o o f. Let M be an arbitrary maximal subgroup of G. Then all maximal subgroups of M are (A)-groups by hypothesis. Theorem 3.1 implies that M is solvable. Hence all proper subgroups of G are solvable. Applying Thompson's theorem (9; see also [7, Bemerkung 7.5, p. 190]) it follows that G is isomorphic to one of the following simple groups:

(1) PSL(3, 3);
(2) The Suzuki group $Sz(r)$, where $r = 2^q$ and q is an odd prime;
(3) PSL(2, p), where p is a prime with $p > 3$ and $p^2 - 1 \not\equiv 0(5)$;
(4) PSL(2, 2^q), where q is a prime;
(5) PSL(2, 3^q), where q is an odd prime.

We claim

(i) G cannot be PSL(3, 3).

Let x be an involution in the centre of a Sylow 2-subgroup of $G =$ PSL(3, 3). Then $C_G(x) \cong$ GL(2, 3) by [10, Lemma 5.1, p. 341]. We know that SL(2, 3) is a proper subgroup of GL(2, 3). Then SL(2, 3) is an (A)-group by hypothesis. Hence if H is a subgroup of SL(2, 3) of order 4, H would be pronormal and subnormal in SL(2, 3) and so $H \lhd$ SL(2, 3). This is impossible because SL(2, 3) has no normal subgroup of order 4.

(ii) G cannot be $Sz(r)$, where $r = 2^q$ and q is an odd prime.

By [10, Theorem 3.3, p. 184], $Sz(r)$ is a Zassenhaus group. Then by [3, Theorem 8.2, p. 41], $Sz(r)$ possesses a Frobenius group N with a cyclic complement H of order $(r - 1)$ and kernel P of order r^2. Since P is non-abelian, it follows that $Z(P)\,H$ is a proper subgroup of N. Hence if $G = Sz(r)$, $Z(P)\,H$ would be an (A)-group by hypothesis. Let x be an involution of $Z(P)$. Then $\langle x \rangle$ is both pronormal and subnormal in $Z(P)\,H$ and so $\langle x \rangle \lhd Z(P)\,H$. Hence $\langle x \rangle H = \langle x \rangle \times H$ and so $H \leqq C_N(x)$. But $C_N(x) \leqq P$ by [3, Theorem 7.6 (iv), p. 38]. This is a contradiction. Thus G can not be $Sz(r)$, where $r = 2^q$ and q is an odd prime.

(iii) G cannot be PSL(2, p), where p is a prime with $p > 5$, $p^2 - 1 \not\equiv 0(5)$, and $p^2 - 1 \equiv 0(16)$.

Suppose that $G =$ PSL(2, p), where p is a prime with $p > 5$, $p^2 - 1 \not\equiv 0(5)$ and $p^2 - 1 \equiv 0(16)$. Then by Dickson's theorem [7, Hauptsatz 8.27, p. 213], Sym(4) is a proper subgroup of $G =$ PSL(2, p). Clearly, Alt(4) is a proper subgroup of Sym(4). By hypothesis, Alt(4) is an (A)-group. This is a contradiction because Alt(4) contains no subgroup of order 6.

(iv) G cannot be $\mathrm{PSL}(2, 2^q)$, where q is an odd prime and $2^q - 1 \neq$ prime. Suppose that $G = \mathrm{PSL}(2, 2^q)$, where q is an odd prime and $2^q - 1 \neq$ prime. Then by (2.6), G possesses a Frobenius group N with kernel P of order 2^q and a cyclic complement H of order $2^q - 1$. P is elementary abelian 2-group. Since $2^q - 1 \neq$ prime, it follows that N possesses a proper subgroup $\langle x \rangle P$, where $\langle x \rangle$ is a proper subgroup of H. By hypothesis, $\langle x \rangle P$ is an (A)-group. Let y be an involution of P. Then $\langle y \rangle$ is both pronormal and subnormal in $\langle x \rangle P$ and so $\langle y \rangle \lhd \langle x \rangle P$. Hence $\langle y \rangle \langle x \rangle = \langle y \rangle \times \langle x \rangle$ and so $x \in C_N(y)$. But $C_N(y) \leqq P$ by [3, Theorem 7.6 (iv), p. 38]. This is a contradiction. Thus G can not be $\mathrm{PSL}(2, 2^q)$, where q is an odd prime and $2^q - 1 \neq$ prime.

(v) G cannot be $\mathrm{PSL}(2, 3^q)$, where q is an odd prime and $(3^q - 1)/2 \neq$ prime.

The proof of (v) is similar to that of (iv).

So the only possibility for G is:

(a) $\mathrm{PSL}(2, p)$, where p is a prime with $p > 3$, $p^2 - 1 \not\equiv 0(5)$ and $p^2 - 1 \not\equiv 0(16)$.
(b) $\mathrm{PSL}(2, 2^q)$, where q is a prime and $2^q - 1 =$ prime.
(c) $\mathrm{PSL}(2, 3^q)$, where q is an odd prime and $(3^q - 1)/2 =$ prime.

But we have seen in the analysis of (a), (b) and (c) that all the second maximal subgroups of G are (A)-groups (see remarks (2.5) and (2.7)). The theorem is proved.

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