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Equivariant Embeddings of Low Dimensional Symmetric Planes

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Abstract. We determine the class of all locally compact stable planes M of positive dimension $d \leqslant 4$ which admit a reflection at each point of some open set $U \subseteq M$. Apart from the expected possibilities (planes defined by real and complex hermitian forms, and almost projective translation planes), one obtains (subplanes of) H. SALZMANN's modified real hyperbolic planes [14; 5.3] and one exceptional plane which was not known before. The case $U = M$ has been treated [9] and is reproved here in a simpler way. The solution to the problem indicated in the title constitutes the main step in the proof of our results.

A *stable plane* is a topological geometry $(M, \mathscr{L})$ with the properties that (i) any two points $p, q \in M$ are joined continuously by a unique line $p \vee q \in \mathscr{L}$, and (ii) the set of pairs of lines intersecting in any open set of points is open; cf. [4] for details. Stable planes are less complete geometrically than *topological projective planes* (= stable planes in which any two lines meet) and are, therefore, more difficult to handle. They have one convenient feature, however; namely, the class of stable planes is closed with respect to taking open subsets. Many geometrically significant subsets of a stable plane are open, such as the set of non-coaffine points [8; § 1], the set of points incident with at least one compact line [4; 1.16], the set of moving points under a group of automorphisms or, with some luck, an orbit of a group. This makes it desirable to find out what can be said about M once an open subset U has been identified as some 'nice' plane.

Here, we shall restrict ourselves to planes of positive dimension $d \leqslant 4$, and U shall be qualified as 'nice' if U is a *hermitian plane*, i. e., if U is isomorphic to an open subplane, defined by a hermitian form, of the real or complex projective plane; cf. [6; § 2], and § 2 of the present paper. Moreover, we shall assume that the plane U has been detected within M by the aid of its motion group (see below); otherwise, our

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question would be far too general. The answer is particularly useful in this situation because the class of (low dimensional) hermitian planes can be characterized in different ways. Indeed, it is almost identical [7, 9] with (i) the class of *symmetric planes*, i. e., of those planes which admit a compatible structure of a symmetric space [6], and with (ii) the class of those planes which admit a reflection at each point. The *motion group* Σ is defined as the group generated by the symmetries of a symmetric plane; for the present considerations, it suffices to think of it as a group of automorphisms which contains a reflection at each point and is minimal with respect to this property. A complete list of the symmetric planes of low dimension is given below (§2); the motion groups can be found in [6; §2].

The proof of our result is based on the observation that for two given lines of a symmetric plane and a given class of involutions in its motion group, it often happens that some involution out of that class either fixes both these lines or interchanges them (§7, Lemma). Also, as indicated above, the classification of symmetric planes [7] and a weak form of the characterization of those planes [9; 1.5] play an essential role in the proof.

§1 Statement of Results; Applications

Let U and M be stable planes, and let $\varphi : U \to M$ be an *embedding* (i. e., an isomorphism onto a subplane of M; for the notion of subplane, cf. §3). φ is called *equivariant* with respect to a group Δ of automorphisms of U if the action of Δ on U^φ extends to an action on M by automorphisms.

Theorem. *Let U be a symmetric plane of positive dimension $d \leqslant 4$, with motion group Σ. Let $\varphi: U \to M$ be a Σ-equivariant open embedding into a stable plane M. Then M is one of the following planes (for definitions, see* §2 *below).*

a) *A symmetric plane;*

b) *an almost projective translation plane (possibly desarguesian);*

c) *a modified real hyperbolic or cylinder plane other than $EH_t(\mathbb{R})$. Moreover, up to an automorphism of M, the image U^φ is the standard copy of U in M.*

Remarks. a) With a little more precision, the last statement means that U^φ can be one of the following.

(i) All of M;

(ii) a connected component of M;

(iii) an affine translation plane obtained by removing a line of M;

(iv) the standard copy of U, defined by a hermitian form, in the desarguesian projective or coaffine plane M;

(v) the interior hyperbolic part of a projective modified real hyperbolic plane M.

However, regarding (ii), not all connected components of M need be symmetric. Also, regarding (iv), not all symmetric planes can be embedded equivariantly into the desarguesian coaffine or projective plane.

Note that we do not attempt to determine φ; our assertions are concerned with U^φ only (cf., however, the Proposition of §3).

b) The restriction to equivariant embeddings is quite necessary, even if one concentrates on embeddings into projective planes in order to avoid trivial counterexamples. For example, the connected real hyperbolic and cylinder planes both embed into the Moulton planes, and also into HILBERT's original example of a nondesarguesian affine plane, where the lines of the real affine plane are interpolated by circular arcs inside some ellipse; the latter plane has $\mathbb{Z}_2$ as its full automorphism group.

Corollary. *Let* $(M, \mathscr{L})$ *be a locally compact stable plane of positive dimension* $d \leqslant 4$. *M belongs to one of the classes* (a), (b), (c) *of the Theorem if and only if M admits a reflection at each point of some open set* $U \subseteq M$.

The Corollary is a generalization of the Main Theorem of [9], where the case $U = M$ was considered. It is proved here using only the main result of the present paper and the results of § 1 of [9]. The proof provides typical examples of the situations described in the introduction. The full information of the Main Theorem of [9] may be recovered from the Corollary, using §2 of [9].

We do use the classification of all symmetric planes of dimension $d \leqslant 4$ in the proof of the Theorem. However, one could extract from the present proof a new and simpler proof of Theorem B of [7] (classification of disconnected symmetric planes).

Finally, the Theorem can be used to simplify the proofs in [10], and it will be used in [11] and in a forthcoming paper on stable planes with a simple group of automorphisms.

§ 2 The Planes

The following set of examples contains a complete list of all symmetric planes of positive dimension $d \leqslant 4$ (cf. [6, 7]), and a number of nonsymmetric planes, namely the nonaffine almost projective translation planes (2.3), and the modified real hyperbolic and cylinder planes (2.8, 2.9). The motion groups of the symmetric planes are described in [6; § 2].

2.1 The *projective plane* $P_2\mathbb{F}$ over the field $\mathbb{F}$ of real or complex numbers.

2.2 The *coaffine plane* $P_2\mathbb{F}\setminus\{\infty\}$ over $\mathbb{F}$, i. e., the projective plane with one point ∞ deleted.

2.3 The *almost projective translation planes*, which are obtained from projective translation planes by removing a closed subset of the translation axis. They include the affine translation planes and, in particular, the desarguesian affine planes $A_2\mathbb{F}$.

2.4 The *united cylinder plane* over $\mathbb{F}$,

$$UC(\mathbb{F}) = \{\langle x_1, x_2, x_3\rangle \in P_2\mathbb{F};\ x_1\bar{x}_1 \neq x_2\bar{x}_2\},$$

considered as a subplane of $P_2\mathbb{F}$ in the sense of § 3. Its connected components are both isomorphic to the subplane

$$C(\mathbb{F}) = \{(x, y);\ |x| < 1\} \leqslant A_2\mathbb{F},$$

called the *(connected) cylinder plane.*

2.5 The *complex Minkowski plane*

$$M(\mathbb{C}) = \{(x, y) \in A_2\mathbb{C};\ x \neq 0\}.$$

The lines of the form $\{x\} \times \mathbb{F}$ in $C(\mathbb{F})$ and $M(\mathbb{C})$ are referred to as the *vertical lines.*

2.6 The *united hyperbolic planes*

$$UH(\mathbb{F}) = \{\langle x_1, x_2, x_3\rangle \in P_2\mathbb{F};\ x_1\bar{x}_1 \neq x_2\bar{x}_2 + x_3\bar{x}_3\}$$

and their connected components, the *interior hyperbolic planes* $IH(\mathbb{F})$, defined by $x_1\bar{x}_1 > x_2\bar{x}_2 + x_3\bar{x}_3$, and the *exterior hyperbolic planes* $EH(\mathbb{F})$, defined by the converse inequality.

2.7 The *complex oval plane*

$$O(\mathbb{C}) = \{\langle x_1, x_2, x_3\rangle \in P_2\mathbb{C};\ x_1^2 + x_2^2 + x_3^2 \neq 0\}.$$

The lines of $EH(\mathbb{F})$ and of $O(\mathbb{C})$ which meet the complement in $P_2\mathbb{F}$ in precisely one point are called the *tangents*.

2.8 The *modified real hyperbolic planes*. Their projective versions are obtained from $P_2\mathbb{R}$ as follows [14; 5.3]: Let Δ be the orientation preserving hyperbolic motion group. Take a line L which meets $IH(\mathbb{R})$, and replace $L \cap EH(\mathbb{R})$ by any orbit of Δ_L which joins the two end points of $L \cap \overline{IH(\mathbb{R})}$. Replace the orbit L^Δ of lines by the orbit L'^Δ of the modified line. Depending on the choice of an orbit of Δ_L, there results one of a one-parameter family of non-isomorphic projective planes $H_t(\mathbb{R})$, $0 < t \in \mathbb{R}$.

The *modified united and exterior hyperbolic planes* $UH_t(\mathbb{R})$ and $EH_t(\mathbb{R})$ are the subplanes of $H_t(\mathbb{R})$ induced on the point sets of $UH(\mathbb{R})$ and $EH(\mathbb{R})$, respectively; cf. [10].

2.9 The *modified real cylinder plane* $MC(\mathbb{R})$ is constructed as follows. Represent $C(\mathbb{R})$ as the right half plane $x > 0$ of $A_2\mathbb{R}$. Form the disjoint union $M = E \cup C(\mathbb{R})$ with a copy E of $\mathbb{R}^2$, and let $\Lambda \cong \mathbb{R}^2$ act on M as the group of all homeomorphisms $\tau_{b,c}$ defined, for $b, c \in \mathbb{R}$, by

$$\tau_{b,c}\colon\ C(\mathbb{R}) \to C(\mathbb{R})\colon (x,y) \to (x, y + bx + c)$$

$$\tau_{b,c}\colon\ E \to E\colon (x,y) \to (x + b, y + c)\,.$$

Consider $C(\mathbb{R})$ as a stable plane with the usual lines, and endow E with the following set of lines: First, the ordinary lines of negative slope s, $-\infty \leqslant s \leqslant 0$; second, the hyperbola branch

$$H = \{(x, -x^{-1});\ 0 < x \in \mathbb{R}\}\,,$$

and its translates H^λ, $\lambda \in \Lambda$. In this way E, too, becomes a stable plane; cf. [14; 2.12]. (The plane E was found by SALZMANN [15].) Next, identify the line H^λ of E with the image X^λ of the x-axis ($y = 0$) of $C(\mathbb{R})$; in other words, form the sum $MC(\mathbb{R})$ of the stable planes $C(\mathbb{R})$ and E with respect to the homeomorphism $f\colon X^\Lambda \to H^\Lambda$ sending X^λ to H^λ, as described in the Appendix.

In order to show that this sum is a stable plane we have to verify, according to the Appendix, that for $p \in C(\mathbb{R})$ the f-image of the set $X^\Lambda \cap \mathscr{L}_p$ covers E simply. We may assume that $p = (x, 0) \in X$, since Λ is transitive on $\{x\} \times \mathbb{R}$. Then

$$\Lambda_p = \{\tau_{b,c};\ bx + c = 0\}$$

acts transitively on $X^{\Lambda} \cap \mathscr{L}_p$, and acts on E as the one-parameter group of translations in the direction of $(1, -x)$. Thus, H^{Λ_p} is, indeed, a simple covering of E.

Remark. Note that *the motion group Σ of $C(\mathbb{R})$ acts on $MC(\mathbb{R})$.* The action is described explicitly in the proof of the Theorem.

The Theorem implies that $MC(\mathbb{R})$ cannot be embedded, equivariantly with respect to Σ, as an open subplane into a bigger stable plane. Actually, our proof will show that more generally, *$MC(\mathbb{R})$ admits no proper open embedding into any stable plane* — even without regard to its group; see case B2ii, step 3, of the proof. This implies that *$MC(\mathbb{R})$ is not isomorphic to (an open subplane of) any known plane*; in fact, disconnected planes have not been considered in the literature.

§ 3 Dense Open Subplanes of Projective Planes

An open set S of points of a stable plane $(M, \mathscr{L})$ can always be made into a stable plane; one simply endows S with the system

$$\mathscr{L}(S) = \{L \cap S;\ L \in \mathscr{L} \text{ and } |L \cap S| \geqslant 2\}$$

of lines; note that we think of lines as sets of points. The topology for $\mathscr{L}(S)$ is obtained by means of the embedding

$$\mathscr{L}(S) \to \mathscr{L} : L \to \langle L \rangle$$

which sends a line L of S to the unique line of M containing L. We call a plane $(S, \mathscr{L}(S))$ of this kind an *open subplane* of $(M, \mathscr{L})$; for a more general notion of subplane, cf. [4; 1.28].

The following result shows that a topological projective plane is uniquely determined by the geometry induced on any of its dense open subsets. No groups are involved.

Proposition. *Let $(P, \mathscr{L})$ and $(Q, \mathscr{M})$ be topological projective planes such that P is not discrete. Let $U \subseteq P$ and $V \subseteq Q$ be open subsets, and let $\varphi: (U, \mathscr{L}(U)) \to (V, \mathscr{M}(V))$ be a topological isomorphism of the associated subplanes.*

If U is dense in P then φ extends uniquely to an isomorphism $\psi: (P, \mathscr{L}) \to (Q, \mathscr{M})$; in particular, V is dense in Q.

Remarks. a) If P and Q are topological manifolds of the same dimension and U is open in P then $V \cong U$ is automatically open in Q; see [2; XVII, 3.1].

b) The assertion is false without the density assumption. This is shown, for example, by the obvious embeddings of $I\!H(\mathbb{R})$ into $H_t(\mathbb{R})$.

Corollary. *If $(U, \mathscr{L}(U))$ is a dense open subplane of a topological projective plane $(P, \mathscr{L})$ then the automorphism group* $\mathrm{Aut}(U, \mathscr{L}(U))$ *consists of those elements of* $\mathrm{Aut}(P, \mathscr{L})$ *which leave U invariant.*

Proof of Proposition. 1) Select different points $x, y \in U$, and let $A = P \setminus (x \vee y)$, $B = Q \setminus (x^\varphi \vee y^\varphi)$. It is clearly sufficient to show that $\varphi|_{U \cap A}$ extends to an Isomorphism $\psi: A \to B$ of topological affine planes; uniqueness of ψ is obvious since U is dense. The line pencil $\mathscr{L}_x$ is contained in $\mathscr{L}(U)$ since U is open and P is not discrete and, thus, has no isolated points. Therefore, we may define

$$p^\psi = (x \vee p)^\varphi \wedge (y \vee p)^\varphi,$$

for any point $p \in A$. The resulting map $\psi: A \to B$ is continuous and extends $\varphi|_{U \cap A}$. Conversely, by

$$q \to (x^\varphi \vee q)^{\varphi^{-1}} \wedge (y^\varphi \vee q)^{\varphi^{-1}}$$

we get a continuous inverse of ψ.

2) Next we show that each triple of collinear points of A can be approximated by a sequence of triples of collinear points of $A \cap U$; this will imply that ψ preserves collinearity. Indeed, let $x_1, x_2, x_3 \in L$ be different points and choose a fourth point $p \in L$. Given neighbourhoods W_i of x_i $(i = 1, 2, 3)$, apply the continuous open map

$$\pi: x \to x \vee p: A \setminus \{p\} \to \mathscr{L}_p$$

to each of them and form the intersection $\mathscr{W}$ of the images $\mathscr{W}_i = W_i^\pi$; it is non-empty, since $L \in \mathscr{W}$. Replacing W_i by $W_i \cap \mathscr{W}^{\pi^{-1}}$ we can adjust these sets so that $\mathscr{W} = \mathscr{W}_i$ for all i. Then the open set $Z_i = W_i \cap U$ is dense in W_i, and $\mathscr{Z}_i = Z_i^\pi$ is open and dense in $\mathscr{W}$. Hence, $\mathscr{Z} = \cap_i \mathscr{Z}_i$ is open and dense in $\mathscr{W}$ and, in particular, is non-empty. On any line $K \in \mathscr{Z}$, we may select a collinear triple (y_1, y_2, y_3) such that $y_i \in Z_i = W_i \cap U$.

3) It remains to show that ψ maps no triangle x_1, x_2, x_3 into one line, K. If that happens, consider a line L which meets at least two sides L_i, L_j of the triangle and contains no vertex x_k. The images $(L_i \wedge L)^\psi$ and $(L_j \wedge L)^\psi$ are different, and lie on K. Thus, $L^\psi \subseteq K$, and $V \cap B \subseteq A^\psi \subseteq K$, a contradiction.

§ 4 A Characterization of Generalized Real Cylinder Planes

The following result is originally due to K. STRAMBACH [18]. We sketch a simplified proof of an extended version of his result, making no special assumptions on the topology of the point set and the lines.

Definition. Consider a point $\infty \in P_2\mathbb{R}$, and let $\mathcal{U} \subseteq \mathcal{L}_\infty = P_1\mathbb{R}$ be an open subset. Let $M_{\mathcal{U}}$ be the open subset

$$M_{\mathcal{U}} = \{p \in P_2\mathbb{R} \setminus \{\infty\};\, p \vee \infty \in \mathcal{U}\}$$

of the real coaffine plane $P_2\mathbb{R} \setminus \{\infty\}$. The subplane defined by this point set is called a *generalized real cylinder plane.* Observe that the group $\Phi \cong \mathbb{R}^2$ of elations of $P_2\mathbb{R}$ with centre ∞ acts on $M_{\mathcal{U}}$.

Proposition. *Let $(M, \mathcal{L})$ be a 2-dimensional stable plane, and let Γ be a connected 2-dimensional abelian group of automorphisms whose orbits on M are all 1-dimensional.*

Then Γ is isomorphic to $\mathbb{R}^2$, all isotropy groups Γ_x are isomorphic to $\mathbb{R}$, and M is isomorphic to the generalized real cylinder plane $M_{\mathcal{X}}$, where

$$\mathcal{X} = \{\Gamma_x;\, x \in M\} \subseteq P_1\mathbb{R}\,.$$

Proof. Γ has a 1-dimensional kernel Γ_x on each orbit x^Γ of points. Thus, x^Γ is contained in a line $F_x = F_x^\Gamma$, and Γ_x acts freely on $M \setminus F_x$. The set $\mathcal{F} = \{F_x;\, x \in M\}$ is a simple covering of M. Consider the action of the connected component $\Delta = \Gamma_x^1$ on the circle $\mathcal{L}_x$ consisting of all lines through x [4; § 1].

(*) Δ is transitive on $\mathcal{L}_x \setminus \mathcal{F}$.

Indeed, if Δ fixes $L \in \mathcal{L}_x \setminus \mathcal{F}$ then x is a boundary point of some orbit $y^\Delta = y^\Gamma$ contained in L. Then $x = x^\Gamma$, a contradiction. It follows easily that Γ is transitive on each $F \in \mathcal{F}$ and sharply transitive on $\mathcal{L} \setminus \mathcal{F}$. In particular, $L \in \mathcal{L} \setminus \mathcal{F}$ meets each $F \in \mathcal{F}$. Moreover, (*) implies that $\Delta \cong \mathbb{R}$. Then the pair (Γ, Γ_x) must be isomorphic to $(\mathbb{R}^2, \mathbb{R})$; otherwise, $x^\Gamma = F_x$ would be compact and would meet all other lines in $\mathcal{F}$ [4; 1.15].

Now $P_2\mathbb{R} \setminus \{\infty\}$ can be coordinatized, using the group Γ acting as the elation group with centre ∞, as follows. The line set is $\Gamma \cup P\Gamma$, where $P\Gamma$ is the set of one-parameter subgroups of Γ. The points are defined by their line pencils, which correspond to the sets

$$\gamma\Delta \cup \{\Delta\} \subseteq \Gamma \cup P\Gamma\,,$$

where $\gamma \in \Gamma$ and $\Delta \in P\Gamma$. The results above show that the given plane M can be coordinatized in the same way; only in M the points $\gamma \Delta \cup \{\Delta\}$ all satisfy $\Delta \in \mathscr{X}$. Thus, an embedding of M onto $M_{\mathscr{X}} \leqslant P_2\mathbb{R} \setminus \{\infty\}$ is obtained.

§ 5 Proof of Theorem: Preliminary Remarks

We shall suppress the embedding φ in our notation; that is, we assume that $U \subseteq M$ and that φ is the inclusion. By [6, 7], the hypothesis that U is a symmetric plane implies that U is one of the planes described in 2.1 through 2.7; in case 2.3, U is affine. Any two points of a stable plane have homeomorphic neighbourhoods. Therefore, M is a locally compact stable plane of positive dimension $d \leqslant 4$. In particular, each line pencil $\mathscr{L}_p$ of a point $p \in M$ is a sphere of dimension $d/2$ [4; 1.19]. If $p \in U$ then the embedding $\mathscr{L}(U) \to \mathscr{L}$ maps the pencil of p in U onto the pencil $\mathscr{L}_p$ in M. Therefore, we shall not distinguish between the two pencils in our notation.

The following assertions are true in the planes U and M [4; 1.15, 1.16]:

(C 1) A line is compact iff it meets each other line.

(C 2) The set of compact lines is open.

The proof of the Theorem is divided into two parts, according as U is dense or not. If U is dense and $M \neq U$, we show first that M is coaffine or projective; then, we apply § 3. If U is not dense, the first steps are to show that $V = M \setminus \bar{U}$ is an orbit of Σ and to determine the isomorphism type of the dense open subplane $U \cup V$ of M; afterwards, M is determined from $U \cup V$ as in the first part of the proof. This makes it necessary to work, in that first part, with the motion group of a connected component of U rather than the whole motion group of U.

The following facts will be used frequently.

Observation 1. Let $L \in \mathscr{L}(U)$, and let $\langle L \rangle$ denote the line of M containing L. For $p \in U \setminus L$, the *growth*

$$L^* := \langle L \rangle \setminus L$$

is homeomorphic to a subset of the 'parallel set'

$$\mathscr{P}(p, L) := \{K \in \mathscr{L}_p; K \cap L = \emptyset\},$$

which depends only on the plane U. Indeed, $\pi: x \to x \vee p$ maps L^* homeomorphically into $\mathscr{P}(p, L)$.

Observation 2. In the same situation, $L^{*\pi} = \mathscr{P}(p, L)$ implies that $\langle L \rangle^\pi = \mathscr{L}_p$, hence that $\langle L \rangle$ is compact. By (C 1), the converse is also true.

Observation 3. If L is a manifold of dimension n and contains a subset S homeomorphic to the n-sphere then $S = L = \langle L \rangle$. This follows from BROUWER's theorem on the invariance of domain [2; XVII, 3.1], applied to $S^\pi \subseteq \langle L \rangle^\pi \subseteq \mathscr{L}_p$. In particular, if L is compact then $L = \langle L \rangle$.

Observation 4. *Let $(A, \mathscr{L})$ be a locally compact connected affine plane and consider any embedding $(A, \mathscr{L}) \leqslant (B, \mathscr{M})$ as a proper open subplane of a stable plane. Then $B \setminus A$ is a line; in particular, B is an almost projective plane in the sense of* [8; 2.6].

Indeed, any line $K \in \mathscr{M} \setminus \mathscr{L}$ must be disjoint from A, since A is open. By Observation 1, the growth of $L \in \mathscr{L}$ satisfies $|L^*| \leqslant 1$. Thus, any two points $x, y \notin A$ are joined by a line $K \notin \mathscr{L}$, and $K \subseteq B \setminus A$. If $B \setminus A$ contains a triangle, then $B \setminus A$ contains an open set.

§ 6 Proof of Theorem, Case A: U Dense in M

Case A 1: *U is affine or coaffine, or projective.*

Suppose first that U is an affine translation plane. By Observation 4, M is an almost projective plane, with line at infinity $M \setminus U$. By [8; 2.7], M embeds, equivariantly with respect to Σ, into the projective hull P of U. Therefore, M is an almost projective translation plane, as stated.

If U is coaffine or projective then either $M = U$, or U is coaffine and M is the unique projective extension $U \cup \{\infty\}$ of U.

Case A 2: *U is neither affine nor coaffine, nor projective.*

We may assume that $M \neq U$. The planes U that remain to be considered are the cylinder planes, the complex Minkowski plane, the hyperbolic planes and the complex oval plane.

1) *If $L \in \mathscr{L}(U)$ and $L \neq \langle L \rangle$ then $\langle L \rangle$ is compact.*

If L is affine (i. e., if $|\mathscr{P}(p, L)| = 1$ for each $p \in U \setminus L$), this follows from Observation 2. That takes care of the vertical lines in the cylinder and Minkowski planes and of the tangents in the exterior hyperbolic and oval planes; cf. §2.

Inspection shows that, in all remaining cases, the orbit L^{Σ} is open and L has at most two connected components. Openness of L^{Σ} implies that L must be dense in $\langle L\rangle$; in particular, $\langle L\rangle$ has at most the same number of components as L. Choose $p \notin L$. We may embed the pair $L \subseteq \langle L\rangle$ into the pencil $S = \mathscr{L}_p$, which is a sphere of dimension $n = 1$ or 2.

If L is an open interval and the closure of L in S is also an interval then $\langle L\rangle$ cannot be a manifold. Therefore, $\bar{L} = S$, and $\langle L\rangle$ must be compact. If L consists of two intervals, then $|L^*| \geqslant 2$ since the reflection at $x \in L$ reverses each component of L. Because $\langle L\rangle$ is a manifold, $\langle L\rangle = S$.

Now let $n = 2$. If $U = M(\mathbb{C})$ or $O(\mathbb{C})$ then Observation 1 shows that $|L^*| \leqslant 2$. If L has two ends, then these are interchanged by the reflection at $x \in L$. Thus, by Observation 2, $\langle L\rangle$ is compact. Inspection shows that in all remaining cases L is either a disc or a disjoint union of two discs, and that Σ_L contains a subgroup $\Phi \cong \mathrm{SO}_2$ acting nontrivially on each component. The action of Φ enforces [13; Theorem 5] that each component of $\langle L\rangle$ is a disc or a cylinder or a sphere, and by [12], Φ acts in the usual way on each of them. We may assume that $\langle L\rangle$ contains no sphere (Observation 3), and each component of the Φ-invariant set L^* must be a point or a circle. This is impossible.

2) *If $L \in \mathscr{L}(U)$ and $\langle L\rangle$ is not compact then U is a cylinder or Minkowski plane, and L is a vertical line.*

In the cylinder and Minkowski planes, $\mathscr{L}(U)$ consists of two Σ-orbits; one of them is the set $\mathscr{F}$ of vertical lines, and the other, $\mathscr{L}(U) \setminus \mathscr{F}$, is dense. If $L = \langle L\rangle$ for each $L \in \mathscr{L}(U) \setminus \mathscr{F}$ then, by (C 2), the same holds for $L \in \mathscr{F}$. In that case, $M = U$. In the interior hyperbolic planes, $\mathscr{L}(U)$ is a single orbit, and the same argument applies.

In $EH(\mathbb{F})$ and $O(\mathbb{C})$, there are two orbits of noncompact lines of U. One of them is the set $\mathscr{T}$ of tangents (or, equivalently, of affine lines); it is contained in the closure of the other orbit. Thus, if the elements of $\mathscr{T}$ satisfy $T^* \neq \emptyset$, the assertion follows by (C 2). If $\langle L\rangle$ is compact for one of the remaining lines, consider $T \in \mathscr{T}$ such that $T \cap L = \emptyset$ in U. By (C 1), $\langle L\rangle \cap \langle T\rangle \neq \emptyset$, and $T^* \neq \emptyset$.

3) *Conclusion*. Consider a point $u \in U$. If the pencil $\mathscr{L}_u$ consists of compact lines then M is projective [8; 1.3]. Otherwise, (2) asserts that of all lines in $\mathscr{L}_u$, only the (unique) vertical line determines a noncompact line of M. This implies that u is a coaffine point of M; see [8; 1.4, 1.6]. Since U is dense in M, all points of M are coaffine [8; 1.9], and M can be embedded into a projective plane $M \cup \{\infty\}$. In both cases, application of § 3 completes the proof.

§ 7 Proof of Theorem, Case B: U not Dense in M

Observation 5. *U must be a connected cylinder or hyperbolic plane.*

Indeed, in all other cases, the standard embedding of U has the property that each line of U is dense in the corresponding line of $P_2\mathbb{F}$. Consequently, $\mathscr{P}(p, L)$ is nowhere dense in $\mathscr{L}_p$ for each $p \notin L$. By Observation 1, this is not compatible with the assumption that $M \setminus U$ contains an open set.

Lemma. *Let $v \in V := M \setminus \bar{U}$, and let $\mathscr{X} \subseteq \Sigma$ be a conjugacy class of involutions. Then $\Sigma_v \cap \mathscr{X} \neq \emptyset$. Moreover, $\mathscr{X}$ contains a reflection at v in the following cases:*

a) *$U = C(\mathbb{C})$ and $\mathscr{X}$ contains the reflections at the vertical lines*

b) *$U = IH(\mathbb{F})$ or $EH(\mathbb{F})$, and $\mathscr{X}$ contains the reflections at all lines or at all compact lines of U, respectively.*

Proof. 1) From the action of Σ on $P_2\mathbb{F}$ it is easily seen that the elements of $\mathscr{X}$ are reflections at points or lines of U. However, not all lines occur as axes. The exceptions are (i) all lines of U if $U = IH(\mathbb{R})$ or $C(\mathbb{R})$ and (ii) the tangents if $U = EH(\mathbb{F})$. If $U = C(\mathbb{C})$ and $\mathscr{X}$ contains the reflections at the nonvertical lines, let $\sigma \in \mathscr{X}$ be the reflection at a line $L \in \mathscr{L}(U)$ such that $v \in \langle L \rangle$; note that L is not vertical since L^* contains an open set. Then σ fixes v, by [5; 1.4]. In this case the result will, however, not be needed in the sequel.

2) Let $U = IH(\mathbb{F})$, and let $\mathscr{X}$ be the class of all reflections at points of U. Choose two lines $K, L \in \mathscr{L}(U)$ such that $v = \langle K \rangle \wedge \langle L \rangle$. There is an element $\sigma \in \mathscr{X}$ which *interchanges* K and L and, hence, fixes v. Indeed, consider the lines K' and L' corresponding to K and L in the standard copy $U' \subseteq P_2\mathbb{F}$ of $IH(\mathbb{F})$, and let $v' = \langle K' \rangle \wedge \langle L' \rangle$. Then v' is an 'exterior' point since K' and L' are not asymptotes of each other. The above assertion is easily checked by examining the action of $\Sigma_{v'}$ on the sub-pencil $\mathscr{L}_{U', v'} \subseteq \mathscr{L}_{v'}$ of lines meeting U'. If $\mathbb{F} = \mathbb{R}$, that action

is equivalent to the action $\{(x \to \pm x + t);\ t \in \mathbb{R}\}$ on the real line. If $\mathbb{F} = \mathbb{C}$ then

$$\Sigma_{v'} = \mathrm{U}_2(\mathbb{C}, 1) \cong (\mathrm{SL}_2\mathbb{R} \times \mathrm{SO}_2)/\langle -1, -1\rangle$$

acts ineffectively, inducing the real hyperbolic motion group; the kernel is the centre of $\mathrm{U}_2(\mathbb{C}, 1)$ (cf. [1]). Observe that every involution of the effective quotient group can be represented by an involution of $\mathrm{U}_2(\mathbb{C}, 1)$.

3) Let $U = IH(\mathbb{C})$, and let $\mathscr{X}$ be the class of reflections at lines of U. In the notation of (2), the reflection $\tau \in \mathscr{X}$ at v' *fixes* K and L and, hence, fixes v. The point v then lies on two fixed lines of τ which are not axes (namely, $\langle K\rangle$ and $\langle L\rangle$), and hence is a centre of τ [5; 1.4].

4) The proofs of (2) and (3) are typical for all the remaining cases. The procedure resembles that of (2) or (3), according as $\mathscr{X}$ consists of reflections at points or lines of U. The reflections at the points of $EH(\mathbb{F})$ have axes as well; nevertheless, they have to be treated as reflections at points. One needs to know the actions of $\Sigma_{v'}$ on $\mathscr{L}_{U', v'}$. If $U = EH(\mathbb{F})$, then $\Sigma_{v'} = \mathrm{U}_2\mathbb{F}$ acts on $\mathscr{L}_{U', v'} = \mathscr{L}_{v'} = P_1\mathbb{F}$ by the usual action. If $U = C(\mathbb{F})$, the action contains a transitive sub-action equivalent to $\{(x \to \pm x + t): \mathbb{F} \to \mathbb{F};\ t \in \mathbb{F}\}$. Moreover, in the complex case, $\Sigma_{v'}$ contains a reflection at v'.

If $\sigma \in \mathscr{X}$ has an axis $A \in \mathscr{L}(U)$ and A is compact $(U = EH(\mathbb{F}))$ or vertical $(U = C(\mathbb{C}))$ then $|A^*| \leqslant 1$ by Observations 3 and 1, and $v \notin \langle A\rangle$ is the centre of σ. This settles the second part of the assertion. □

Case B 1: *$U = EH(\mathbb{F})$ or $C(\mathbb{C})$ or $IH(\mathbb{C})$.*

According to the Lemma, Σ contains a class $\mathscr{X}$ of involutions which contains a reflection at each point $v \in V = M \setminus \bar{U}$. Because the centre of a reflection is unique, Σ is transitive on V, and the centralizer $C(\sigma)$ of $\sigma \in \mathscr{X}$ fixes the centre $v \in V$. Conversely, Σ_v centralizes σ. Indeed, if Σ contains several reflections at v then the results [8; 3.21 and 3.2] imply that V is dense in M. By [6; Theorem A], V and $U \cup V$ are symmetric planes. Σ contains the symmetries of $U \cup V$; hence, Σ is the motion group of $U \cup V$, and application of part A yields the assertion.

Case B 2: *$U = C(\mathbb{R})$.*

1) Represent U as the right half plane (defined by $x > 0$) of the real affine plane $\mathbb{R}^2$. Then Σ acts as the product $\Sigma^1 \cdot \langle \sigma_{(1,0)}\rangle$, where

$$\Sigma^1 = \{\tau_{a,b,c};\ a, b, c \in \mathbb{R}, 0 < a\}$$

$$\tau_{a,b,c}\colon (x, y) \to (a^2 x,\ a y + a^2 b x + c)$$

$$\sigma_{(1,0)}\colon (x, y) \to (x^{-1},\ - x^{-1} y).$$

Observe that $\sigma_{(1,0)}$ is the reflection at the point $(1, 0)$. The isotropy group of the line X defined by $y = 0$ is $\Sigma_X = \Sigma_X^1 \cdot \langle\sigma_{(1,0)}\rangle$, where

$$\Sigma_X^1 = \{\tau_{a,0,0};\ 0 < a \in \mathbb{R}\}.$$

The vertical lines $F \in \mathscr{F}$ of U are the lines $x = \text{const.}$; they do not meet $V = M \setminus \bar{U}$, by Observation 1.

2) According to the Lemma, each point $v \in V$ is fixed by the reflection σ_u at some point $u \in U$. By [5; 1.4], σ_u has an axis $A_u \in \mathscr{L}_v$. The isotropy group Σ_u is the centralizer of σ_u; it acts transitively on the set $\mathscr{L}_u \setminus \mathscr{F}$ and fixes A_u. Therefore, it acts transitively on A_u, and A_u is homeomorphic to $\mathbb{R}$. Since all reflections σ_u are conjugate under Σ^1, we infer that Σ^1 is transitive on V.

In order to determine the action of Σ on V, we have to determine the isotropy group Σ_v of $v \in V$. We may assume that Σ_v contains the reflection $\sigma_{(1,0)}$. On the Lie algebra of Σ_v, the involution $\sigma_{(1,0)}$ induces either the identity id, or -id. In the first case, Σ_v^1 is centralized by $\sigma_{(1,0)}$, and $\Sigma_v^1 = \Sigma_{(1,0)}^1$. This is impossible since $\Sigma_{(1,0)}^1$ acts freely on $\mathscr{L}_u \setminus \mathscr{F}$.

Now assume that $\sigma_{(1,0)}$ induces -id on the Lie algebra of Σ_v. By [7; 1.7], each one-parameter group Φ with this property fixes a line $L \in \mathscr{L}_{(1,0)}$. If $L \in \mathscr{F}$, then Σ_L^1 is the normal subgroup

$$\Lambda = \{\tau_{1,b,c};\ b, c \in \mathbb{R}\}$$

of Σ, on which $\sigma_{(1,0)}$ acts by

$$\sigma_{(1,0)}\colon \tau_{1,b,c} \to \tau_{1,-c,-b}.$$

Then, Φ is uniquely determined. In all other cases, $\Phi = \Sigma_L^1$. Up to conjugacy, we get the following two possibilities:

(B 2 i) $\qquad \Sigma_v = \{\tau_{1,b,b};\ b \in \mathbb{R}\} \cdot \langle\sigma_{(1,0)}\rangle$

(B 2 ii) $\qquad \Sigma_v = \Sigma_X.$

3) In case (B 2 i), all orbits of the normal subgroup $\Lambda \cong \mathbb{R}^2$ on the plane $U \cup V$ are 1-dimensional. By § 4, $U \cup V \cong UC(\mathbb{R})$. If $M \neq U \cup V$ then M is the real coaffine or projective plane, by case A; observe that reflections at the points of V are not used in the proof of case A.

4) *Case* (B 2ii): If $\Sigma_v = \Sigma_X$ then the normal subgroup $\Lambda \cong \mathbb{R}^2$ acts on V sharply transitively, and the action of Σ on V may be described as the action of Λ on itself by translation, extended by Σ_X acting on Λ by conjugation. In coordinate description, the elements of Σ act on $V = \mathbb{R}^2$ as follows.

$$\tau_{a,b,c}: (v, w) \to (a^{-1} v + b, a w + c)$$
$$\sigma_{(1,0)}: (v, w) \to (-w, -v)$$

In particular, the axis of $\sigma_{(1,0)}$ is the line L defined by $w = -v$. Thus the set $\mathcal{N} = L^\Sigma$ of all ordinary lines of negative slope belongs to $\mathcal{L}(V)$. For topological reasons, the sets defined by $v = \text{const.}$ or $w = \text{const.}$ are lines as well.

Now, consider $p \in U$ and $q \in V$, and let $K = (p \vee q) \cap U$. Then K is not a vertical line of U, and we may assume that $K = X$. Then the line $L = (p \vee q) \cap V$ of V satisfies $\Sigma_L = \Sigma_X$, and L is a union of orbits of Σ_X. The orbits

$$Z = \{(0, w);\ 0 < w\} \cup \{(v, 0);\ v < 0\}$$

and $-Z$ cannot be contained in any line. The remaining orbits of Σ_X are hyperbolae and one point. Since L meets each line of $\mathcal{N}$ at most once, L must be one of the hyperbolae

$$L_s = \{(a^{-1} s, -a s);\ 0 < a \in \mathbb{R}\},\ 0 \neq s \in \mathbb{R}$$

contained in the second or fourth quadrant. Together with the lines in $\bar{\mathcal{N}}$ and in $\mathcal{L}(U)$, the orbit

$$(L_s \cup X)^\Sigma = (L_s \cup X)^\Lambda$$

forms a stable plane $MC_s(\mathbb{R})$. Now for $0 \neq t \in \mathbb{R}$, the map

$$\varphi_t: \begin{cases} U \to U: (x, y) \to (x, t y) \\ V \to V: (v, w) \to (t v, t w) \end{cases}$$

induces an automorphism of U, fixing X, and leaves $\bar{\mathcal{N}}$ invariant. Moreover, φ_t normalizes Λ and maps L_s onto L_{st}. Therefore, it maps $(L_s \cup X)^\Lambda$ onto $(L_{st} \cup X)^\Lambda$, and induces an isomorphism between $MC_s(\mathbb{R})$ and $MC_{st}(\mathbb{R})$. This shows that $U \cup V$ is isomorphic to the modified cylinder plane $MC(\mathbb{R}) = MC_1(\mathbb{R})$.

5) Now assume that $U \cup V \cong MC(\mathbb{R})$ is a proper subset of M. As announced in § 2, we shall obtain a contradiction without using Σ.

A point $x \in M \setminus (U \cup V)$ can be joined to $u \in U$; therefore, $x \in K^*$ for some line $K \in \mathcal{L}(U \cup V)$ meeting U. If K is a vertical line of U then the parallel set $\mathscr{P}(p, K)$ consists of one line or a closed interval of lines, according as $p \in U \setminus K$ or $p \in V$. Therefore, $\langle K \rangle \neq K$ contradicts Observations 1 and 2.

Suppose now that K is one of the disconnected lines. We may assume that $K = L_1 \cup X$, in the notation of (4). For all points $p = (a, b)$ of the quadrant $Q \subseteq V$ defined by $a \leqslant 0 \leqslant b$, the set $\mathscr{P}(p, K)$ consists of the horizontal line $H_b = \{(r, b); r \in \mathbb{R}\}$ and the vertical line V_a. One of these lines must be $p \vee x$; let us assume that $p \vee x = \langle H_b \rangle$. By continuity, the lines $q \vee x$ are all horizontal for $q \in Q$, and $\mathscr{L}_x$ contains the set $\mathscr{H}_0 = \{H_b; b \geqslant 0\}$. If H_b^* contains another point $y \neq x$ then y lies on no vertical line, and repetition of the argument shows that $\mathscr{L}_y$ contains some set $\mathscr{H}_t$ of horizontal lines, a contradiction. Now, since the line $\langle H_b \rangle = H_b \cup \{x\}$ is a manifold, it must be compact. This implies that $|\mathscr{P}(p, H_b)| = 1$ for each $p \in V \setminus H_b$, which is evidently false.

Case B 3: $U = IH(\mathbb{R})$.

We know from the Lemma that each $v \in V = M \setminus \bar{U}$ is fixed by the reflection σ at some point $u \in U$. The group $\Phi = \Sigma_u \cong \mathrm{SO}_2$ acts transitively on $\mathscr{L}_u$, with kernel $\langle \sigma \rangle$. Hence, $\Phi_v = \langle \sigma \rangle$, and $K = v^{\Phi} \subseteq V$ is a circle consisting of fixed points of σ. This implies (cf. Observation 3) that K is a line of M. In particular, each point of V lies on some compact line, and V is connected; cf. (C 1). The subgroup $\Sigma_v \leqslant \Sigma$ has positive dimension; it contains σ, but does not contain Φ. The only subgroups of Σ with this property are the isotropy groups of points of $EH(\mathbb{R})$, and the action of Σ on V is equivalent to the action of Σ on $EH(\mathbb{R})$. It is now easy to determine the remaining lines of $U \cup V$, using that action. This has been carried out, in a practically identical situation, in step (4) of the proof of Theorem 2 in [10]. The result is that $U \cup V \cong UH(\mathbb{R})$ or $UH_t(\mathbb{R})$. By case A, M is equal to $U \cup V$ or is the (unique) projective extension of $U \cup V$. (In case A, we have assumed that $U \cup V$ is a symmetric plane, but the proof carries over without change to the present situation. In particular, no use was made of reflections at the points of V.)

§ 8 Proof of Corollary

Let $\Sigma \leqslant \mathrm{Aut}(M, \mathscr{L})$ be the closure of the group generated by all reflections at points of U. We may assume that $U = U^{\Sigma}$. Denote by C_U the set of coaffine points of the stable plane U; cf. [8; § 1]. If U is not projective and $U \neq C_U$ then we may assume that $C_U = \emptyset$, since C_U is always closed [8; 1.9]. By [9; 1.5], U is a symmetric plane, whose motion group is contained in $\Sigma \leqslant \mathrm{Aut}(M, \mathscr{L})$, and we may apply the Theorem.

If $U = C_U$ (i. e., if U is a coaffine plane) or if U is projective, then Σ is a Lie group [16; 3.9]. By [9; 1.4], Σ has an open orbit V in U. If V is neither coaffine nor projective, apply to V the arguments of the first paragraph. Otherwise, $|M \setminus V| \leqslant 1$, and V is a desarguesian projective or coaffine plane by [16; 5.4], [17], and [14; 5.1, 5.7].

§ 9 Appendix: The Sum of Two Stable Planes

We describe here a general construction which allows to embed two given stable planes as open and closed, disjoint subplanes into a third plane, which is obtained from the given ones by means of a suitable identification of lines. H. Groh [3] has given a similar construction for flat planes; however, he glues the two components together along a line whereas, in our case, they have no boundary in common. In higher dimensions it is impossible, anyway, that a line should be the boundary of an open subset.

Our result is that, somewhat surprisingly, the trouble with constructing sums of stable planes lies in the definition of a suitable incidence structure, not in the verification of its contintuity and stability properties.

Definition. Let $(M_1, \mathscr{L}_1)$ and $(M_2, \mathscr{L}_2)$ be stable planes, and let $\mathscr{U}_k \subseteq \mathscr{L}_k$ be open subsets. A map $f: \mathscr{U}_1 \to \mathscr{U}_2$ is called *admissible* if

(i) f is a homeomorphism, and

(ii) for all $p \in M_1$, the set $(\mathscr{U}_{1p})^f$ of lines covers M_2 simply, i. e., each $q \in M_2$ lies on the image L^f of exactly one line $L \in \mathscr{U}_1 \cap \mathscr{L}_p$.

Definition. Let $(M_k, \mathscr{L}_k)$ $(k = 1, 2)$ be stable planes, and let $f: \mathscr{U}_1 \to \mathscr{U}_2$ be an admissible map between open subsets $\mathscr{U}_k$ of $\mathscr{L}_k$. Define

$$(M_1, \mathscr{L}_1) +_f (M_2, \mathscr{L}_2) = (M, \mathscr{L}),$$

where $M = M_1 + M_2$ is the sum of the topological spaces M_1, M_2, and where $\mathscr{L}$ is the set

$$\mathscr{L} = [(\mathscr{L}_1 \cup \mathscr{L}_2) \setminus (\mathscr{U}_1 \cup \mathscr{U}_2)] \cup \{L \cup L^f;\ L \in \mathscr{U}_1\}$$

of subsets of M, topologized as the amalgamated sum $\mathscr{L} = \mathscr{L}_1 +_f \mathscr{L}_2$; i. e., $\mathscr{L}$ is the quotient of the sum $\mathscr{L}_1 + \mathscr{L}_2$ modulo the identification $L \sim L^f$ for $L \in \mathscr{U}_1$.

$(M, \mathscr{L})$ is called the *sum of the planes* $(M_k, \mathscr{L}_k)$ *(with respect to the admissible map* f*)*.

Remark. Let X_1, X_2 be topological spaces, and let $f: U_1 \to U_2$ be a continuous map between subsets $U_k \subseteq X_k$. When the amalgamated sum $S = X_1 +_f X_2$ is considered, it is usually assumed that the sets $U_k \subseteq X_k$ are closed. We have assumed, instead, that U_k is *open* in X_k, and thus we do not even know that S is Hausdorff if both X_1 and X_2 are Hausdorff. However, it is easily verified that openness of U_k, together with the additional assumption that f is a homeomorphism, ensures that *the natural inclusion* $i_k: X_k \to S$ *is an open embedding*, and this is all we need.

Proposition. *The sum of two locally compact stable planes of positive dimension, as defined above, is a stable plane.*

Proof. We use the notation of the definitions and of the Remark above. It is evident that different points $p, q \in M$ are joined by a unique line. Stability of intersections is easily deduced from stability in $(M_k, \mathscr{L}_k)$, using the Remark.

We now prove continuity of the join operation $\vee$. We split the map $\vee$ up into its restrictions

$$\vee : M_k \times M_k \to \mathscr{L}_k^{i_k} \subseteq \mathscr{L}$$

(which is, by the Remark, just the join operation of $(M_k, \mathscr{L}_k)$) and

$$\vee : M_1 \times M_2 \to \mathscr{L}_1^{i_1} \cap \mathscr{L}_2^{i_2} =: \mathscr{L}_{12}.$$

The set $\mathscr{L}_{12} = \mathscr{U}_k^{i_k}$ is, again by the Remark, open in $\mathscr{L}$ and homeomorphic to $\mathscr{U}_k$. Thus, $\mathscr{L}_{12}$ is a separable metric space [4; 1.9], and continuity of $\vee$ may be proved using sequence arguments. So, let $p_{kn} \to p_k$ be convergent sequences ($k = 1, 2$), and consider the lines $L_n = p_{1n} \vee p_{2n} = K_n \cup K_n^f$. We have to show that

$$L_n \to p_1 \vee p_2 = K \cup K^f \in \mathscr{L}_{12}.$$

By the Remark, this is equivalent to showing that $K_n \to K$ or $K_n^f \to K^f$. Assume that this is not so. By [4; 1.17], we may select a subsequence $\{L_m\}_{m \in \mathbb{N}}$ such that $\{K_m\}$ converges to $A \in \mathscr{L}_{p_1} \setminus \{K\}$ and $\{K_m^f\}$ converges to $B \in \mathscr{L}_{p_2} \setminus \{K^f\}$. For m large, this implies that K_m meets K and K_m^f

meets K^f, by stability. In other words, the lines $K_m \cup K_m^f$ and $K \cup K^f$ have two points of intersection, a contradiction.

It remains to show that $\mathscr{L}$ is Hausdorff. This follows from the fact that M is Hausdorff, by virtue of the continuity and stability properties of $(M, \mathscr{L})$.

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