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Frobenius Subgroups of Free Products of Prosolvable Groups

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Abstract. In this paper we establish the existence of profinite Frobenius subgroups in a free prosolvable product $A \amalg B$ of two finite groups A and B. In this way the classification of solvable subgroups of free profinite groups is completed.

1. Introduction

Let $\mathscr{C}$ be a class of finite groups closed under the operations of taking subgroups, quotients and extensions (e.g., the class of all finite groups). Let A and B be finite groups in $\mathscr{C}$, and let $G = A \amalg B$ be their free pro-$\mathscr{C}$-product, i.e., G is the coproduct of A and B in the category of pro-$\mathscr{C}$-groups. In [8] a description of the possible structure of a solvable subgroup H of G is given; if H is infinite, it must be of one of the following types: (i) $H \approx \hat{\mathbb{Z}}_\pi \rtimes \hat{\mathbb{Z}}_\sigma$, where π and σ are sets of prime numbers, $\hat{\mathbb{Z}}_\pi \approx \prod_{p \in \pi} \hat{\mathbb{Z}}_p$, and $\hat{\mathbb{Z}}_p$ is the additive group of p-adic integers; (ii) $H \approx \hat{\mathbb{Z}}_\pi \rtimes C_2$, where $2 \in \pi$, C_2 is the group of order 2, and it acts on $\hat{\mathbb{Z}}_\pi$ by inversion, i.e., H is the dihedral pro-π-group; (iii) $H \approx \hat{\mathbb{Z}}_\pi \rtimes C$, where C is a finite cyclic group whose order is not divisible by the primes in π, and the action of C on $\hat{\mathbb{Z}}_\pi$ is elementwise fixed-point-free, i.e., H is a profinite Frobenius group with cyclic kernel and cyclic complement. In addition, it is proved in [8] that the groups of types (i) and (ii) actually arise as subgroups of free pro-C-products.

Our first result is a description of the profinite Frobenius subgroups of a free product (Theorem 3.1 and Corollary 3.3). For the group G above the only infinite profinite Frobenius groups are solvable of the form $\hat{\mathbb{Z}}_\pi \rtimes C$, as in (iii) above.

The main purpose of this paper is to show that the groups of type (iii) described above, do also arise as subgroups of free products of finite groups (Theorem 5.5), in fact we prove that if $H \approx \hat{\mathbb{Z}}_\pi \rtimes C$ is as in (iii), then H is contained in any free prosolvable product $G = A \amalg B$ where A and B are finite solvable groups, C is a subgroup of A and B is non-trivial. It turns out that this is equivalent to showing that every element c of A normalizes an infinite procyclic subgroup of G. In contrast to this result, note that if $\Gamma = A * B$ is the free product of A and B as abstract groups, and $1 \neq c \in A$ is an element of order different from 2, then c cannot normalize an infinite cyclic subgroup of Γ: otherwise, c would centralize an infinite cyclic subgroup of Γ, and this is not possible in a free product.

The basic result of the paper is actually about finite solvable groups (Theorem 4.2): If C is a cyclic group of automorphisms of a finite solvable group F, N is a C-invariant normal subgroup of F such that F/N is cyclic, and the induced action of C on F/N is without non-trivial fixed points, then there exists a C-invariant cyclic subgroup Z of F with $ZN = F$, and C acts on Z without non-trivial fixed points.

2. Notation

Generally, we use the notation of [19] and [15]. By $\mathscr{C}$ we mean a full class of finite groups, i.e., a non-empty class of finite groups such that (i) if $G \in \mathscr{C}$, and H is a subgroup of G then $H \in \mathscr{C}$; (ii) if $G \in \mathscr{C}$ and H is a homomorphic image of G then $H \in \mathscr{C}$; (iii) if $1 \to K \to G \to H \to 1$ is an exact sequence of groups and K, $H \in \mathscr{C}$, then $G \in \mathscr{C}$. In this paper we are mainly interested in the case when $\mathscr{C}$ is the class of all finite solvable groups. A pro-$\mathscr{C}$-group is a projective limit of groups in $\mathscr{C}$. All homomorphisms of pro-$\mathscr{C}$-groups are assumed to be continuous; all subgroups of a pro-$\mathscr{C}$-group are assumed to be closed. If G is a group, $H \leqslant G$ will indicate that H is a subgroup of G; and $H < G$, that H is a proper subgroup of G. By $G = K \rtimes H$ we mean that the group G is a semidirect product of the normal subgroup K and the subgroup H. If G is a pro-$\mathscr{C}$-group, $\pi(G)$ denotes the set of prime numbers dividing the order of G, i.e., the set of prime numbers dividing the orders of all finite quotients of G; and $d(G)$ denotes the smallest cardinality of a set of generators of G converging to 1 (cf. [15], p. 60). For a prime number p, $\hat{\mathbb{Z}}_p$ denotes the additive group of p-adic integers. If π is a set of primes, then we put

$\hat{\mathbb{Z}}_\pi = \prod_{p \in \pi} \hat{\mathbb{Z}}_p$; if π is the set of all prime numbers, we set $\hat{\mathbb{Z}} = \hat{\mathbb{Z}}_\pi$. Finally, a cyclic group of order n is denoted by C_n.

3. Frobenius Subgroups of Free Products

Recall that a profinite Frobenius group H is a profinite group of the form $H = K \rtimes T$, where T (a proper non-trivial subgroup of H) acts on K elementwise fixed-point-free (i.e., for $1 \neq t \in T$ and $1 \neq k \in K$, one has $[t, k] \neq 1$), and the sets of primes $\pi(K)$ and $\pi(T)$ are disjoint (cf. [2]); K is the so called kernel of the Frobenius group, and T a complement. In this section we describe which profinite Frobenius groups could arise as subgroups of a free product of profinite groups. We show that such groups, if not contained in a conjugate of a free factor, must be solvable of the form $\hat{\mathbb{Z}}_\pi \rtimes C$, where $\hat{\mathbb{Z}}_\pi$ is the kernel and C is finite cyclic. As we will see in Th. 5.5, such groups $\hat{\mathbb{Z}}_\pi \rtimes C$, can always be embedded in a free prosolvable product of appropriate finite solvable groups.

Theorem 3.1. *Let $A_1, \ldots, A_n$ be finite groups in $\mathscr{C}$, $G = A_1 \amalg \ldots \amalg A_n$ their free pro-$\mathscr{C}$-product, and let H be a profinite Frobenius subgroup of G. If H is finite, then it is conjugate to a subgroup of one of the groups A_i; and if H is infinite, then $H \approx \hat{\mathbb{Z}}_\pi \rtimes C$ where C is a finite cyclic group whose order is not divisible by the primes in π, and C acts on $\hat{\mathbb{Z}}_\pi$ elementwise fixed-point-free.*

Proof. If H is finite, then by Th. 2 in [6], H is a conjugate of a subgroup of some A_i. Assume then that H is infinite; say that $H = K \rtimes T$, where K is the kernel of the Frobenius group, and $T(\neq \{1\}, H)$ a complement. By Th. 3.6 and Cor. 3.7 in [2], T is a finite group and K is a nilpotent profinite group. Hence K is the direct product of its p-Sylow subgroups, $K = \prod K_p$, where K_p is the p-Sylow subgroup of K, and p runs though the set of prime numbers. Consider the cartesian subgroup L of the free product G, i.e., the kernel of the homomorphism $G \to A_1 \times \ldots \times A_n$ that sends A_i to A_i identically for each i, then L is a normal subgroup of finite index of G, and $L \cap A_i = \{1\}$, for each $i = 1, \ldots, n$; therefore L is a free pro-$\mathscr{C}$-group (cf. [3], Th. 5.5). Since the index of L in G is finite and K is an infinite group, it follows that there is some prime number q such that K_q is infinite. By Prop. 2.1 in [8], either $K_q = \hat{\mathbb{Z}}_q$ or $K_q \approx \hat{\mathbb{Z}}_2 \rtimes C_2$ (the dihedral pro-2-group). However, the second alternative cannot occur:

if $K_q \approx \hat{\mathbb{Z}}_2 \rtimes C_2$, then the kernel subgroup $R \approx \hat{\mathbb{Z}}_2$ of K would be normalized by the elements of T; and since $\mathrm{Aut}(\hat{\mathbb{Z}}_2) \approx C_2 \times \hat{\mathbb{Z}}_2$ (cf. [18], p. 17), and $2 \nmid |T|$, one would have that T centralizes some non-trivial elements of K, a contradiction. So $K_q \approx \hat{\mathbb{Z}}_q$. Now, in the free product G, no element of finite order can have an infinite centralizer (as can be easily deduced from Th. 2 in [6]); and therefore if $p \neq q$, K_p must be torsion-free, if $p \neq q$; thus, by Prop. 2.1 in [8], $K_p = \{1\}$ or $K_p \approx \mathbb{Z}_p$ for all prime numbers p. I.e., $K \approx \hat{\mathbb{Z}}_\pi$, where π is the set of prime numbers p for which $K_p \neq \{1\}$. Finally, consider the homomorphism $\varphi\colon T \to \mathrm{Aut}(K_q) \approx \mathrm{Aut}(\hat{\mathbb{Z}}_q) \approx C_{q-1} \times \hat{\mathbb{Z}}_q$ (note that $2 \notin \pi$), induced by conjugation. If $1 \neq t \in T$ and $\varphi(t) = 1$, then t would centralize the infinite group $\hat{\mathbb{Z}}_q$, but this is not possible as we have pointed out above, since t is of finite order, thus φ is an injection, and T is a subgroup of C_{q-1}, and so cyclic. □

Next we extend the above result to a general free pro-$\mathscr{C}$-product $G = \coprod_X A_x$ of pro-C-groups A_x, indexed by a topological space X in the sense of [4], [13], [14] or [3]. It is not difficult to prove that G is a projective limit of pro-$\mathscr{C}$-groups $G = \varprojlim G_i$ over a directed set I with canonical epimorphisms $\psi_i\colon G \to G_i$ for $i \in I$, and $\psi_{ij}\colon G_i \to G_j$ for $i \geqslant j$, such that (1) each G_i is a free pro-$\mathscr{C}$-product $G_i = \coprod G_{ik}$ of a finite number of finite groups $G_{jk} \in \mathscr{C}$; (2) for every $i \in I$ and every $x \in X$, $\psi_i(A_x) \leqslant G_{ik}$ for some $k \in \{1, \ldots, n_i\}$; (3) if $i \geqslant j$ (in I), then ψ_{ij} maps every G_{ik} into some G_{jl}. See [16] for an explicit proof.

Lemma 3.2. *Let $G = \coprod A_x$ be a free pro-$\mathscr{C}$-product of pro-$\mathscr{C}$-groups A_x in the sense of* [4], [13], [3] or [14], *and let H be a subgroup of G. Then H is a conjugate of a subgroup of one of the free factors A_x of G if and only if the group $H_i = \psi_i(H)$ is finite for each $i \in I$, where ψ_i is the map defined above.*

Proof. Since $\psi_i(A_x)$ is finite for each i, if H is conjugate to a subgroup of some $A_x, \psi_i(H)$ will also be finite for each $i \in I$. Conversely, assume that for every $i \in I$, the group H_i is finite, then we may assume that each $H_i \neq \{1\}$, by taking a cofinal subset of I if necessary. Then H_1 is conjugate to a subgroup of a unique free factor $G_{ik(i)}$ of G_i (cf. Th. 2, [6]). Define $X_i = \{x \in G_i \mid H_i^X \leqslant G_{ik(i)}\}$; then X_i is obviously non-empty, and we assert that it is a compact set. For, let

$1 \neq h \in H$, and let $\varrho: G_i \to G_i$ be the continuous mapping given by $x \mapsto h^x$; then

$$\varrho^{-1}(G_i) = \{x \in G_i \mid h^x \in G_{ik(i)}\} = X_i,$$

where the last equality follows from Th. 2 in [6]; hence X_i is closed in G_i, and so compact. Since $\psi_{ij}(X_i) \subset X_j$, we have a projective system (X_i, ψ_{ij}); and since each X_i is compact and non-empty, $\varprojlim X_i$ is non-empty. Let $y \in \varprojlim X_i$. Then $H_y = \varprojlim H_i^{\psi_i(y)} \leqslant \varprojlim G_{ik(i)} = A_x$, where A_x is one of the free factors of G, as desired. □

Corollary 3.3. *Let* $G = \coprod A_x$ *be a free pro-*$\mathscr{C}$*-product of pro-*$\mathscr{C}$*-groups* A_x *in the sense of* [4], [13], [14] or [3]. *Let* H *be a profinite Frobenius subgroup of* G. *Then either* H *is conjugate to a subgroup of one of the free factors* A_x, *or* H *is of the form* $H \approx \hat{\mathbb{Z}}_\pi \rtimes C$, *where* C *is a finite cyclic group that acts on* $\hat{\mathbb{Z}}_\pi$ *elementwise fixed-point-free, and* π *is some set of primes.*

Proof. Put $H_i = \psi_i(H)$, for $i \in I$. Assume first that H_i is finite for each $i \in I$; then by Lemma 3.2, H is conjugate to a subgroup of some A_x. Hence, suppose that H_k is infinite for some $k \in I$; then we may assume that for all $i \in I$, H_i is infinite, by choosing a cofinal subset of I if necessary. Say $H = K \rtimes T$, where K is the kernel of H as a Frobenius group, and T a complement. Since T is finite, we may assume that $T \approx T_i = \psi_i(T)$, by taking a cofinal subset of I if necessary. Similarly, we may assume that $\psi_i(K) \neq \{1\}$, for all $i \in I$. Then $\psi_i(H) = K_i \rtimes T_i$ is a profinite Frobenius group, with kernel $K_i = \psi_i(K)$ (cf. [2], Lemma 1.3). By Th. 3.1, $K_i \approx \hat{\mathbb{Z}}_{\pi_i}$, and T_1 is cyclic. Thus $K = \varprojlim K_i \approx \hat{\mathbb{Z}}_\pi$, and T is cyclic, as desired. □

4. Lifting Frobenius Groups

Lemma 4.1. *Let* $S = Q \rtimes T$ *be a finite Frobenius group with cyclic kernel* $Q = \langle q \rangle$ *and cyclic complement* $T = \langle t \rangle$. *Say* $t^{-1} q t = q^\alpha$ *for some natural number* α, *and let* β *and* γ *be the orders of* Q *and* T, *respectively. Let* V *be a finite dimensional vector space over a field* F (*if* char $F = p > 0$, *we assume in addition that* $p \nmid \beta$). *Let* $\varrho: S \to \mathrm{GL}(V)$ *be an irreducible representation of* S *on* V. *We think of the elements of* $\mathrm{GL}(V)$ *as matrices with respect to a fixed basis of* V. *Then*

(i) *The eigenvalues of* $\varrho(q)$ *in the algebraic closure* $\bar{F}$ *of* F, *are primitive* β*-th roots of unity.*

(ii) *If* ζ *is an eigenvalue of* $\varrho(q)$, *then* $\zeta, \zeta^{\alpha}, \zeta^{\alpha^2}, \ldots, \zeta^{\alpha^{\gamma-1}}$ *are all distinct.*

(iii) *There is an invertible matrix* A *over* $\bar{F}$ *such that* $A^{-1}\varrho(q)A$ *is a diagonal block matrix* $\operatorname{diag}(B_1, \ldots, B_\gamma)$, *where* $B_j = \operatorname{diag}(\zeta_1 I_{\mu_1}, \ldots, \zeta_l I_{\mu_l})^{\alpha^{j-1}}$, *the* ζ_i*'s are eigenvalues of* $\varrho(q)$, *and* I_μ *represents an identity matrix of degree* μ.

(iv) $A^{-1}\varrho(t)A = MP$, *where* P *is block permutation matrix of the form*

$$P = \begin{pmatrix} 0 & 0 & \ldots & 0 & I \\ I & 0 & \ldots & 0 & 0 \\ 0 & I & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & I & 0 \end{pmatrix},$$

M *is a diagonal block matrix* $M = \operatorname{diag}(M_1, \ldots, M_\gamma)$, *and the square matrices* I, M_j, B_j *have the same size.*

(v) $\operatorname{Ker}(I + \varrho(t) + \varrho(t)^2 + \ldots + \varrho(t)^{\gamma-1}) = \operatorname{Im}(\varrho(t) - I)$.

Proof. (i) For $d \mid \beta$, define $V_d = \{v \in V \mid (\varrho(q)^d - 1)v = 0\}$. Then V_d is an S-invariant subspace of V, for $v \in V_d$ implies $(\varrho(q)^d - I)(\varrho(q)v) = \varrho(q)(p(q)^d - I)v = 0$, i.e., $\varrho(q)v \in V_d$; and also,

$$(\varrho(q)^d - I)\varrho(t)v = \varrho(t)(\varrho(q)^{\alpha d} - I)v =$$
$$= \varrho(t)[\varrho(q)^{(\alpha-1)d} + \varrho(q)^{(\alpha-2)d} + \ldots + \varrho(q)^d + I](\varrho(q)^d - I)v = 0,$$

i.e., $\varrho(t)v \in V_d$. Since ϱ is irreducible, it follows that $V_d = 0$ or $V_d = V$. Finally, if in addition $d \neq \beta$, we must have $V_d = 0$, since ϱ is faithful, hence if $d \neq \beta$, $\varrho(q)^d - I$ is invertible. Now, one has

$$X^\beta - 1 = \prod_{d \mid \beta} \psi_d(X),$$

where $\psi_d(X)$ is the d-th cyclotomic polynomial over F, i.e., the product $\prod (X - \zeta)$, where ζ runs through the primitive d-th roots of 1 in $\bar{F}$ (this is valid even if $\operatorname{char} F = p$, since $p \nmid \beta$). Then $0 = \varrho(q)^\beta - 1 = \prod_{d \mid \beta} \psi_d(\varrho(q))$. Since as observed above, $\psi_d(\varrho(q))$ is invertible if $d < \beta$, we deduce that $\psi_\beta(\varrho(q)) = 0$. I.e., the minimal polynomial of $\varrho(q)$ divides $\psi_\beta(X)$, and thus the eigenvalues of $\varrho(q)$ are primitive β-th roots of unity as desired.

(ii) Let ζ be an eigenvalue of $\varrho(q)$. If $\zeta^{\alpha^i} = \zeta^{\alpha^j}$, $0 \leqslant i, j < \gamma$, then $\alpha^j - \alpha^i$ is a multiple of β. It follows that for every eigenvalue ξ of $\varrho(q)$ one has $\xi^{\alpha^i} = \xi^{\alpha^j}$. Since $\varrho(q)$ is diagonalizable, we get $\varrho(q)^{\alpha^i} = \varrho(q)^{\alpha^j}$; i.e., $q^{t^i} = q^{t^j}$. But since T acts on Q fixed point free, we obtain $i = j$.

(iii) Since $\varrho(q)$ and $\varrho(q^\alpha)$ are conjugate, they have the same eigenvalues, and clearly, if ζ is an eigenvalue, ζ and ζ^α have the same multiplicity. Let $\zeta_1, \ldots, \zeta_l$ be representatives of the different orbits in the set of eigenvalues of $\varrho(q)$ under the action of T (the action is $\zeta^t = \zeta^\alpha$). Let μ_i be the multiplicity of ζ_i. Put

$$B_1 := \operatorname{diag}(\zeta_1 I_{\mu_1}, \ldots, \zeta_l I_{\mu_l}), \text{ and } B_j := B_1^{\alpha^{j-1}}.$$

Note that then $B := \operatorname{diag}(B_1, \ldots, B_\gamma)$ and $\varrho(q)$ are conjugate. Finally, choose a matrix A such that $A^{-1}\varrho(q)A = B$.

(iv) Set $L := A^{-1}\varrho(t)A$. Then $L^{-1}BL = B^\alpha = \operatorname{diag}(B_2, B_3, \ldots, B_\gamma, B_1)$. Consider the block permutation matrix P as defined above. Then $L^{-1}BL = P^{-1}BP$. Put $M := LP^{-1}$, so that $BM = MB$. Since B is diagonal and the entries in B_i and B_j $(i \neq j)$ are different by part (ii), it follows that $M = \operatorname{diag}(M_1, \ldots, M_\gamma)$ where M_i is a square matrix of the same size as B_1 $(i = 1, \ldots, \gamma)$.

(v) From the proof of (iv) we have $L = MP$. By part (iv), statement (v) is equivalent to $\operatorname{Ker}(I + L + L^2 + \ldots + L^{\gamma-1}) = \operatorname{Im}(L - I)$. Note first that

$$L = \begin{pmatrix} 0 & 0 & \ldots & 0 & M_\gamma \\ M_1 & 0 & \ldots & 0 & 0 \\ 0 & M_2 & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & M_{\gamma-1} & 0 \end{pmatrix},$$

$$L^2 = \begin{pmatrix} 0 & 0 & \dots & 0 & M_\gamma M_{\gamma-1} & 0 \\ 0 & 0 & \dots & 0 & 0 & M_1 M_\gamma \\ M_2 M_1 & 0 & \dots & 0 & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & M_{\gamma-1} M_{\gamma-2} & 0 & 0 \end{pmatrix},$$

$I = L^\gamma = \operatorname{diag}(M_\gamma M_{\gamma-1} \dots M_1, \dots, M_{\gamma-1} M_{\gamma-2} \dots M_\gamma)$. It follows that $M_1 M_2 \dots M_\gamma = I$, and $\bar{L} := I + L + L^2 + \dots + L^{\gamma-1} =$

$$\begin{pmatrix} I & M_\gamma M_{\gamma-1} \dots M_2 & M_\gamma M_{\gamma-1} \dots M_3 & \dots & M_\gamma \\ M_1 & I & M_1 M_\gamma \dots M_3 & \dots & M_1 M_\gamma \\ \dots & \dots & \dots & \dots & \dots \\ M_{\gamma-1} M_{\gamma-2} \dots M_1 & M_{\gamma-1} M_{\gamma-2} \dots M_2 & M_{\gamma-1} M_{\gamma-2} \dots M_3 & \dots & I \end{pmatrix}.$$

It is immediate that the rank of $\bar{L}$ is μ ($:=$ degree of M_i), and therefore $\dim(\operatorname{Ker} \bar{L}) = (\gamma - 1)\mu$. On the other hand, from the identity $(L - I)\bar{L} = L^\gamma - I = 0$, we obtain $\operatorname{Ker} \bar{L} \supseteq \operatorname{Im}(L - I)$. Therefore to finish the proof it suffices to show that $\operatorname{Rank}(L - I) \geqslant (\gamma - 1)\mu$, which is obvious. □

Theorem 4.2. *Let $G = F \rtimes C$ be a finite solvable group, which is the semidirect product of a cyclic subgroup C and a normal subgroup F. Let N be a subgroup of F which is normal in G and such that F/N is cyclic. Assume that the induced action of C on F/N is efpf (elementwise fixed point free; i.e., if $x \neq 1$ is in C, and y is F with $yN \neq N$, then $(yN)^x \neq yN$). Then F contains a cyclic subgroup D such that:*

(i) *D is normalized by C, and C acts on D efpf, i.e., DC is a Frobenius group with Frobenius kernel D and complement C;*

(ii) *$\pi(D) = \pi(F/N)$; and*

(iii) *$DN = F$.*

Proof. The proof of the theorem is by contradiction. If the theorem is false, consider a group G as above for which the conclusion of the theorem fails, and such that $|N|$ is minimal, and among those, $|G|$ is minimal; we refer to such a group G, as a "minimal counterexample". Note that then $N \neq 1$. We shall establish a series of claims that will lead to a final contradiction.

Claim 1. N is a minimal normal subgroup of G. For, let $1 \neq M < N$, with M normal in G. Then G/M, F/M, N/M, and $CM/M \approx C$ satisfy the hypotheses of the theorem. Therefore, since $|N/M| < |N|$, there exists $D_1 \leqslant F$ containing M, such that D_1/M is cyclic, D_1 is normalized by C, the action of C on D_1/M is efpf, $\pi(D_1/M) = \pi(F/N)$ and $D_1 N = F$. Since $|M| < |N|$, we can again apply the theorem to C, D_1 and M, to get a cyclic subgroup D of D_1 such that C normalizes D, C acts efpf on D, $\pi(D) = \pi(D_1/M) = \pi(F/N)$, and $DM = D_1$; and hence $DN = D_1 N = F$. This means that G is not a counterexample, contradicting our hypotheses. Thus N is minimal normal in G, as claimed.

Claim 2. N is an elementary abelian p-group, for some prime number p. This is a well-known fact for minimal normal subgroups of finite solvable groups.

Claim 3. p divides the order of G/N. Suppose not. Then by Schur—Zassenhaus' lemma (cf. [17], 9.3.6), there exist a Hall subgroup L_1 of G, with $G = N \rtimes L_1$. Since G is solvable and $|C|$ divides $|L_1|$, there is some $g \in G$ with $C \leqslant L_1{}^g$. Put $L := L_1{}^g$. Then $G = N \rtimes L$, and $L \approx G/N$ is a Frobenius group with isolated subgroup C. Note that the Frobenius kernel of L is a normal subgroup D of L isomorphic to F/N. We will prove now that $D \leqslant F$. Note $F = F \cap G = F \cap (NL) = N(F \cap L)$. Since D and $F \cap L$ are Hall subgroups of L of the same order, they are conjugate, and hence $D = F \cap L$ since $F \cap L$ is normal in L. So D is a subgroup of F satisfying the conclusions of the theorem, and hence G is not a counterexample. A contradiction. Thus $p \in \pi(G/N)$, as desired.

Claim 4. The prime p divides $|C|$. Suppose not. Then, by claim 3, p must divide $|F/N|$. Let P be the unique p-Sylow subgroup of F. Since P is characteristic in F, it is normal in G. Denote by P^* the Frattini subgroup of P. We will consider two cases.

Case 1. $P^* \cap N \neq 1$. Since N is a minimal normal subgroup of M, $P^* \cap N = N$, and so $N \leqslant P^*$. Then P/P^* is a subquotient of F/N, and therefore it is cylic. It follows that P is cyclic. By Schur—Zassenhaus' lemma there is a complement Q of P in F. The action of Q on F by conjugation, induces an action on F/N that is trivial since F/N is cyclic. So the induced action on F/P^*, and hence on P/P^*, is also trivial. Therefore Q acts trivially on P via conjugation (cf. [9], Satz 3.18, p. 275). This means that $F = P \times Q$, and Q is characteristic in F. Consequently, $Q \lhd G$, Q is cyclic, and since C acts efpf on F/N, the action of C on Q is also efpf. Since C acts efpf on P/N, it must act efpf on P/P^* (cf. [17], 12.6.6), and thus on P (cf. [9], Satz 3.18, p. 275). Hence C acts efpf on the cyclic group F, and therefore G is not a counterexample. A contradiction.

Case 2. $P^* \cap N = 1$. By Schur—Zassenhaus' lemma we may choose a complement R of P in G, and replacing it by a conjugate if necessary, we will assume that $C \leqslant R$. The action of R on P by conjugation induces an action of R on P/P^*. Note that NP^*/P^* is an $F_p R$-submodule of the $F_p R$-module P/P^* (F_p is the field with p elements), and so by Maschke's theorem (cf. [9], Satz 17.7, p. 123), there is an R-invariant subgroup U of P such that $U \geqslant P^*$ and U/P^* is a complement of NP^*/P^* in P/P^*. Since $U/P^* \approx P/NP^*$ and P/N is cyclic, then U/P^* is cyclic. Say $U = \langle u, P^* \rangle$, with $u \in U$, and $u^p \in P^*$. Then $P = \langle u, P^*, N \rangle$. Observe that U is a normal subgroup of G. We now have two possibilities, either $\langle u, P^* \rangle \cap N \neq 1$ or $\langle u, P^* \rangle \cap N = 1$. The first alternative implies that $N \leqslant \langle u, P^* \rangle$, for N is minimal normal in G; then $P = \langle u, P^* \rangle$, and hence P is cyclic; but since by assumption $N \neq 1$ and $P^* \cap N = 1$, we deduce $P^* = 1$, so that $|P| = p$; this, however, is not possible since $P > N \neq 1$. Therefore, we are left with the other alternative, $\langle u, P^* \rangle \cap N = 1$. It then follows that $U = \langle u, P^* \rangle$ is cyclic, since P/N is cyclic, and also that $P = U \times N$. Observe that $F = F \cap G = F \cap PR = P(F \cap R)$. Set $Q := F \cap R$. Then Q is normal in R. We then have that C normalizes the cyclic group $UQ \approx F/N$, and C acts on it efpf, since $UQ \cap N = 1$. Moreover, $UQN = F$. Thus G is not a counterexample, against our assumption. Therefore, p divides $|C|$ as asserted.

Claim 5. The centralizer $E := C_G(N)$ *of* N *in* G *is* N. Since G/N is a Frobenius group with Frobenius kernel F/N and $E \lhd G$, one has that either $E/N < F/N$ or $E/N \geqslant F/N$ (cf. [9], Satz 8.16, p. 506). Since N is a normal Hall subgroup of F, there exists a subgroup Q of F with $F = N \rtimes Q$. Remark that $Q \approx F/N$ is cyclic. If $E/N \geqslant F/N$, i.e., $E \geqslant F$, then $F = N \times Q$; consequently $D := Q$ satisfies conditions (i), (ii) and (iii) of the theorem, and hence G is not a counterexample. Thus

$E < F$. Note that $C_G(N) = C_F(N) = N \times C_Q(N)$. Then, to prove that $E = N$, it suffices to show that $M := C_Q(N) = 1$. Remark that M is normal in G. Consider the group $\bar{G} := G/M = (F/M) \rtimes (CM/M)$. If $M \neq 1$, then $\bar{N} := NM/M < \bar{F} := F/M$. Moreover, $\bar{G}/\bar{N} \approx G/NM$ is Frobenius with (cyclic) kernel $\bar{F}/\bar{N} \approx F/NM$ (cf. [17], 12.6.6). Since $|\bar{N}| = |N|$ and $|\bar{G}| < |G|$, $\bar{G}$ is not a counterexample to the theorem. Hence there exists a cyclic subgroup $\bar{D} \leqslant \bar{F}$ on which $\bar{C} := CM/M$ operates efpf, $\overline{DN} = \bar{F}$ and $\pi(\bar{D}) = \pi(\bar{F}/\bar{N})$. Let D be a preimage of $\bar{D}$ under the canonical map $F \to F/M$. Clearly C normalizes D. Note that $\pi(D) = \pi(F/NM) \cup \pi(M) = \pi(Q)$, so that $D \cap N = 1$, and hence D is cyclic. Also C acts efpf on D, and $DN = F$. Thus G would not be a counterexample. Therefore $M = 1$, as desired.

Claim 6. $G = N \rtimes S$, where S is a Frobenius group with kernel isomorphic to F/N. By a result of Baer, since N is a minimal normal subgroup of a solvable group G, there exists a subgroup S of G such that $G = C_G(N)S$ and $S \cap N = 1$ (cf. [9], p. 688). By claim 5, $C_G(N) = N$, and so $G = NS$. Obviously $S \approx G/N$, which by assumption is a Frobenius group with kernel F/N.

Claim 7. The representation via conjugation of S on the F_p-vector space N, is irreducible and faithful. The representation is irreducible since N is minimal normal in G. And it is faithful since $C_S(N) = 1$, according to claim 5.

After establishing these series of claims, we are in a position to finalize the proof of the theorem. Set $Q := F \cap S$. Then $F = N \rtimes Q$, and Q is the kernel of S as a Frobenius group. Say $S = Q \rtimes T$ where $T \approx C$ is cyclic. Note that NC and NT are Hall subgroups of G of the same order, and hence they are conjugate. Say $NC = (NT)^g$, with $g \in G$. Substituting S by S^g, we may assume that $NC = NT$. Let $C = \langle x \rangle$. Then $x = tn$, for some $n \in N$ and $t \in T$. Clearly $\langle t \rangle = T$. Denote by γ the order of t. Then

$$1 = x^\gamma = (tn)^\gamma = t^\gamma n^{t^{\gamma-1}} n^{t^{\gamma-2}} \dots n^t n = n^{t^{\gamma-1}} n^{t^{\gamma-2}} \dots n^t n.$$

Now, according to claim 7, the hypotheses of Lemma 4.1 hold (N plays the role of V.) Thus by part (v) of Lemma 4.1, there exists some $m \in N$ such that $n = t^{-1} m t m^{-1}$. Then $x = tn = tt^{-1} m t m^{-1} = = m t m^{-1}$. Therefore $C = \langle x \rangle = m^{-1} \langle t \rangle m = m^{-1} T m$ normalizes the cyclic group $D := m^{-1} Q m \leqslant F$ which satisfies the conditions (i), (ii) and (iii) of the theorem. Thus G cannot be a counterexample, and the theorem is proved. □

Next we shall extend Theorem 4.2 to the case when F is an infinite prosolvable group. We need first some auxiliary results that may be well-known to specialists, but for which there is no easily accessible reference.

Lemma 4.3. *Let G be a profinite group, H an open subgroup of G, and let $\{U_i \mid i \in I\}$ be a collection of open normal subgroups of G. Assume that $\bigcap_{i \in I} U_i \leqslant H$. Then, there exists a finite subset J of I such that $\bigcap_{j \in J} U_j \leqslant H$.*

Proof. Otherwise, consider the closed subsets of G, $G - H$ and $\bigcap_{k \in K} U_k$ for all finite subsets K of I. Since we may assume that $H \neq G$, those sets have the finite intersection property, and therefore, since G is compact, $(G - H) \cap (\bigcap_{j \in J} U_j) \neq \emptyset$, ([10], p. 136), a contradiction. □

Lemma 4.4. *Let G be a profinite group, and let $\{U_i \mid i \in I\}$ be a collection of open normal subgroups of G such that for every finite subset J of I, there exists some $r \in I$ with $U_r \subset \bigcap_{j \in J} U_j$. Let $H = \bigcap_{i \in I} U_i$, and let K be any closed subgroup of G. Then $\bigcap_{i \in I} U_i K = HK$.*

Proof. Since HK is the intersection of all the open subgroups of G containing HK ([15], p. 11), it suffices to prove that $\bigcap_{i \in I} U_i K \leqslant V$ for every open subgroup V of G containing HK. By Lemma 4.3 and our hypothesis, there is some $r \in I$ such that $U_r \leqslant V$. It follows that $\bigcap_{i \in I} U_i K \leqslant V$. □

Proposition 4.5. *Let $G = F \rtimes C$ be a prosolvable group which is a semidirect product of a finite cyclic group C and a normal prosolvable group F. Assume that there exists a normal subgroup N of G such that $F/N \approx \hat{\mathbb{Z}}_\pi$ for a certain set of primes π, and the induced action of C on F/N by conjugation is efpf. Then F contains a cyclic subgroup $Z \approx \hat{\mathbb{Z}}_\pi$ such that $ZN = F$, C normalizes Z, and the action of C on Z by conjugation is efpf.*

Proof. First we make two remarks. Note that it suffices to find a subgroup $Z \approx \hat{\mathbb{Z}}_\pi$ of F which is normalized by C and such that $ZN = F$. This is so since on such a group Z, the group C will act automatically efpf: $Z/Z \cap N = ZN/N \approx F/N \approx \hat{\mathbb{Z}}_\pi \approx Z$; therefore from the structure of procyclic groups ([15], p. 56) one has $Z \cap N = 1$; hence if $1 \neq x \in C$, $z \in Z$ and $x^{-1} z x = z$, one has $z \in N$ since C acts efpf on F/N by hypothesis, and thus $z = 1$. Our second remark is that one may assume that F, and hence G, is (topologically) finitely generated. For let $f \in F$ be such that fN is a generator of F/N; consider the group $\bar{G} := \langle C, f^x \mid x \in C \rangle$ and let $\bar{F} := \langle f^x \mid x \in C \rangle$ and $\bar{N} := \bar{F} \cap N$. Then $\bar{N}$ is normal in $\bar{G}$, and $\bar{N} < \bar{F} < \bar{G} < G$ with $\bar{G} = \bar{F} \rtimes C$, $\bar{F}/\bar{N} \approx \hat{\mathbb{Z}}_\pi$ and the action of C on $\bar{F}/\bar{N}$ is efpf. If there exists some $\bar{Z} < \bar{F}$ normalized by C, $\bar{Z} \approx \hat{\mathbb{Z}}_\pi$ and $\bar{Z}\bar{N} = \bar{F}$, then $\bar{Z}N = F$. Therefore from now on we will assume that G is topologically finitely generated.

Next we prove the following assertion. Let M be an open normal subgroup of G contained in F; then there exists a procyclic subgroup Z_M of F, such that MZ_M is normalized by C, $MZ_M N = F$ and $\pi(Z_M) = \pi(F/MN)$. To prove this claim, consider the exact sequence

$$1 \to MN/M \to F/M \to F/MN \to 1.$$

Then $CM/M \approx C$ acts by conjugation efpf on the finite cyclic group $(F/M)/(NM/M) \approx F/NM$ (cf. [2], Th. 3.6 and Cor. 1.4). Hence, by Theorem 4.2, there exists a subgroup R of F containing M such that R/M is C-invariant and procyclic, $RN = F$, and $\pi(R/M) = \pi(F/MN)$. Let rM generate R/M and such that $\pi(\langle r \rangle) = \pi(R/M)$, and put $Z_M := \langle r \rangle$. It is plain that Z_M satisfies the conditions required in the assertion.

Now consider a sequence of open normal subgroups $F > M_1 > M_2 > \dots$ of G such that $\bigcap_i M_i = 1$. By the above assertion, for each i there exists some procyclic subgroup Z_i of F such that $Z_i M_i$ is C-invariant, $\pi(Z_i) = \pi(F/NM_i)$, and $Z_i M_i N = F$. Let z_i be a generator of Z_i. Since G is a compact metric space, taking a subsequence of (z_i) if necessary, we may assume that $\lim z_i = z$, for some element $z \in F$. Set $Z := \langle z \rangle$. Then for each $k \in \mathbb{N}$, there is some $l(k) \in \mathbb{N}$ with $z_l \in z M_k$ if $l \geqslant l(k)$, and so $z_l M_l \leqslant Z M_k$ if

$l \geqslant \max\{k, l(k)\}$. Hence $F = Z_l M_l N \leqslant Z M_k N \leqslant F$, and thus $Z M_k F = F, \forall k \in \mathbb{N}$. Therefore $F = \bigcap_k Z M_k N = Z N$, by Lemma 4.4.

To see that Z is C-invariant, note that $x^{-1} z_l x \in Z_l M_l \leqslant Z M_k$ if $x \in C$, $l \geqslant \max\{k, l(k)\}$; therefore $x^{-1} z x = \lim x^{-1} z_l x \in Z M_k$, for every $k \in \mathbb{N}$; hence $x^{-1} z x \in \bigcap_k Z M_k = Z$, by Lemma 4.4.

Let Z_π be the π-Hall subgroup of Z. Clearly Z_π is also C-invariant, and $Z_\pi N = F$. Hence we will assume from now on that $Z = Z_\pi$. To see that $Z \approx \hat{\mathbb{Z}}_\pi$, it remains to prove only that Z is torsion-free. Now, since $Z/Z \cap N \approx ZN/N = F/N \approx \hat{\mathbb{Z}}_\pi$, we have $Z \approx \hat{\mathbb{Z}}_\pi \oplus t(Z)$ where $t(Z)$ is the torsion part of Z. Since $\pi(t(Z)) \subset \pi$ and Z is cyclic, one must have that $t(Z)$ is trivial, i.e., Z is torsion-free. □

5. Frobenius Subgroups of Free Prosolvable Products

In this section we show that every Frobenius profinite group with cyclic kernel and cyclic complement can be realized as a subgroup of a free prosolvable product of finite solvable groups.

Lemma 5.1. *Let A and B be non-trivial pro-$\mathscr{C}$-groups with $|A| + |B| \geqslant 5$, and let $G = A \amalg B$ be their free pro-$\mathscr{C}$-product. Then for every natural number n, there exists an open subgroup H of G containing A such that $d(H) \geqslant n$.*

Proof. Let U and V be open normal subgroups of A and B respectively. Consider the canonical epimorphism $\varphi: G = A \amalg B \to T = A/U \amalg B/V$, if H is an open subgroup of T containing A/U, then $\varphi^{-1}(H)$ is open in G and contains A. Thus to prove the lemma, we may assume that A and B are finite.

Let K be the cartesian subgroup of G, i.e., the kernel of the homomorphism $G \to A \times B$ that sends A to A, and B to B, respectively. Since K is an open normal subgroup of G and $K \cap A = K \cap B = \{1\}$, K is free pro-C by the Kurosh subgroup theorem; moreover the rank of K is $(|A| - 1)(|B| - 1) \geqslant 2$ (cf. [1]): in fact K is free pro-C on the basis $\{[a, b] \mid 1 \neq a \in A, 1 \neq b \in B\}$ as can be easily deduced from the analogous result for free products of abstract groups (cf., e.g., [11], p. 196). Let K' denote the derived group of the cartesian subgroup K of G. Since K' has infinite index in K, it follows that K' is not finitely generated, i.e., $d(K') = \infty$ (cf. [12], Th. 3.5). Therefore $d(K'A) = \infty$, since K' has finite index in $K'A$. Hence there is an open normal subgroup U of G with $d(K'A\,U/U) \geqslant n$, and so $d(K'A\,U) \geqslant n$. Thus set $H := K'A\,U$. □

Lemma 5.2. *Let A, B be finite groups in $\mathscr{C}$ and let $G = A \amalg B$ be their free pro-$\mathscr{C}$-product. Then G contains a subgroup of the form $H = A \amalg \hat{\mathbb{Z}}_{\mathscr{C}}$, with $\hat{\mathbb{Z}}_{\mathscr{C}} = \prod \mathbb{Z}_p$, where p ranges over the primes that divide the order of some group in $\mathscr{C}$.*

Proof. Let $n/2 > \delta := \max\{d(U) \mid U \leqslant A \text{ or } U \leqslant B\}$. By Lemma 4.1, there exists some open subgroup $L \leqslant G$ with $L \geqslant A$ and $d(L) \geqslant n$. By the Kurosh subgroup theorem (which is valid for L) one has (cf. [1])

$$L = (\coprod A^s \cap L) \amalg (\coprod B^t \cap L) \amalg F,$$

where F is a certain free pro-$\mathscr{C}$ group, and s and t range through sets of representatives of the double cosets of A and L in G, and of B and L in G. We may assume that the representative of the double coset $A\,L$ is 1, and hence $A = A \cap L$ is one of the free factors of L in the above decomposition. If $F \neq 1$, we are done, since F is a free pro-$\mathscr{C}$-product of copies of $\hat{\mathbb{Z}}_{\mathscr{C}}$, and so $A \amalg \hat{\mathbb{Z}}_{\mathscr{C}}$ is naturally contained in G. Hence assume that $F = 1$. Next note that

$$d(L) \leqslant \sum_s d(A^s \cap L) + \sum_t d(B^t \cap L) \leqslant (|I| + |J|)\,\delta,$$

where I (respectively J) denote the set of those indexes i (respectively j) with $A^s \cap L \neq 1$ (respectively $B^t \cap L \neq 1$). Since n was chosen so that $n > 2\,\delta$, one has $|I| + |J| > 2$. Therefore G contains a subgroup of the form $R = A \amalg A_1 \amalg A_2$, where A_1 and A_2 are non-trivial finite groups. Let K be the cartesian subgroup of R, and consider the open subgroup $M := A\,K$ of R. Again by Kurosh,

$$M = (\coprod_\alpha A^\alpha \cap M) \amalg (\coprod_\beta A_1^\beta \cap M) \amalg (\coprod_\gamma A_2^\gamma \cap M) \amalg \Phi, \qquad (*)$$

with Φ a free pro-$\mathscr{C}$-group whose rank is (cf. [1])

$$d(\Phi) = 1 - |R/M| + (|R/M| - |A \backslash R/M|) + (|R/M| -$$
$$- |A_1 \backslash R/M|) + (|R/M| - |A_2 \backslash R/M|),$$

where $A \backslash R/M$ denotes the set of double cosets of G with respect to the subgroups A, M; etc. Since M is normal in R and $M \geqslant A$, one deduces that $|A \setminus R/M| = |R/M|$, $|A_1 \setminus R/M| = |A_2|$, $|A_2 \setminus R/M| = |A_1|$, and therefore $d(\Phi) = 1 + |A_1||A_2| - |A_2| - |A_1| = (|A_1| - 1)(|A_2| - 1) \geqslant 1$. Since $A \leqslant M$, we may assume that A is one of the free factors of M in the expression (*). It then follows that $A \amalg \hat{\mathbb{Z}}_{\mathscr{C}}$ is naturally embedded in M, and thus in G. □

Next we proof a generalization of a result of D. HARAN and A. LUBOTZKY ([5], Proposition 4).

Lemma 5.3. *Let $\mathscr{C}$ be a full class of finite groups, A and B pro-$\mathscr{C}$-groups, and A' and B' closed subgroups of A and B respectively. Then the free pro-$\mathscr{C}$-product $G' = A' \amalg B'$ is canonically embedded in $G = A \amalg B$.*

Proof. Since $G = \varprojlim((A/U) \amalg (B/V))$, with U and V ranging through the open normal subgroups of A and B respectively, we may assume that A and B are finite (in $\mathscr{C}$). Consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \rightarrow & K & \rightarrow & G = A \amalg B & \xrightarrow{\varphi} & A \times B & \rightarrow & 1 \\ & & {\scriptstyle\alpha}\uparrow & & {\scriptstyle\beta}\uparrow & & {\scriptstyle\gamma}\uparrow & & \\ 1 & \rightarrow & K' & \rightarrow & G' = A' \amalg B' & \underset{\varphi'}{\rightarrow} & A' \times B' & \rightarrow & 1 \end{array}$$

with exact rows (φ sends A, B identically to A, B respectively, and similarly for φ'). Recall that K (the cartesian subgroup of $G = A \amalg B$) is freely generated by the elements of the form $[a, b]$ for $a \neq 1 \neq b$, and similarly for K'. The map β is induced by the inclusions $A' \hookrightarrow A$ and $B' \hookrightarrow B$, and α and γ are given by $\alpha([a', b']) = [a', b']$ and $\gamma(a', b') = (a', b')$. Clearly α and γ are monomorphisms. Thus β is also a monomorphism. □

Corollary 5.4. *If G, A, B, A', B' are as above, the closed subgroup H of G generated by A' and B' is $A' \amalg B'$; moreover $H \cap A = A'$, and $H \cap B = B'$.*

Proof. The first assertion follows immediately from the above lemma. For the second assertion, note first that by a standard limit argument, the diagram in the proof of Lemma 5.3 is still valid even if the groups A and B are not finite. Now if $x \in H \cap A$, then $\varphi(x) \in \varphi(H) \cap A = A'$, i.e., $x \in A$, since φ is the identity on A. □

Theorem 5.5. *Let A, B be finite solvable groups. Assume that C is a cyclic subgroup of A, $\sigma \neq \emptyset$ is a set of primes such that $\sigma \cap \pi(C) = \emptyset$, $p \in \pi(C)$ and if $q \in \sigma$ then $p \mid (q-1)$. Consider the prosolvable product $G = A \amalg_s B$ of the groups A and B, and let K be its cartesian subgroup. Then K contains a procyclic subgroup Z such that $\sigma = \pi(Z)$ and Z is normalized by C. Moreover, the action of C on Z by conjugation is elementwise fixed-point-free (efpf), i.e., $H = ZC = Z \rtimes C$ is a profinite Frobenius group with kernel Z and complement C.*

Proof. It suffices to prove the first statement, for if the elements $1 \neq x \in C$ and $1 \neq y \in Z$ commute, then every element of the infinite subgroup $\langle y \rangle$ of Z generated by y also commutes with x; however the centralizer in G of a non-trivial element of A must be contained in A (cf. [6], Th. 2), and hence must be finite.

By Lemma 5.2, G contains a subgroup $A \amalg_s L$, where $L \approx \hat{\mathbb{Z}}_\sigma$. Hence by Lemma 5.3, G contains a subgroup $H = L \amalg_s C$. Recall that for a prime number $q \neq 2$, $\mathrm{Aut}(\hat{\mathbb{Z}}_q) \approx C_{q-1} \times \hat{\mathbb{Z}}_q$ (cf., [18], p. 17); so if $q \in \sigma$ and $p \in \pi(C)$, there are non-trivial actions $C_q \to \mathrm{Aut}(\hat{\mathbb{Z}}_p)$, and in fact any two of them define isomorphic semidirect products $\hat{\mathbb{Z}}_p \rtimes C_q$; moreover since the action of C_q on $\hat{\mathbb{Z}}_p$ is efpf, $\hat{\mathbb{Z}}_p \rtimes C_q$ is a profinite Frobenius group (cf. [2], Th. 3.6), it follows that there is a (unique) profinite Frobenius group $\Gamma = L \rtimes C$ with kernel L and complement C. Let $\varphi\colon H \to \Gamma$ be the homomorphism that sends L to L identically, and C to C identically, and let $N := \mathrm{Ker}(\varphi)$. Denote by F the normal subgroup of H generated by L. Then $N < F$, and the action of C on $F/N \approx \hat{\mathbb{Z}}_\sigma$ induced by conjugation within H, is efpf. Therefore by Proposition 4.5, there exists a subgroup Z_1 of F normalized by C, such that $Z_1 \approx \hat{\mathbb{Z}}_\sigma$. Put $Z := Z_1 \cap K$. Then Z is still normalized by C, and $Z \approx \hat{\mathbb{Z}}_\sigma$, since Z has finite index in Z_1. □

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