Z. Phys. B - Condensed Matter 52, 259–266 (1983)

Condensed Matter
Zeitschrift für Physik B

Eigenvalues and Their Connection to Transition Rates for the Brownian Motion in an Inclined Cosine Potential

H.D. Vollmer and H. Risken
Abteilung für Theoretische Physik der Universität Ulm,
Federal Republic of Germany

Received May 9, 1983

For certain parameters the motion of particles in an inclined cosine potential is bistable, i.e. particles are either in a locked or a running state. Fluctuations will cause transitions between these two states. First the connection of the transition rates with the lowest non-zero eigenvalue and the stationary solution of the Fokker-Planck equation is given. Then the eigenvalues of the Fokker-Planck equation for this Brownian motion problem are calculated using the matrix continued fraction method. Finally explicit results for these (generally complex) eigenvalues as a function of the averaged angle of inclination are shown for three typical friction constants and various temperatures.

1. Introduction

The problem of Brownian motion in a cosine potential $f(x)=-d\cos x$ with an additional constant force F (assumed to be non-negative) i.e. the motion in the inclined washboard potential

$$V(x)=-d\cos x-Fx \tag{1.1}$$

arises for instance in connection with Josephson tunneling junctions [1–5], phase locked loops [6–10], motion of electric dipoles in an external field [11–14], and superionic conductors [15–19]. Whereas in the last two cases the constant force F is usually small compared to the amplitude of the periodic force, F and d are of the same order in the first two cases. For $F<d$ the potential (1.1) has minima whereas for $F>d$ no minima are present. For $F<d$ the following bistability can occur: In the stationary state the particles may either sit in these minima (locked state) or they may move down the corrugated plane (running state). In the last case a stationary solution is only possible if the friction is not too large and the force F is not too small because otherwise the particles cannot overcome the next maximum. In Fig. 1 this bistability region (II) is shown [5, 20–22]. With the inclusion of white noise, the equation of motion then takes the form (the total force is given by $-dV/dx=-d\sin x+F$)

$$\ddot{x}+\gamma\dot{x}+d\sin x=F+\Gamma(t) \tag{1.2}$$

where $\Gamma(t)$ is a δ-correlated Langevin force with

$$\langle\Gamma(t)\rangle=0;\quad \langle\Gamma(t)\,\Gamma(t')\rangle=2\gamma\Theta\delta(t-t'). \tag{1.3}$$

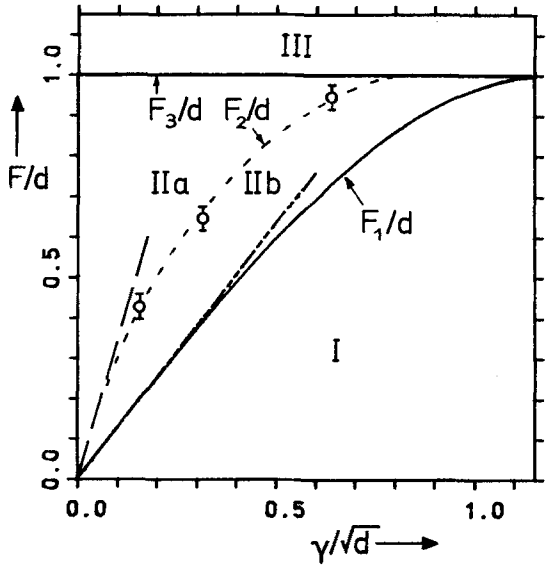

Fig. 1. Phase diagram of the different solutions without noise. Region I: Locked solution, III: running solution, II: coexistence of locked and running solutions (depending on initial condition). If noise is added the bistability region splits into a running (IIa) and a locked (IIb) one in the zero temperature limit, see [22]. The circles indicate extrapolated values of the critical force calculated with the method of [28]

In (1.2, 3) γ is the damping constant and Θ the noise power of the Langevin force. Here we have normalized the mass of the particles to unity, or in other words d and F are the potential amplitude and force per mass m and Θ is given by $\Theta = kT/m$ where T is the temperature and k Boltzmann's constant.

In the bistability region the noise force in (1.2) causes transitions between the locked and the running state. To calculate transition rates we proceed as follows. First we set up the Fokker-Planck equation for the probability density $W(x, v, t)$ in phase space. The Fokker-Planck equation corresponding to (1.2, 3) reads [23, 24]

$$\frac{\partial W}{\partial t} = \mathbf{L}_{\mathrm{FP}} W, \tag{1.4}$$

$$\mathbf{L}_{\mathrm{FP}} = -\frac{\partial}{\partial x} v + \frac{\partial}{\partial v}(\gamma v + d \sin x - F) + \gamma \Theta \frac{\partial^2}{\partial v^2}. \tag{1.5}$$

The separation ansatz $W(x, v, t) = \exp(-\lambda_\mu t)\, \psi_\mu(x, v)$ leads to the eigenvalue equation

$$\mathbf{L}_{\mathrm{FP}} \psi_\mu = -\lambda_\mu \psi_\mu. \tag{1.6}$$

This equation together with proper boundary conditions determine the eigenvalues. For the boundary condition we require that the eigenfunctions are periodic in x

$$\psi_\mu(x, v) = \psi_\mu(x + 2\pi, v) \tag{1.7}$$

and that the eigenfunctions decrease sufficiently fast for $v \to \pm\infty$ (natural boundary conditions for v). If for instance x is an angle variable and if we do not distinguish whether an additional full rotation has been made or not the boundary condition (1.7) is appropriate. If one is only interested in expectation values which are periodic in x, it is also sufficient to use (1.7) for the distribution function. If the diffusion of particles is investigated, we have to apply more general boundary conditions. Because $\mathbf{L}_{\mathrm{FP}}$ is periodic in x we may use

$$\psi_\mu(x, v, k) = e^{ikx} u_\mu(x, v) \tag{1.7a}$$

with periodic $u_\mu(x, v)$ (Floquet's theorem). The eigenvalue will then depend on k.

As will be explained in Chap. 3 the eigenvalues of (1.6, 7) can be determined by the matrix continued fraction method. As it turns out, the lowest non-zero eigenvalue is well separated from the other (heigher) eigenvalues in the bistability region for low noise strength Θ. This enables us in Chap. 2 to express the transition rates in terms of this lowest non-zero eigenvalue and the stationary distribution of (1.5). In Chap. 4 the results for the generally complex eigenvalues are presented. For three typical friction constants γ the eigenvalues are shown as a function of the force for various noise strengths θ. A comparison with the eigenvalues of the Smoluchowski equation is also discussed. Transition rates for the above problem have been obtained by Ben Jacob et al. [25] and Büttiker et al. [26] in the zero temperature limit $\Theta \to 0$. Whereas in [25] the transition rate out of the running state only was calculated, the transition out of the locked state only was obtained in [26]. (In [26] the investigation was restricted to the zero friction limit.) The methods used in [25, 26] are not suitable for finite temperatures whereas the present one will not work in the zero temperature limit $\Theta \to 0$. In a recent investigation [27] we treated the above problem in the zero friction limit $\gamma \to 0$. In this limit we are able to obtain numerical results for finite Θ as well as analytical results for the zero temperature limit $\Theta \to 0$.

2. Transition Rates

For low noise strength Θ and after some initial disturbances have decayed the distribution function in phase space will be a quasistationary solution centered around the locked and the running solutions without noise. (Here we assume that we are in the bistability region.) Therefore we make the ansatz

$$W(x, v, t) = p^L(t)\, W_{St}^L(x, v) + p^R(t)\, W_{St}^R(x, v). \tag{2.1}$$

$$\int_{\mathrm{L.R.}} W_{St}^L(x, v)\, dx\, dv = \int_{\mathrm{R.R.}} W_{St}^R(x, v)\, dx\, dv = 1 \tag{2.2}$$

where $W_{St}^L(x, v)$ and $W_{St}^R(x, v)$ are the normalized quasistationary distributions near the locked and the running solution respectively. The indices L.R. and R.R. in (2.2) indicate that we have to integrate the distribution functions over a region near the locked and running solution (both regions are different areas of the phase space and therefore do not overlap). In (2.1) $p^L(t)$ ($p^R(t)$) is the probability to find the particles in the locked (running) state. The time dependence of these probabilities follow from the master equation ($p^L + p^R = 1$)

$$\begin{aligned} \dot{p}^L &= r(R \to L)\, p^R - r(L \to R)\, p^L \\ \dot{p}^R &= -r(R \to L)\, p^R + r(L \to R)\, p^L, \end{aligned} \tag{2.3}$$

where the transition rate from the locked to the running state is denoted by $r(L \to R)$ and vice versa.

As in all rate equation treatments we have to assume that there is a strong localization near both states and that the result does not depend strongly

on the way in which the phase space is cut into two regions. This assumption requires for instance that the probability density near the cut is sufficiently small. In an exact treatment the rates may be calculated from the currents crossing the cut in either direction.

The general solution of (2.3) is given by

$$p^L(t)=p^L_{St}+Ae^{-\lambda t}$$
$$p^R(t)=p^R_{St}-Ae^{-\lambda t} \tag{2.4}$$

where p^L_{St} and p^R_{St} are the stationary solutions of (2.3)

$$p^L_{St}=r(R\to L)/\lambda, \quad p^R_{St}=r(L\to R)/\lambda \tag{2.5}$$

the non-zero eigenvalue λ reads

$$\lambda=r(L\to R)+r(R\to L) \tag{2.6}$$

and the constant A is the difference of the initial probability and the stationary probability

$$A=p^L(0)-p^L_{St}=-(p^R(0)-p^R_{St}). \tag{2.7}$$

Insertion of (2.4) in (2.1) leads to

$$W(x,v,t)=p^L_{St}W^L_{St}(x,v)+p^R_{St}W^R_{St}(x,v)$$
$$+A(W^L_{St}(x,v)-W^R_{St}(x,v))\,e^{-\lambda t}. \tag{2.8}$$

This result must be compared with the general solution of the Fokker-Planck Eq. (1.4, 5). In terms of eigenvalues and eigenfunctions this solutions reads

$$W(x,v,t)=W_{St}(x,v)+\sum_{\mu=1}^{\infty}A_\mu\psi_\mu(x,v)\,e^{-\lambda_\mu t}. \tag{2.9}$$

In (2.9) we have split off the stationary distribution ($\lambda_0=0$); the coefficients A_n are determined by the initial condition of $W(x,v,0)$. As we will see in the next chapter the eigenvalue λ_1 is well separated from the other (in general complex) eigenvalues

$$\lambda_1 \ll \mathrm{Re}\,\lambda_\mu, \quad \mu\geqq 2 \tag{2.10}$$

in the bistability region for low noise strength Θ. Whereas for small times $t\lesssim t_0=(\mathrm{Min}\,\mathrm{Re}\,\lambda_\mu)^{-1}$, $\mu\geqq 2$, large number of terms of (2.9) have to be taken into account we need only the first two

$$W(x,v,t)=W_{St}(x,v)+A_1\psi_1(x,v)\,e^{-\lambda_1 t}$$
$$\text{for } t\gg t_0. \tag{2.11}$$

A comparison of this expression with (2.8) leads to

$$\lambda_1=\lambda=r(L\to R)+r(R\to L), \tag{2.12}$$
$$W_{St}=p^L_{St}W^L_{St}+p^R_{St}W^R_{St}, \tag{2.13}$$
$$\psi_1=(A/A_1)(W^L_{St}-W^R_{St}). \tag{2.14}$$

(The result (2.14) is consistent with the requirement $\int\psi_0^+\psi_1\,dx\,dv=0$. Here $\psi_0^+=1$ is the eigenfunction of $\mathbf{L}^+_{\mathrm{FP}}$ with the eigenvalue zero). Because of (2.2, 5) we can thus express the transition rates in terms of the stationary solution $W_{St}(x,v)$ and the eigenvalue λ_1 of the Fokker-Planck equation,

$$r(R\to L)=\lambda_1 p^L_{St}=\lambda_1\int_{\mathrm{L.R.}}W_{St}(x,v)\,dx\,dv, \tag{2.15a}$$

$$r(L\to R)=\lambda_1 p^R_{St}=\lambda_1\int_{\mathrm{R.R.}}W_{St}(x,v)\,dx\,dv. \tag{2.15b}$$

The mobility times the damping constant was calculated in [22, 28]. Denoting the mobility for noiseless particles in the running state by $(\gamma\mu)^R$ ($\gamma\mu$ for the locked state is zero) we have

$$(\gamma\mu)_{St}=(\gamma\mu)^R\,p^R_{St} \tag{2.16}$$

and we may thus write instead of (2.15)

$$r(R\to L)=\lambda_1(1-(\gamma\mu)_{St}/(\gamma\mu)^R), \tag{2.17a}$$

$$r(L\to R)=\lambda_1(\gamma\mu)_{St}/(\gamma\mu)^R. \tag{2.17b}$$

In Fig. 2 $(\gamma\mu)_{St}$ as well as $(\gamma\mu)^R$ are shown as a function of the force F for various temperatures Θ. Finally concerning the transition rates we would like to make the following remarks:

(i) As seen from (2.5, 6) the transition rates determine the decay constant and the stationary solution and vice versa. Therefore we can use either set. If one is interested in the decay constant one needs the combination (2.6) of both transition rates which, however, can be obtained directly by the method outlined in Chap. 3. If one is interested in the individual transition rates we show that we can use the Fokker-Planck approach similarly to calculate $p^L_{St}=(1-p^R_{St})$ in addition to λ.

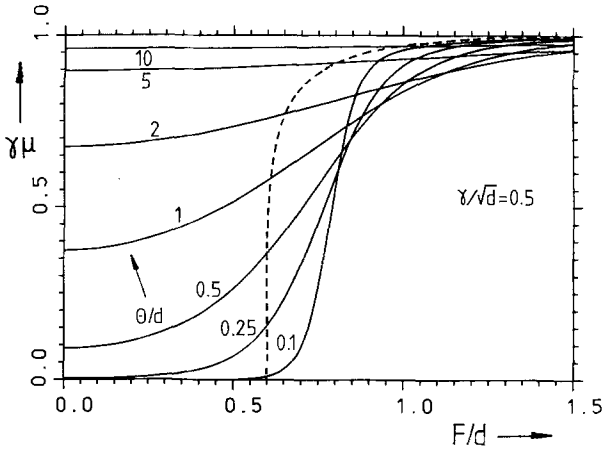

Fig. 2. The stationary mobility times γ ($\gamma\mu=\gamma\langle v\rangle/F$) is shown as a function of the external force F for various temperatures Θ/d and for $\gamma/\sqrt{d}=0.5$. The stationary mobility $(\gamma\mu)^R$ of the noiseless running particles is shown by the broken line

(ii) As mentioned previously the transition rates should be derived from the solution of the Fokker-Planck equation for an appropriate boundary value problem in both regions simultaneously. When calculating transition rates for each region separately, care should be taken that the distribution function in each region is consistent with the solution of the full problem.

(iii) It should be stressed that the rate equations (2.3) are valid only if the times are sufficiently large ($t \gg t_0$) and if the lowest non-zero eigenvalue is well separated from the higher ones, see (2.10). (For degenerate eigenvalues this must hold for the whole group).

(iv) There are examples where only a few terms of the expansion (2.9) enter in the final results, e.g. [29]. In contrast to the rate Eqs. (2.3), the Fokker-Planck equation may then still be used for intermediate times and arbitrary barrier heights.

3. Determination of the Eigenvalues

The procedure for determining the eigenvalues of the Fokker-Planck operator (1.5) and for similar operators was discussed in several papers [30–33]. The main steps are: First we expand the eigenfunctions $\psi(x, v) = \psi_\mu(x, v)$ into two complete sets satisfying the boundary conditions for x and v, i.e.

$$\psi(x,v) = \exp[(-v^2/2 + d\cos x)/(2\Theta)] \cdot \sum_{n=0}^{\infty} \sum_{q=-\infty}^{\infty} c_n^q \psi_n(v)\, \phi^q(x). \tag{3.1}$$

Here $\psi_n(v)$ are the Hermite functions

$$\psi_n(v) = H_n(v/\sqrt{2\Theta}) \exp(-v^2/(4\Theta))/\sqrt{n!\, 2^n \sqrt{2\pi\Theta}} \tag{3.2}$$

and $\phi^q(x)$ are the exponential functions

$$\phi^q(x) = (2\pi)^{-1/2} \exp(iqx). \tag{3.3}$$

By inserting (3.1) into (1.6) and using the normalization

$$\int_{-\infty}^{\infty} \psi_n(v)\, \psi_m(v)\, dv = \delta_{nm}, \tag{3.4a}$$

$$\int_0^{2\pi} (\phi^p(x))^*\, \phi^q(x)\, dx = \delta_{pq} \tag{3.4b}$$

we get a system of coupled differential equations for the expansion coefficients c_n^p. Next we introduce the column vector $\mathbf{c}_n$ and the matrices $\mathbf{D}$ and $\hat{\mathbf{D}}$

$$\mathbf{c}_n = (c_n^q), \quad \mathbf{D} = (D^{pq}), \quad \hat{\mathbf{D}} = (\hat{D}^{pq}) \tag{3.5}$$

where the elements c_n^q are the expansion coefficients of (3.1) truncated at $q = \pm Q$ and where the matrix elements D^{pq} and $\hat{D}^{pq}$ are given by

$$D^{pq} = \sqrt{\Theta}\,[ip\,\delta_{pq} + \mathrm{id}(\delta_{p,q+1} - \delta_{p,q-1})/(4\Theta)], \tag{3.6}$$

$$\hat{D}^{pq} = \sqrt{\Theta}\,[(ip - F/\Theta)\,\delta_{pq} - \mathrm{id}(\delta_{p,q+1} - \delta_{p,q-1})/(4\Theta)]. \tag{3.7}$$

The system of coupled differential equations for the expansion coefficients c_n^q truncated at $q = \pm Q$ can then be cast into the tridiagonal recurrence relation for the column vectors $\mathbf{c}_n$

$$\sqrt{n+1}\,\mathbf{D}\mathbf{c}_{n+1} + (n\gamma - \lambda)\,\mathbf{c}_n + \sqrt{n}\,\hat{\mathbf{D}}\mathbf{c}_{n-1} = \mathbf{0}. \tag{3.8}$$

The final step consists in eliminating all $\mathbf{c}_n$ with $n \geqq 1$ by iteration. This leads to

$$[\lambda\mathbf{I} + \tilde{\mathbf{K}}_0(-\lambda)]\,\mathbf{c}_0 = \mathbf{0} \tag{3.9}$$

where $\tilde{\mathbf{K}}_0(s)$ is the following matrix continued fraction ($\mathbf{I}$ is the unit matrix)

$$\tilde{\mathbf{K}}_0(s) = \mathbf{D}[(s+\gamma)\,\mathbf{I} - 2\mathbf{D}[(s+2\gamma)\,\mathbf{I} - 3\mathbf{D}[(s+3\gamma)\,\mathbf{I} - \ldots]^{-1}\hat{\mathbf{D}}]^{-1}\hat{\mathbf{D}}]^{-1}\hat{\mathbf{D}}. \tag{3.10}$$

Nontrivial solutions can occur only if the corresponding determinant vanishes i.e.

$$D_0(\lambda) = \mathrm{Det}[\lambda\mathbf{I} + \tilde{\mathbf{K}}_0(-\lambda)] = 0. \tag{3.11}$$

By this transcendental equation the eigenvalues $\lambda = \lambda_\mu$ are determined. Thus one has to calculate the continued fraction (3.10) (truncated at an appropriate step) and then determine the roots of (3.11) by some root finding technique. For complex eigenvalues the roots of the complex Eq. (3.11) must be found.

Eigenvalues of the Smoluchowski Equation

For large damping constants we may use the Smoluchowski equation for the probability density $W = W(x, t)$

$$\frac{\partial W}{\partial t} = \mathbf{L}_S W, \tag{3.12}$$

$$\mathbf{L}_S = \frac{1}{\gamma}\left[\frac{\partial}{\partial x}(d\sin x - F) + \Theta\frac{\partial^2}{\partial x^2}\right] \tag{3.13}$$

instead of the Kramers Eq. (1.4, 5). The derivation of (3.12, 13) from (1.4, 5) and various inverse friction expansions are treated in a number of papers, see for instance [34–37, 32]. The distribution function in position only i.e. $W(x, t)$ is the integral of the distribu-

tion function $W(x, v, t)$ with respect to v. The separation ansatz $W(x,t)=\exp(-\lambda_\mu t)\psi_\mu(x)$ leads to

$$\mathbf{L}_S\psi_\mu=-\lambda_\mu\psi_\mu. \tag{3.14}$$

To solve (3.14) we make the ansatz

$$\psi(x)=\sum_{q=-\infty}^{\infty} c_q\phi^q(x) \tag{3.15}$$

where ϕ^q in (3.15) are the expansion functions (3.3) of a Fourier series. Insertion of (3.15) into the Fokker-Planck Eq. (3.12, 13) leads to the two sided tridiagonal recurrence relation

$$(\gamma\lambda-iqF-\Theta q^2)c_q+(dq/2)(c_{q-1}-c_{q+1})=0. \tag{3.16}$$

For $\lambda=0$, i.e. for the stationary solution, (3.16) for $q=0$ vanishes identically. In this case the probability current $S=\gamma^{-1}[(F-d\sin x)W-\Theta\partial W/\partial x]$ is constant and (3.16) has to be replaced by

$$(-iF-\Theta q)c_q+(d/2)(c_{q-1}-c_{q+1})=-i\gamma S\delta_{q0}. \tag{3.16a}$$

For $\lambda\neq 0$, (3.16) for $q=0$ leads to $c_0=0$. Thus (3.16) splits into two systems one for $q>0$ and the other for $q<0$. For real λ, ψ_μ must be real too and therefore we have $c_q=c^*_{-q}$. Both systems are thus equivalent and can be used to find the real eigenvalues. For complex λ one is the complex conjugate of the other one. Thus one system determines λ the other λ^*.

For real and complex eigenvalues $\lambda\neq 0$ we can eliminate all c_q with $q>1$ finally leading to

$$[\gamma\lambda-iF-\Theta-(d/2)\tilde{K}_1(-\lambda)]c_1=0 \tag{3.17}$$

where $\tilde{K}_1(s)$ is given by the ordinary continued fraction

$$\tilde{K}_1(s)=\cfrac{1}{2(\gamma s+2iF+2^2\Theta)/(2d)+\cfrac{1}{2(\gamma s+3iF+3^2\theta)/(3d)+\ldots}}. \tag{3.18}$$

Solutions of (3.16) with $c_1\neq 0$ can occur only if

$$D_1(\lambda)=\gamma\lambda-iF-\Theta-(d/2)\tilde{K}_1(-\lambda)=0. \tag{3.19}$$

From this equation the eigenvalues $\lambda\neq 0$ can be determined.

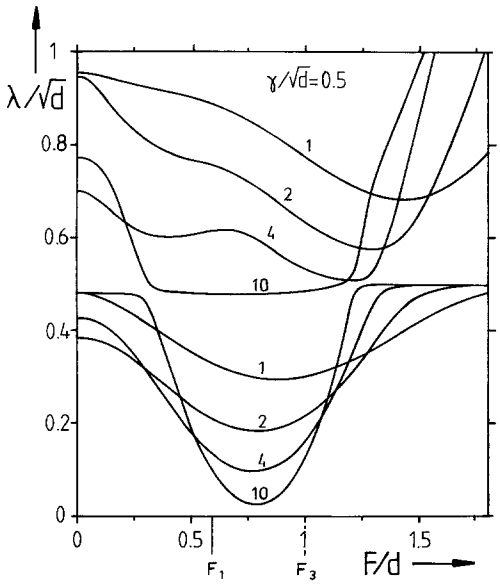

Fig. 3. The two lowest nonzero real eigenvalues λ as a function of the external force F for $\gamma/\sqrt{d}=0.5$ and for various d/Θ values. The critical forces F_1, F_3 are indicated

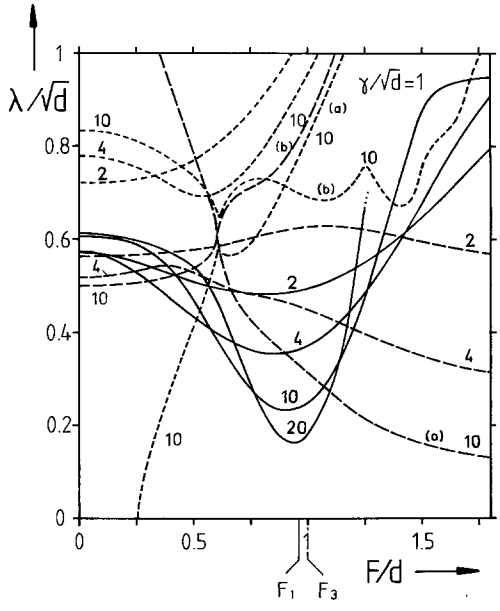

Fig. 4. Some of the lowest non-zero real (full lines) and complex (real parts --- imaginary parts ---) eigenvalues λ as a function of the external force F for $\gamma/\sqrt{d}=1$ and for various d/Θ values. The critical forces F_1, F_3 according to Fig. 1 are indicated. The letters a and b distinguish different eigenvalues for the same d/Θ ratio. (The eigenvalue for $d/\Theta=20$ could not be completed with a 16 digit arithmetic)

4. Results and Discussion

In Figs. 3–5 the results for the eigenvalues of the Fokker-Planck operator (1.5) are shown. As may be easily seen from (1.5, 6) by dividing (1.6) by $\sqrt{d}$ and by using the normalized velocity $\tilde{v}=v/\sqrt{d}$, the eigenvalue divided by $\sqrt{d}$ is a function of $\gamma/\sqrt{d}$, Θ/d and F/d only, i.e.

$$\lambda/\sqrt{d}=f_{\mathrm{FP}}(\gamma/\sqrt{d}, \Theta/d, F/d). \tag{4.1}$$

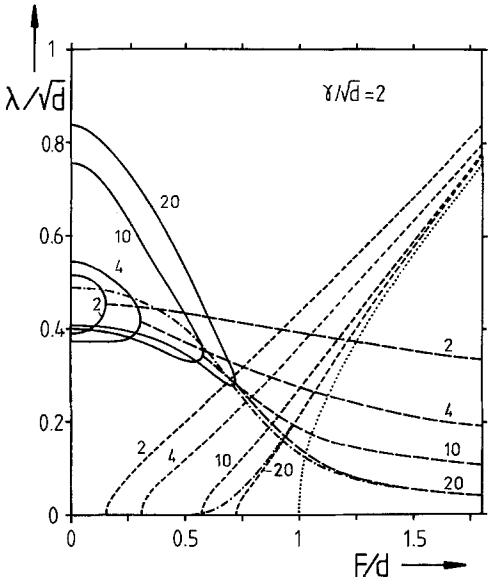

Fig. 5. The two lowest non-zero eigenvalues as a function of the external force F for $\gamma/\sqrt{d}=2$ and different d/Θ ratios indicated in the same style as in Fig. 4. The dotted line is the limit result (4.5) for the imaginary part, the line —·—·— is the Smoluchowski result of Fig. 6 for $d/\Theta=20$

Therefore we have plotted in Figs. 3–5 $\lambda/\sqrt{d}$ as a function of F/d for typical $\gamma/\sqrt{d}$ values and for various noise powers Θ/d. These results have been obtained by the method described in Chap. 3. The truncation values Q and N of the indices q and n respectively in the expansion (3.1) are determined by the requirement that an increase of Q and N does not change the final result within a given accuracy. Generally the following features have been observed. For decreasing noise power Θ/d, Q and N increase whereas for decreasing damping constants γ essentially N increases. (For this reason the method does not work in the limits $\Theta/d\to 0$ or $\gamma/\sqrt{d}\to 0$.) Typical values for the parameters in Fig. 3 are $Q=5$, $N=32$ for $d/\Theta=1$ and $Q=11$ and $N=128$ for $d/\Theta=10$. To compare the results of the Fokker-Planck equation with those of the Smoluchowski equation, we have plotted the eigenvalues of the Smoluchowski operator (3.13) in Fig. 6. As is easily seen by dividing (3.14) by d, $\lambda\gamma$ divided by d is a function of F/d and Θ/d only i.e.

$$\lambda\gamma/d=f_S(\Theta/d, F/d). \tag{4.2}$$

Because the eigenvalues of the Smoluchowski operator agree with the lowest ones of the Fokker-Planck equation for large damping constants f_{FP} and f_S should be connected by

$$f_{FP}(\gamma/\sqrt{d}, \Theta/d, F/d)=(\sqrt{d}/\gamma)\, f_S(\Theta/d, F/d)$$
$$\text{for } \mathrm{Re}\{\gamma/\sqrt{d}\}\lesssim\mathcal{O}(1) \text{ and } \gamma/\sqrt{d}\gg 1. \tag{4.3}$$

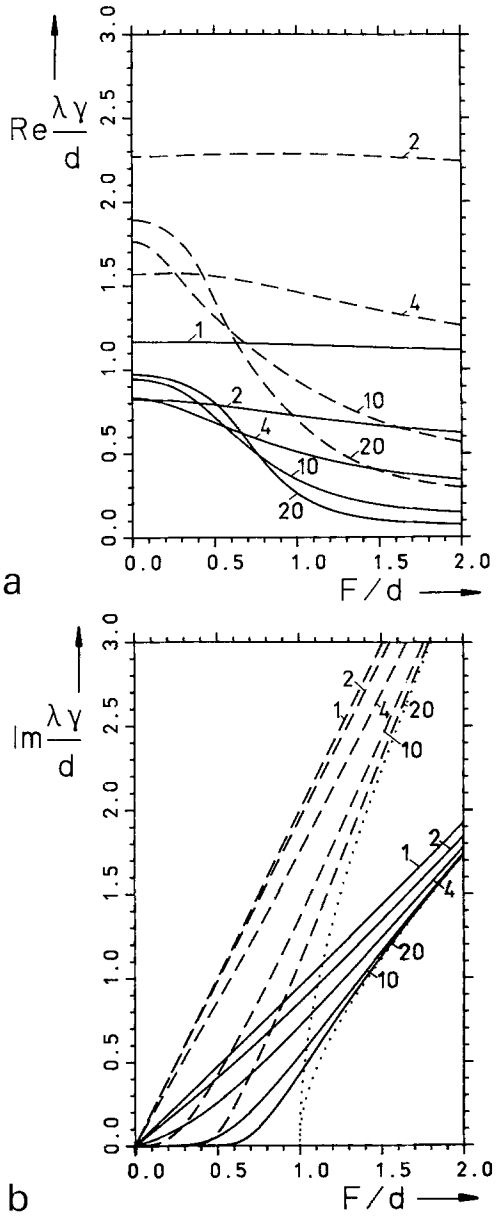

Fig. 6. The real (a) and imaginary (b) parts of the two lowest non-zero eigenvalues λ as a function of F/d for various d/Θ ratios in the Smoluchowski limit. In Fig. b the limit result according to (4.5) and its second harmonic are dotted in

This agreement is confirmed by the comparison in Fig. 5 where the curve $d/\Theta=20$ of Fig. 6 is plotted in.

We now discuss the results by going from small to large friction constants. In Fig. 3 the two lowest non-zero real eigenvalues are shown for the friction constant $\gamma/\sqrt{d}=0.5$, which lies in the middle of the bistability region $0\leqq\gamma/\sqrt{d}<1.193\ldots$ see Fig. 1. (First results of Fig. 3 have been reported recently [38].) The most remarkable feature of Fig. 3 is that the lowest non-zero eigenvalue tends to zero for

decreasing noise power Θ/d for forces F/d in the bistability region ($F_1 < F < F_3$). The explanation for this behaviour is as follows. Without any Langevin force (i.e. $\Theta/d=0$) (1.2) has two stable solutions, a running and a locked one. For finite noise strength Θ/d one gets transitions between these two solutions. For $\Theta/d \to 0$ the transition rates and therefore also the eigenvalue (see (2.12)) must vanish.

For finite but small noise power Θ the stationary solution W_{St} of the Fokker-Planck equation and some stationary expectation values show a sharp transition at a critical force F_2 ($F_1 < F_2 < F_3$) as already discussed in [22]. In terms of the stationary probabilities p_{St}^L and p_{St}^R defined in Chap. 2 this means that p_{St}^L goes from a value near 1 to approximately zero by changing F from a value below F_2 to a value above F_2 thus making a phase transition of first order at F_2. As is seen in Fig. 3 the lowest non-zero eigenvalue is smallest at approximately $F = F_2$. Thus a very long time is needed to establish the stationary solution near this critical force. This is similar to the well known critical slowing which we find here for a transition which is reminiscent of a first order phase transition. In the limit $\Theta/d \to 0$ one expects that the $\lambda/\sqrt{d}$ curve as a function of F makes a sharp transition to zero if F reaches the first critical force F_1 and stays zero till F reaches the third critical force F_3 where $\lambda/\sqrt{d}$ will jump to a finite value which is approximately given by $\gamma/\sqrt{d}$. This value is obtained as follows: For large forces $F \gg d$ one can neglect the periodic force in (1.2) and thus obtains the damping constant $\lambda = \gamma$ of the Brownian motion without a potential. (The constant force can be absorbed in the shifted velocity $v_s = v - F$, i.e. one obtains for v_s the equation $\dot{v}_s + \gamma v_s = \Gamma(t)$ of free Brownian motion.) For low forces F and low noise powers Θ/d the particles oscillate in a well which is approximately parabolic. The first non-zero real eigenvalue in such a parabolic potential is given by γ [33, Eq. (3.33)]. (The deviations from this value for finite Θ/d stem from the deviations from the parabolic form). As seen from Fig. 3 in the bistability region the next real eigenvalue takes over the value $\lambda \approx \gamma$ of the lowest non-zero eigenvalue outside the bistability region for very low noise power Θ. Complex eigenvalues are also obtained. They are not plotted in Fig. 3 in order not to overload this Figure. The real parts of these complex eigenvalues did not show signs of critical slowing down in the bistability region as was the case for the real eigenvalues.

In Fig. 4 real and complex eigenvalues are shown for $\gamma/\sqrt{d}=1$, i.e. just near the end of the bistability region at $\gamma/\sqrt{d} \approx 1.193\ldots$, see Fig. 1. The lowest non-zero real eigenvalue still shows the critical slowing down in the bistability region; however, it does not reach the low values as in Fig. 3 even for the lower $\Theta/d=0.05$. For large forces the real parts of the complex eigenvalues decrease for increasing F whereas the imaginary parts increase with increasing F as in Figs. 5 and 6. Thus at $\gamma/\sqrt{d}=1$ the eigenvalue dependence shows features of the eigenvalues for small and large $\gamma/\sqrt{d}$. Therefore the eigenvalues show a complicated structure in this intermediate region.

In Fig. 5 at $\gamma/\sqrt{d}=2$ we are well outside the bistability region and the dependence of the eigenvalues on the force simplifies again. For $F > d$ only a running solution occurs for zero noise ($\Theta/d=0$). This running solution shows oscillation in time with frequency components being multiples of a fundamental frequency ω. If we neglect the second time derivative in (1.2) the time to travel the distance 2π is given by

$$T = \int_0^{2\pi} dx/v = \gamma \int_0^{2\pi} dx/(F - d\sin x) = 2\pi\gamma/\sqrt{F^2 - d^2}. \quad (4.4)$$

Thus the fundamental frequency ω or beat frequency [39] reads

$$\mathrm{Im}\,\lambda/\sqrt{d} = \omega/\sqrt{d} = 2\pi/(T\sqrt{d}) = \sqrt{(F/d)^2 - 1}/(\gamma/\sqrt{d}). \quad (4.5)$$

The imaginary parts of the eigenvalues with low real parts agree approximately with (4.5) for $F/d > 1$ and low Θ/d. For smaller forces F/d the imaginary part of the eigenvalues disappears and two real eigenvalues appear instead of the two complex conjugate ones. (Because $\mathbf{L}_{FP}$ is real the complex conjugate of a complex eigenvalue is also an eigenvalue of $\mathbf{L}_{FP}$.) The bend in the real part of the eigenvalue at $F \approx d$ for low Θ may be considered as a rudiment of the critical slowing down of the lowest non-zero real eigenvalue in the bistability region.

Finally in Fig. 6 the eigenvalues of the Smoluchowski equation are shown. For large F/d the eigenvalues behave similarly as in Fig. 5. In contrast to Fig. 5, however, the imaginary parts do not disappear suddenly for decreasing F. The imaginary parts are zero for $F=0$ only, though they have extremely low values for small Θ/d and $F/d < 1$. The real parts behave for large F similar to those in Fig. 5. For smaller F/d no bifurcation as in Fig. 6 is observed.

5. Conclusion

As shown the eigenvalues for the Brownian motion in an inclined cosine potential can be determined

inside and outside the bistability region by the matrix continued fraction method. In the bistability region we get the typical critical slowing down of a phase transition. (Eigenfunctions may be obtained by upiteration of (3.8) as done in [40] for the stationary state. This upiteration is unstable. A stable upiteration is the iteration according to (2.19) of [31].) Though the method does not work in the limit $\Theta \to 0$, relatively low Θ/d values can be handled. The value $\Theta/d=0.05$ in Fig. 4 for low F/d for instance, corresponds to an energy difference ΔE of the potential of 40 kT. In the bistability region of Fig. 3 the energy difference is still of the order of 10 kT for $F=F_2$ and $\Theta/d=0.1$. The method is of course not restricted to a cosine potential. For more complicated periodic potentials the matrices $\mathbf{D}$ and $\hat{\mathbf{D}}$ have more non-zero elements but the other expressions remain the same. By taking the Fourier (Laplace) transform of the Fokker-Planck equation, the Fourier (Laplace) transform of the distribution function and also of the correlation functions can be obtained by continued fractions as done in [41] for the cosine potential with $F=0$ (leading to matrix continued fractions) and in [42] for the Smoluchowski equation with $F\neq 0$ (leading to ordinary continued fractions). As mentioned in the introduction, a wave vector dependence of the eigenvalues $\lambda(k)$ must be taken into account if the diffusion of particles in the corrugated plane is investigated. This can also be done by the matrix continued fraction method. (The matrix $\mathbf{D}$ and $\hat{\mathbf{D}}$ will then also depend on k). In the zero friction limit, the matrix continued fraction method will not work. Here we can transform the Fokker-Planck equation to a differential equation, where mainly the energy enters as the variable. This equation can then be solved in the stationary state $\lambda_0=0$ [43] as well as for other eigenvalues [27].

References

1. Ivanchenko, V.M., Zil'berman, L.A.: Sov. Phys. JETP **28**, 1272 (1969)
2. Ambegaokar, V., Halperin, B.I.: Phys. Rev. Lett. **22**, 1364 (1969)
3. Lee, P.A.: J. Appl. Phys. **42**, 325 (1971)
4. Kurkijärvi, K., Ambegaokar, V.: Phys. Lett. A**31**, 314 (1970)
5. Barone, A., Paterno, G.: Physics and applications of the Josephson effect. New York: John Wiley (1982)
6. Stratonovich, R.L.: Radiotekh. Elektron. **3**, (No. 4) 497 (1958)
7. Stratonovich, R.L.: Topics in the theory of random noise. Vol. II, Chap. 9. New York: Gordon and Breach 1967
8. Viterbi, A.J.: Proc. IEEE, **51**, 1737 (1963)
9. Viterbi, A.J.: Principles of coherent communication. New York: McGraw-Hill 1966
10. Lindsey, W.C.: Synchronization systems in communication and control. Englewood Cliffs, NJ: Prentice Hall 1972
11. Coffey, W.T.: Adv. Mol. Relaxation Interact. Processes **17**, 169 (1980)
12. Wyllie, G.: Phys. Rep. **61**, 329 (1980)
13. McConnel, J.: Rotational Brownian motion and dielectric theory. London: Academic Press 1980
14. Praestgaard, E., Kampen, N.G. van: Mol. Phys. **43**, 33 (1981)
15. Fulde, P., Pietronero, L., Schneider, W.R., Strässler, S.: Phys. Rev. Lett. **35**, 1776 (1975)
16. Dieterich, W., Peschel, I., Schneider, W.R.: Z. Phys. B - Condensed Matter **27**, 177 (1977)
17. Risken, H., Vollmer, H.D.: Z. Phys. B - Condensed Matter **31**, 209 (1978)
18. Geisel, T.: Physics of superionic conductors. Salomon, M.B. (ed.), p. 201. Berlin, Heidelberg, New York: Springer-Verlag 1979
19. Dieterich, W., Fulde, P., Peschel, I.: Adv. Phys. **29**, 527 (1980)
20. McCumber, D.E.: J. Appl. Phys. **39**, 3113 (1968)
21. Schneider, T., Stoll, E.P., Morf, R.: Phys. Rev. B**18**, 1417 (1978)
22. Vollmer, H.D., Risken, H.: Z. Phys. B - Condensed Matter **37**, 343 (1980)
23. Chandrasekhar, S.: Rev. Mod. Phys. **15**, 1 (1943)
24. Wang, M.C., Uhlenbeck, G.E.: Rev. Mod. Phys. **17**, 323 (1945)
25. Ben Jacob, E., Bergman, D.J., Matkowsky, B.J., Schuss, Z.: Phys. Rev. A**26**, 2805 (1982)
26. Büttiker, M., Harris, E.P., Landauer, R.: Thermal activation in extremly underdamped Josephson junction circuits. Preprint (1983)
27. Jung, P.: Diplom Thesis, Ulm (1983)
28. Risken, H., Vollmer, H.D.: Z. Phys. B - Condensed Matter **33**, 297 (1979)
29. Risken, H., Vollmer, H.D.: Z. Phys. **201**, 323 (1967)
30. Risken, H., Vollmer, H.D., Denk, H.: Phys. Lett. **78**A, 22 (1980)
31. Risken, H., Vollmer, H.D.: Z. Phys. B - Condensed Matter **39**, 339 (1980)
32. Risken, H., Vollmer, H.D., Mörsch, M.: Z. Phys. B - Condensed Matter **40**, 343 (1981)
33. Vollmer, H.D., Risken, H.: Physica **110**A, 106 (1982)
34. Wilemski, G.: J. Stat. Phys. **14**, 153 (1976)
35. Titulaer, U.M.: Physica **91**A, 321 (1978)
36. Chaturvedi, S., Shibata, F.: Z. Phys. B - Condensed Matter **35**, 297 (1979)
37. San Miguel, M., Sancho, J.M.: J. Stat. Phys. **22**, 605 (1980)
38. Risken, H., Vollmer, H.D.: Matrix continued fraction solution of the Kramers equation with applications to the Brownian motion in periodic potentials. Talk given at the international symposium "Synergetics and Cooperative Phenomena in Solids and Macromolecules", Tallin, UdSSR, Fall (1982)
39. Cresser, J.D., Louisell, W.H., Meystre, P., Schleich, W., Scully, M.O.: Phys. Rev. A**25**, 2214 (1982)
40. Vollmer, H.D., Risken, H.: Z. Phys. B - Condensed Matter **34**, 313 (1979)
41. Risken, H., Vollmer, H.D.: Mol. Phys. **46**, 555 (1982)
42. Cresser, J.D., Hammonds, D., Louisell, W.H., Meystre, P., Risken, H.: Phys. Rev. A**25**, 2226 (1982)
43. Risken, H., Vollmer, H.D.: Z. Phys. B - Condensed Matter **35**, 177 (1979)

H.D. Vollmer
H. Risken
Abteilung für Theoretische Physik
Universität Ulm
Oberer Eselsberg
D-7900 Ulm
Federal Republic of Germany