Integration of E-Functions and Related Series

By

F. M. Ragab and M. A. Simary

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§ 1. Introductory. In papers [1], [2], [3] the sums of a number of series of products of E-functions have been found. For the definitions and properties of the E-functions the reader is referred to [4, pp. 348—358]. In § 3 further series of this type is given. The proof is based on an integral of E-function with respect to its parameters and will be established in § 2. In § 4 series involving Whittaker functions can be obtained. Similar integrals were given in [6], [7], [8].

Ihe following formulae will be made use of in the proofs.

If $p \leqq q$, $\mathfrak{z} \neq 0$, [4, p. 352]

$$E(p; \alpha_1 : q; \varrho_s : \mathfrak{z}) = \frac{\Gamma(\alpha_1) \ldots \Gamma(\alpha_p)}{\Gamma(\varrho_1) \ldots \Gamma(\varrho_q)} F \begin{pmatrix} p; \alpha_r : -1/\mathfrak{z} \\ q; \varrho_s \end{pmatrix}, \tag{1}$$

this formula also holds when $p = q + 1$, provided that $|\mathfrak{z}| > |$.

If $R(\varrho_{q+1}) > R(\alpha_{p+1}) > 0$, [4, p. 395]

$$\int_0^1 \lambda^{\alpha_{p+1}-1} (1-\lambda) \varrho^{q+1-\alpha_{p+1}-1}\ E(p; \alpha_r : q; \varrho_s : \mathfrak{z}/\lambda)\, d\lambda = \Gamma(\varrho_{q+1} - \alpha_{p+1}) \,.$$
$$. \, E(p+1; \alpha_r : q+1; \varrho_s : \mathfrak{z}). \tag{2}$$

If $R(\alpha_{p+1}) > 0$, [4, p. 394]

$$\int_0^\infty e^{-\lambda}\, \lambda^{\alpha_{p+1}-1} E(p; \alpha_r : q; \varrho_s\, \mathfrak{z}/\lambda)\, d\lambda = E(p+1; \alpha_r : q; \varrho_s : \mathfrak{z}). \tag{3}$$

If $|\mathfrak{z}| < 1$, [5, p. 185, 186]

$$\left[{}_2F_1 \begin{pmatrix} \alpha,\ \beta;\ \ ;\ \mathfrak{z} \\ \alpha + \beta + 1/2 \end{pmatrix} \right]^2 = {}_3F_2 \begin{pmatrix} 2\,\alpha\ ,\ \alpha+\beta\ ,\ 2\,\beta\ ;\ \mathfrak{z} \\ \alpha+\beta+1/2,\ 2\,\alpha + 2\,\beta \end{pmatrix}. \tag{4}$$

$${}_2F_1 \begin{pmatrix} \alpha\ ,\beta\ ;\mathfrak{z} \\ \alpha+\beta-1/2 \end{pmatrix} {}_2F_1 \begin{pmatrix} \alpha\ ,\beta\ ;\mathfrak{z} \\ \alpha+\beta+1/2 \end{pmatrix} = {}_3F_2 \begin{pmatrix} 2\,\alpha\ ,2\,\beta\ ,\alpha+\beta\ ;\mathfrak{z} \\ 2\,\alpha+2\,\beta-1,\ \alpha+\beta+1/2 \end{pmatrix}. \tag{5}$$

$${}_2F_1 \begin{pmatrix} \alpha\ ,\beta\ ;\mathfrak{z} \\ \alpha+\beta-1/2 \end{pmatrix} {}_2F_1 \begin{pmatrix} \alpha-1\ ,\beta\ ;\mathfrak{z} \\ \alpha+\beta-1/2 \end{pmatrix} = {}_3F_2 \begin{pmatrix} 2\,\alpha-1,\ 2\,\beta,\ \alpha+\beta-1\ ;\mathfrak{z} \\ 2\,\alpha+2\,\beta-2,\ \alpha+\beta-1/2 \end{pmatrix}. \tag{6}$$

If $|\operatorname{amp} \mathfrak{z}| < \pi$, [4, p. 374]

$$E(p; \alpha_r : q; \varrho_s : \mathfrak{z}) = \frac{1}{2\pi i} \int \frac{\Gamma(\zeta) \Pi \Gamma(\alpha_r - \zeta)}{\Pi \Gamma(\varrho_s - \zeta)} \mathfrak{z}^{\zeta} d\zeta, \tag{7}$$

where the contour of integration is taken up the η-axis with loops, if necessary, to ensure that the pole at the origin lies to the left and the poles at $\alpha_1, \alpha_2, \ldots, \alpha_p$ to the right of the contour. Zero and negative integral values of the parameters are excluded. If $p < q + 1$ the contour is bent to the left at both ends. When $p > q + 1$ the formula is valid for $|\operatorname{amp} \mathfrak{z}| < \frac{1}{2}(p - q + 1)\pi$.

Finally, [4, p. 351]

$$E(\alpha, \beta :: \mathfrak{z}) = \Gamma(\alpha) \Gamma(\beta) \mathfrak{z}^{\frac{1}{2}(\alpha+\beta-1)} e^{\frac{1}{2}\mathfrak{z}} W_{\frac{1}{2}(1-\alpha-\beta), \frac{1}{2}(\beta-\alpha)}(\mathfrak{z}). \tag{8}$$

§ 2. Integration of E-functions with respect to their parameters. The formulae to be proved are

$$\frac{1}{2\pi i} \int \frac{\Gamma(\zeta) \Gamma(\alpha - \zeta) \Gamma(\beta - \zeta)}{\Gamma \alpha + \beta + \frac{1}{2} - \zeta)} \mathfrak{z}^{\zeta} E\begin{pmatrix} \alpha, \beta, \alpha_1 - \zeta, \ldots, \alpha_p - \zeta : \mathfrak{z} \\ \alpha+\beta+\frac{1}{2}, \varrho_1 - \zeta, \ldots, \varrho_q - \zeta \end{pmatrix} d\zeta =$$

$$= \frac{2\sqrt{\pi}\, \Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\frac{1}{2}) \Gamma(\beta+\frac{1}{2})} E\begin{pmatrix} (2\alpha, \alpha+\beta, 2\beta, \alpha_1, \ldots, \alpha_p : \mathfrak{z} \\ \alpha+\beta+\frac{1}{2}, 2\alpha+2\beta, \varrho_1, \ldots, \varrho_q \end{pmatrix}. \tag{9}$$

$$\frac{1}{2\pi i} \int \frac{\Gamma(\zeta) \Gamma(\alpha-\zeta) \Gamma(\beta-\zeta)}{\Gamma(\alpha + \beta - \frac{1}{2} - \zeta)} \mathfrak{z}^{\zeta} E\begin{pmatrix} \alpha, \beta, \alpha_1 - \zeta, \ldots, \alpha_p - \zeta : \mathfrak{z} \\ \alpha+\beta+\frac{1}{2}, \varrho_1 - \zeta, \ldots, \varrho_q - \zeta \end{pmatrix} d\zeta =$$

$$= \frac{\sqrt{\pi}\, \Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\frac{1}{2}) \Gamma(\beta+\frac{1}{2})} E\begin{pmatrix} (2\alpha, 2\beta, \alpha+\beta, \alpha_1, \alpha_2, \ldots, \alpha_p : \mathfrak{z} \\ 2\alpha+2\beta-1, \alpha+\beta+\frac{1}{2}, \varrho_1, \varrho_2, \ldots, \varrho_q \end{pmatrix}. \tag{10}$$

$$\frac{1}{2\pi i} \int \frac{\Gamma(\zeta) \Gamma(\alpha - \zeta) \Gamma(\beta - \zeta)}{\Gamma(\alpha + \beta - \frac{1}{2} - \zeta)} \mathfrak{z}^{\zeta} E\begin{pmatrix} \alpha-1, \beta, \alpha_1 - \zeta, \ldots, \alpha_p - \zeta : \mathfrak{z} \\ \alpha+\beta-\frac{1}{2}, \varrho_1 - \zeta, \ldots, \varrho_q - \zeta \end{pmatrix} d\zeta =$$

$$= \frac{\sqrt{\pi}\, \Gamma(\alpha-1) \Gamma(\beta)}{\Gamma(\alpha-\frac{1}{2}) \Gamma(\beta+\frac{1}{2})} E\begin{pmatrix} (2\alpha-1, 2\beta, \alpha+\beta-1, \alpha_1, \ldots, \alpha_p : \mathfrak{z} \\ 2\alpha+2\beta-2, \alpha+\beta-\frac{1}{2}, \varrho_1, \ldots, \varrho_q \end{pmatrix}. \tag{11}$$

In (9), (10), and (11) $p \geqq q$, $|\operatorname{amp} \mathfrak{z}| < \frac{1}{2}(p-q+2)\pi$, $R(\varrho_n - \alpha_n) > 0$ $(n = 1, \ldots, q)$, $R(\alpha_n) > 0$ $(n = 1, \ldots, p)$, α and β being such that the E-functions exist. The contour of integration is the same as in (7).

To prove (9) apply (1) to (4) it can be deduced that

$$\left[E\begin{pmatrix} \alpha, \beta \ ; \mathfrak{z} \\ \alpha+\beta+\frac{1}{2} \end{pmatrix} \right]^2 = \frac{2\sqrt{\pi}\, \Gamma(\alpha) \Gamma(\beta)}{\Gamma(\alpha+\frac{1}{2}) \Gamma(\beta+\frac{1}{2})} E\begin{pmatrix} 2\alpha, \alpha+\beta, 2\beta; \mathfrak{z} \\ \alpha+\beta+\frac{1}{2}, 2\alpha+2\beta \end{pmatrix}. \tag{12}$$

From (2) and (3) it follows that the left-hand side of (9) is equal to

$$\frac{1}{2\pi i}\int \frac{\Gamma(\zeta)\,\Gamma(\alpha-\zeta)\,\Gamma(\beta-\zeta)}{\Gamma(\alpha+\beta+\tfrac{1}{2}-\zeta)}\,\mathfrak{z}^{\zeta}\,[\prod_{n=1}^{q}\Gamma(\varrho_n-\alpha_n)]^{-1}\prod_{n=1}^{q}\int_0^1 \lambda_n^{\alpha_n-\zeta-1}(1-\lambda_n)\,\varrho^{n-\alpha_n-1}d\lambda_n$$

$$.\prod_{n=q+1}^{p}\int_0^{\infty} e^{-\lambda_n}\,\lambda_n^{\alpha_n-\zeta-1}\,d\lambda_n\,E\begin{pmatrix}\alpha,\beta:\mathfrak{z}/\lambda_1.\lambda_2\ldots\lambda_p\\ \alpha+\beta+\tfrac{1}{2}\end{pmatrix}d\zeta.$$

Here change the order of the factors and get

$$[\prod_{n=1}^{q}\Gamma(\varrho_n-\alpha_n)]^{-1}\prod_{n=1}^{q}\int_0^1 \lambda_n^{\alpha_n-1}\,(1-\lambda_n)\,\varrho^{n-\alpha_n-1}\,d\lambda_n\,.$$

$$.\prod_{n=q+1}^{p-1}\int_0^{\infty} e^{-\lambda_n}\,\lambda_n^{\alpha_n-1}\,d\lambda_n\int_0^{\infty} e^{-\lambda_p}\;\lambda_p^{\alpha_p-1}\,E\begin{pmatrix}\alpha,\beta:\mathfrak{z}/(\lambda_1\,\lambda_2\ldots\lambda_p)\\ \alpha+\beta+\tfrac{1}{2}\end{pmatrix}d\,\lambda_p\,.$$

$$.\frac{1}{2\pi i}\int\frac{\Gamma(\zeta)\,\Gamma(\alpha-\zeta)\,\Gamma(\beta-\zeta)}{\Gamma(\alpha+\beta+\tfrac{1}{2}-\zeta)}\left(\frac{\mathfrak{z}}{\lambda_1\ldots\lambda_p}\right)^{\zeta}d\zeta.$$

On substituting from (7) for the last integral the expression becomes

$$[\prod_{n=1}^{q}\Gamma\,(\varrho_n-\alpha_n)]^{-1}\prod_{n=1}^{q}\int_0^1\lambda_n^{\alpha_n-1}(1-\lambda_n)\,\varrho^{n-\alpha_n-1}d\lambda_n\prod_{n=q+1}^{p-1}\int_0^{\infty}e^{-\lambda_n}\,\lambda_n^{\alpha_n-1}\,d\lambda_n\,.$$

$$.\int_0^{\infty}e^{-\lambda_p}\,\lambda_p^{\alpha_p-1}\,E\begin{pmatrix}\alpha,\beta:\mathfrak{z}/\lambda_1\ldots\lambda_p\\ \alpha+\beta+\tfrac{1}{2}\end{pmatrix}\,E\,\begin{pmatrix}(\alpha,\beta:\mathfrak{z}/\lambda_1\ldots\lambda_p\\ \alpha+\beta+\tfrac{1}{2}\end{pmatrix}d\lambda_p.$$

Now substitute from (12) and, on integrating, using (2) and (3), formula (9) is obtained.

The restrictions on the ϱ's can be removed as the paths of integration from 0 to 1 can be replaced by contours which start from 0, pass round the point 1 and return to 0.

To prove (10) apply (1) to (5) and proceed as before.

To prove (11) apply (1) to (6) and proceed also as before.

§ 3. Series of products of E-functions. The formulae to be proved are

$$\sum_{r=0}^{\infty}\frac{\mathfrak{z}^{-2r}}{r!(\gamma;r)}\left[E\begin{pmatrix}\alpha+r,\beta+r,\gamma+r:\mathfrak{z}\\ \alpha+\beta+\tfrac{1}{2}+r\end{pmatrix}\right]^2=$$

$$=\frac{2\sqrt{\pi}\,\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\tfrac{1}{2})\,\Gamma(\beta+\tfrac{1}{2})}\,\Gamma(\gamma)\,E\begin{pmatrix}(2\,\alpha,\alpha+\beta,2\,\beta.\gamma;\mathfrak{z}\\ \alpha+\beta+\tfrac{1}{2},2\,\alpha+2\,\beta\end{pmatrix},\qquad(13)$$

where $|\operatorname{amp}\mathfrak{z}|<{}^3/_2\,\pi$, $R(\gamma)>0$, $R(\alpha)>0$, $R(\beta)>0$.

$$\sum_{r=0}^{\infty}\frac{\mathfrak{z}^{-2r}}{r!\,(\gamma;r)}\,E\begin{pmatrix}\alpha+r,\beta+r,\gamma+r:\mathfrak{z}\\ \gamma+\beta-\tfrac{1}{2}+r\end{pmatrix}E\begin{pmatrix}\alpha+r,\beta+r,\gamma+r:\mathfrak{z}\\ \alpha+\beta+\tfrac{1}{2}+r\end{pmatrix}=$$

$$=\frac{\sqrt{\pi}\,\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\tfrac{1}{2})\,\Gamma(\beta+\tfrac{1}{2})}\,\Gamma(\gamma)\,E\begin{pmatrix}2\,\alpha,2\,\beta,\alpha+\beta,\gamma;\mathfrak{z}\\ 2\,\alpha+2\,\beta-1,\alpha+\beta+\tfrac{1}{2}\end{pmatrix},\qquad(14)$$

where $|\operatorname{amp} z| < {}^3/_2\pi$, $R(\gamma) > 0$, $R(\beta) > 0$, $R(\alpha) > 0$, $R(2\alpha + 2\beta) > 1$.

$$\sum_{r=0}^{\infty} \frac{z^{-2r}}{r!\,(\gamma; r)} E\begin{pmatrix}\alpha + r, \beta + r, \gamma + r; z\\ \alpha + \beta - \tfrac{1}{2} + r\end{pmatrix} E\begin{pmatrix}\alpha - 1 + r, \beta + r, \gamma + r; z\\ \alpha + \beta - \tfrac{1}{2} + r\end{pmatrix} =$$

$$= \frac{\sqrt{\pi}\,\Gamma(\alpha - 1)\,\Gamma(\beta)}{\Gamma(\alpha - \tfrac{1}{2})\,\Gamma(\beta + \tfrac{1}{2})}\,\Gamma(\gamma)\,E\begin{pmatrix}2\alpha - 1, 2\beta, \alpha + \beta - 1, \gamma; z\\ 2\alpha + 2\beta - 2, \alpha + \beta - \tfrac{1}{2}\end{pmatrix}, \quad (15)$$

where $|\operatorname{amp} z| < {}^3/_2\pi$, $R(\gamma) > 0$, $R(\alpha) > 1$, $R(\beta) > 0$.

To prove (13) substitute from (7) for the E-functions on the left of (13) and get

$$\sum_{r=0}^{\infty} \frac{z^{-2r}}{r!\,(\gamma; r)} \cdot \frac{1}{2\pi i} \int \frac{\Gamma(\zeta)\,\Gamma(\alpha + r - \zeta)\,\Gamma(\beta + r - \zeta)\,\Gamma(\gamma + r - \zeta)}{\Gamma(\alpha + \beta + \tfrac{1}{2} + r - \zeta)} z^{\zeta}\, d\zeta\,.$$

$$\cdot \frac{1}{2\pi i} \int \frac{\Gamma(\xi)\,\Gamma(\alpha + r - \xi)\,\Gamma(\beta + r - \xi)\,\Gamma(\gamma + r - \xi)}{\Gamma(\alpha + \beta + \tfrac{1}{2} + r - \xi)} z^{\xi}\, d\,\xi.$$

Here replace ζ and ξ by $\zeta + r$ and $\xi + r$, and interchange the order of summation and integration, so getting

$$\frac{1}{2\pi i} \int \frac{\Gamma(\zeta)\,\Gamma(\alpha - \zeta)\,\Gamma(\beta - \zeta)\,\Gamma(\gamma - \zeta)}{\Gamma(\alpha + \beta + {}^1/_2 - \zeta)} z^{\zeta}\, d\,\xi\,.$$

$$\cdot \frac{1}{2\pi i} \int \frac{\Gamma(\xi)\,\Gamma(\alpha - \xi)\,\Gamma(\beta - \xi)\,\Gamma(\gamma - \xi)}{\Gamma(\alpha + \beta + \tfrac{1}{2} - \xi)} z^{\zeta}\, F\begin{pmatrix}\zeta, \xi, 1\\ \gamma\end{pmatrix} d\,\xi.$$

On applying Gauss's theorem this becomes

$$\frac{\Gamma(\gamma)}{2\pi i} \int \frac{\Gamma(\zeta)\,\Gamma(\alpha - \zeta)\,\Gamma(\beta - \zeta)}{\Gamma(\alpha + \beta + \tfrac{1}{2} - \zeta)} z^{\zeta}\, E\begin{pmatrix}\alpha, \beta, \gamma - \zeta : z\\ \alpha + \beta + \tfrac{1}{2}\end{pmatrix} d\,\zeta;$$

and from (9), with $p = 1$, $q = 0$, the result follows.

To prove (14) proceed as before and apply the formula (10). Also to prove (15) proceed as before and apply formula (11).

§ 4. Series involving Whittaker functions. The formulae to be proved are

$$\sum_{r=0}^{\infty} \frac{[\Gamma(\alpha + r)\,\Gamma(\beta + r)]^2}{r!\,(\alpha + \beta + \tfrac{1}{2}; r)} \cdot [W_{\frac{1}{2}(1 - \alpha - \beta - 2r),\, \frac{1}{2}(\beta - \alpha)}(z)]^2 =$$

$$= e^{-z}\, z^{1 - \alpha - \beta} \cdot \frac{2\sqrt{\pi}\,\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha + \tfrac{1}{2})\,\Gamma(\beta + \tfrac{1}{2})} \cdot \Gamma(\alpha + \beta + \tfrac{1}{2})\,E\begin{pmatrix}2\alpha, \alpha + \beta, 2\beta; z\\ 2\alpha + 2\beta\end{pmatrix}, \quad (16)$$

the conditions are as in (13).

$$\sum_{r=0}^{\infty} \frac{\Gamma(\alpha+r)\,\Gamma(\beta+r)\,\mathfrak{z}^{-r}}{r!\,(\alpha+\beta-\tfrac{1}{2};r)} W_{\frac{1}{2}(1-\alpha-\beta-2r),\,\frac{1}{2}(\beta-\alpha)}(\mathfrak{z}) . E\begin{pmatrix}\alpha+r, \beta+r, \alpha+\beta-\tfrac{1}{2}+r : \mathfrak{z}\\ \alpha+\beta+\tfrac{1}{2}+r\end{pmatrix}$$

$$= e^{-\frac{1}{2}\mathfrak{z}}\,\mathfrak{z}^{\frac{1}{2}(1-\alpha-\beta)} . \frac{\sqrt{\pi}\,\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\tfrac{1}{2})\,\Gamma(\beta+{}^1/_2)} . \Gamma(\alpha+\beta-{}^1/_2)\, E\begin{pmatrix}2\alpha, 2\beta, \alpha+\beta, \alpha+\beta-{}^1/_2; \mathfrak{z}\\ 2\alpha+2\beta-1,\ \alpha+\beta+{}^1/_2\end{pmatrix} \tag{17a}$$

$$\sum_{r=0}^{\infty} \frac{\Gamma(\alpha+r)\,\Gamma(\beta+r)\,\mathfrak{z}^{-r}}{r!\,(\alpha+\beta+{}^1/_2;r)} W_{\frac{1}{2}(1-\alpha-\beta-2r),\,\frac{1}{2}(\beta-\alpha)}(\mathfrak{z}) .$$

$$. E\begin{pmatrix}\alpha+r, \beta+r, \alpha+\beta+{}^1/_2+r : \mathfrak{z}\\ \alpha+\beta-{}^1/_2+r\end{pmatrix} = e^{-\frac{1}{2}\mathfrak{z}}\,\mathfrak{z}^{\frac{1}{2}(1-\alpha-\beta)} .$$

$$. \frac{\sqrt{\pi}\,\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+{}^1/_2)\,\Gamma(\beta+{}^1/_2)} . \Gamma(\alpha+\beta+{}^1/_2) . E\begin{pmatrix}2\alpha, 2\beta, \alpha+\beta : \mathfrak{z}\\ 2\alpha+2\beta-1\end{pmatrix}, \tag{17b}$$

the conditions are as in (14).

$$\sum_{r=0}^{\infty} \frac{\Gamma(\alpha+r)\,\Gamma(\alpha-1+r)\,[\Gamma(\beta+r)]^2}{r!\,(\alpha+\beta-{}^1/_2;r)} . W_{\frac{1}{2}(1-\alpha-\beta-2r),\,\frac{1}{2}(\beta-\alpha)}(\mathfrak{z}) .$$

$$. W_{\frac{1}{2}(2-\alpha-\beta-2r),\,\frac{1}{2}(\beta-\alpha+1)}(\mathfrak{z}) = e^{-\mathfrak{z}}\,\mathfrak{z}^{{}^3/_2-\alpha-\beta} . \frac{\sqrt{\pi}\,\Gamma(\alpha-1)\,\Gamma(\beta)}{\Gamma(\alpha-{}^1/_2)\,\Gamma(\beta+{}^1/_2)} .$$

$$. \Gamma(\alpha+\beta-{}^1/_2) . E\begin{pmatrix}2\alpha-1, 2\beta, \alpha+\beta-1; \mathfrak{z}\\ 2\alpha+2\beta-2\end{pmatrix}, \tag{18}$$

the conditions are as in (15).

To prove (16) replace γ in (13) by $\alpha+\beta+{}^1/_2$ and apply (8).

To prove (17a) and (17b) replace γ in (14) by $\alpha+\beta-{}^1/_2$ and $\alpha+\beta+{}^1/_2$ and apply (8).

Finally to prove (18) replace γ in (15) by $\alpha+\beta-{}^1/_2$ and apply (8).

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Cairo University and Cairo High Institute of Technology.