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Generalization of the Equations for Frame-Type Structures; a Variational Approach*

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With 8 Figures

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Summary

The differential equations for frame-type structures with elastically deformable joints, derived recently by A. D. Kerr and A. M. Zarembski [1], are generalized first by including the translational inertia terms. The corresponding variational principle is then derived formally, and the mechanical meaning of each term is established. The variational principle is then generalized by including a geometrical non-linearity, the effect of thermal and variable axial forces, and the variation of sectional properties. The corresponding differential equations are derived and the admissible boundary and matching conditions are discussed. As examples, formulations for two problems are presented.

1. Introduction

Frame-type structures, that are long and repetitive in design, appear in various areas of engineering, as railroad tracks, tall buildings, etc.

The analysis of such structures may be divided into two groups: those that determine the "local" response of the individual beams and those that determine the "global" response of the entire structure.

The methods of analysis of the "local" behavior, for long as well as for short structures, are well known. They are described in the standard books on structural mechanics. The finite element method, which utilizes the results of the "local" analysis to piece together the "global" response of the structure, can be used to determine numerically the response of such structures for a given set of parameters. However, this method is less convenient for a general discussion of the response of such structures, for a wide range of parameters.

The analyses of the "global" response of long repetitive structures generally use differential equations, which describe the behavior of the structure as one

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with "continuous" properties. The derivation of these "global" response equations are often heuristic and their accuracy is strongly affected by simplifying assumptions (or unintended omissions) made by the respective investigator. For examples of this approach refer to [2], [3], [4]. Equations of this type, which may be easily solved analytically, are useful for quick calculations needed for preliminary design purposes and as a check of more elaborate and costly computer calculations used in the final design.

A rigorous method for deriving the governing differential equations for long repetitive structures consists of first formulating the corresponding *difference equations* and then by a limit and averaging process obtain the corresponding *differential equations*. This approach, which was utilized in the mathematics literature at various occasions, was demonstrated by F. B. Hildebrand [5] for deriving the dynamic response equations of a stretched string. Recently, this approach was utilized by A. D. Kerr and A. M. Zarembski [1] for deriving the governing equations for a long repetitive frame-type structure (cross-tie track), where the joints have a torsional stiffness, as shown in Fig. 1. This derivation yielded differential equations with well defined coefficients in terms of the geometrical and mechanical parameters of the structure.

The obtained differential equations for the long frame-type structure shown in Fig. 1, as derived in [1], are

$$\left.\begin{aligned} &2EI\hat{v}^{\mathrm{IV}} + (\hat{N} - \varkappa)\,\hat{v}'' + (2\varkappa/h)\,\tilde{u}' = q \\ &-hEA\tilde{u}'' + (2\varkappa/h)\,\tilde{u} - \varkappa\hat{v}' = 0 \\ &2EA\hat{u}'' = 0. \end{aligned}\right\} \tag{1.1}$$

In the above equations $\hat{v}(x)$ and $\hat{u}(x)$ are respectively the lateral and axial displacements of the reference axis x, $\tilde{u}(x)$ is the axial displacement of the chord axes[1] due to lateral bending deformations, $(\ )' = d(\ )/dx$, $\hat{N}$ is a constant axial compression force in both chords, q is the distributed lateral load (and/or lateral resistance) which acts on the structure, E is Young's modulus of the chord material, I is the moment of inertia of one chord with respect to its centroidal axis that is normal to the plane of the structure, A is the cross-sectional area of one chord, h is the distance between the chord axes,

$$\varkappa = \frac{12K^*s^*}{6K^* + s^*} \tag{1.2}$$

$$K^* = \frac{E_0I_0}{ah}; \qquad s^* = \frac{s}{a}. \tag{1.3}$$

[1] In a cross-tie railroad track a chord represents a rail. In a tall structure, it represents a column.

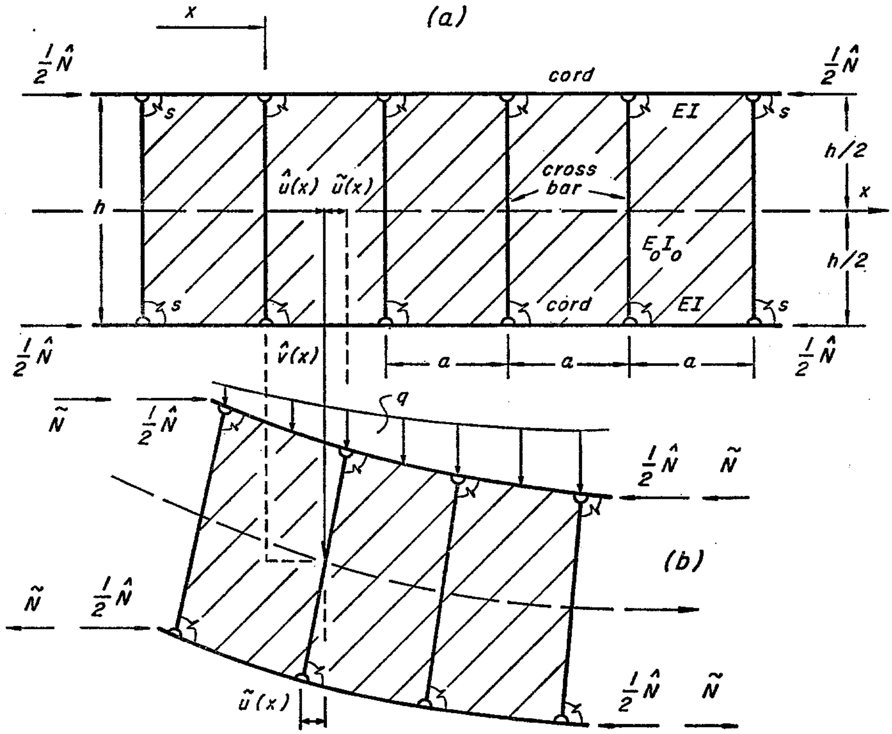

Fig. 1. Analytical model of frame-type structure
a) undeformed state, *b*) deformed state of equilibrium

$E_0 I_0$ is the bending stiffness of each cross-bar in the lateral plane, s is the rotational stiffness of a joint connection between chord and cross-bar, and a is the center to center cross-bar spacing.

As shown in [1], the internal forces in the structure may be expressed in terms of the displacements $\hat{v}(x)$, $\hat{u}(x)$ and $\tilde{u}(x)$. Namely, the axial compression force in both chords is

$$\hat{N} = -2EA\hat{u}'. \tag{1.4}$$

The additional axial force in each chord, caused by the lateral bending of the entire structure, is

$$\tilde{N}(x) = -EA\tilde{u}'. \tag{1.5}$$

The bending moment of the long frame-type structure is

$$\hat{M}(x) = M_b(x) + \tilde{M}(x) = -2EI\hat{v}'' - hEA\tilde{u}' \tag{1.6}$$

and the corresponding shearing force is

$$\hat{V}(x) = -2EI\hat{v}''' - (\hat{N} - \varkappa)\,\hat{v}' - (2\varkappa/h)\,\tilde{u}. \tag{1.7}$$

In the expression (1.6) for the bending moment in the structure, the first term

$$M_b = -2EI\hat{v}'' \tag{1.6.1}$$

represents the bending moment of the two chords and the second term

$$\tilde{M} = h\tilde{N} = -hEA\tilde{u}' \tag{1.6.2}$$

is the bending moment absorbed by the axial forces $\tilde{N}$, as shown in Fig. 2.

The method used to derive the governing differential equations for the frame-type structure did not provide a general procedure for choosing the necessary boundary or matching conditions. They may be prescribed heuristically as it is usually done in classical beam theory. However, since the theory is new it is necessary, in order to avoid improper formulations, to determine these conditions using variational calculus [6].

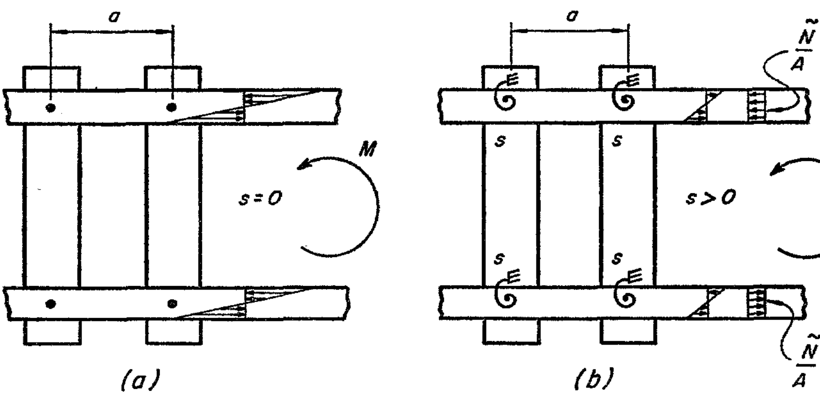

Fig. 2. Effect of rotational joint stiffness, s, on the cord stresses (note: $s^* = s/a$)

The purpose of the present paper is to derive the corresponding variational formulation for the frame-type structure under consideration, and to generalize the obtained results. At first the differential equations obtained by Kerr and Zarembski [1] are generalized by including the essential translational inertia term. The corresponding variational principle is then derived formally using these equations, and the mechanical meaning of each obtained term is established. The variational principle is then generalized by including a geometrical non-linearity, the effect of thermal and variable axial forces, and the variation of sectional properties. The corresponding differential equations of motion are then derived and the admissible boundary and matching conditions are discussed.

2. Derivation of Hamilton's Principle for the Differential Equations in (1.1) Including Inertia

For the problem under consideration, it is assumed that the rotational inertia effects are negligible. In the first equation in (1.1), the translational inertia term is included by utilizing D'Alemberts principle. The dynamic equivalent of the third equation in (1.1) is not included in the following derivation, since the equations in (1.1) were derived under the a priori assumption that $\hat{N} =$ constant[2].

The resulting equations of motion for $\hat{v}(x, t)$ and $\tilde{u}(x, t)$ are:

$$\left.\begin{aligned} &2EI\hat{v}_{,xxxx} + (\hat{N} - \varkappa)\,\hat{v}_{,xx} + (2\varkappa/h)\,\tilde{u}_{,x} + m\hat{v}_{,tt} = q \\ &-hEA\tilde{u}_{,xx} + (2\varkappa/h)\,\tilde{u} - \varkappa\hat{v}_{,x} = 0 \end{aligned}\right\} \tag{2.1}$$

where $(\ )_{,x} = \partial(\ )/\partial x$ and $m = 2m_c + hm_0/a$ is the mass of the frame-type structure per unit length of axis.

The corresponding variational principle is obtained formally, using the above equations ([5, Sections 2.8 and 2.14]). First, we multiply each of the above equations by the corresponding displacement variation (in order to form the so called virtual work equations), sum them, integrate the resulting equation over the space domain (x_1, x_2), and then over the time domain (t_1, t_2). The result is

$$\begin{aligned} \int_{t_1}^{t_2}\int_{x_1}^{x_2} &\{[2EI\hat{v}_{,xxxx} + (\hat{N} - \varkappa)\,\hat{v}_{,xx} + (2\varkappa/h)\,\tilde{u}_{,x} + m\hat{v}_{,tt} - q]\,\delta\hat{v} \\ &+ [-EA\tilde{u}_{,xx} + (2\varkappa/h^2)\,\tilde{u} - (\varkappa/h)\,\hat{v}_{,x}]\,2\delta\tilde{u}\}\,dx\,dt = 0. \end{aligned} \tag{2.2}$$

Note, that the second equation in (2.1), being the rotational equation of motion of the cross-bars, was multiplied by the variation of the corresponding rotation $\delta(2\tilde{u}/h) = (2/h)\,\delta\tilde{u}$.

Next, Eq. (2.2) is transformed using integration by parts. All terms are integrated with respect to x, except for the term with $\hat{v}_{,tt}$ which is integrated with respect to t. To illustrate this procedure note that

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2} \hat{N}\hat{v}_{,xx}\delta\hat{v}\,dx\,dt = \int_{t_1}^{t_2} [\hat{N}\hat{v}_{,x}\delta\hat{v}]_{x_1}^{x_2}\,dt - \int_{t_1}^{t_2}\int_{x_1}^{x_2} \hat{N}\hat{v}_{,x}\delta\hat{v}_{,x}\,dx\,dt, \tag{2.3}$$

where $\hat{N}\hat{v}_{,x}\delta\hat{v}_{,x} = \delta(\hat{N}\hat{v}_{,x}^2/2)$, and that

$$\begin{aligned} \int_{t_1}^{t_2}\int_{x_1}^{x_2} m\hat{v}_{,tt}\delta\hat{v}\,dx\,dt &= \int_{x_1}^{x_2}\int_{t_1}^{t_2} m\hat{v}_{,tt}\delta\hat{v}\,dt\,dx \\ &= \int_{x_1}^{x_2} [m\hat{v}_{,t}\delta\hat{v}]_{t_1}^{t_2}\,dx - \int_{x_1}^{x_2}\int_{t_1}^{t_2} m\hat{v}_{,t}\delta\hat{v}_{,t}\,dt\,dx \end{aligned} \tag{2.3.1}$$

[2] This assumption is dropped in Section 3.

where $m\hat{v}_t$, $\delta\hat{v}_{,t} = \delta(m\hat{v}_{,t}^2/2)$. Also

$$\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left[-\varkappa\hat{v}_{,xx}\delta\hat{v} + \frac{2\varkappa}{h}\,\tilde{u}_{,x}\delta\hat{v} + \frac{4\varkappa}{h^2}\,\tilde{u}\delta\tilde{u} - \frac{2\varkappa}{h}\,\hat{v}_{,x}\delta\tilde{u}\right] dx\,dt$$

$$= \int_{t_1}^{t_2}\int_{x_1}^{x_2}\left[-\varkappa\left(\hat{v}_{,x} - \frac{2\tilde{u}}{h}\right)_{,x}\delta\hat{v} - \varkappa\left(\hat{v}_{,x} - \frac{2\tilde{u}}{h}\right)\frac{2}{h}\,\delta\tilde{u}\right] dx\,dt$$

$$= \int_{t_1}^{t_2}\int_{x_1}^{x_2}\left[\varkappa\left(\hat{v}_{,x} - \frac{2\tilde{u}}{h}\right)\delta\hat{v}_{,x} - \varkappa\left(\hat{v}_{,x} - \frac{2\tilde{u}}{h}\right)\frac{2}{h}\,\delta\tilde{u}\right] dx\,dt - \int_{t_1}^{t_2}\left[\varkappa\left(\hat{v}_{,x} - \frac{2\tilde{u}}{h}\right)\delta\hat{v}\right]_{x_1}^{x_2} dt$$

$$= \int_{t_1}^{t_2}\int_{x_1}^{x_2}\left[\varkappa\left(\hat{v}_{,x} - \frac{2\tilde{u}}{h}\right)\left(\delta\hat{v}_{,x} - \frac{2}{h}\,\delta\tilde{u}\right)\right] dx\,dt - \int_{t_1}^{t_2}\left[\varkappa\left(\hat{v}_{,x} - \frac{2\tilde{u}}{h}\right)\delta\hat{v}\right]_{x_1}^{x_2} dt, \qquad (2.3.2)$$

where the integrand of the double integral may be written as $\delta[\varkappa(\hat{v}_{,x} - 2\tilde{u}/h)^2/2]$.

Performing the other integrations, as indicated above, and noting that for the problem under consideration the variations and integrations are interchangeable, Eq. (2.2) may be written as follows:

$$\delta\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left\{\frac{2EI\hat{v}_{,xx}^2}{2} + \frac{2EA\tilde{u}_{,x}^2}{2} + \frac{\varkappa(\hat{v}_{,x} - 2\tilde{u}/h)^2}{2} - \frac{\hat{N}\hat{v}_{,x}^2}{2} - q\hat{v} - \frac{m\hat{v}_{,t}^2}{2}\right\} dx\,dt$$
$$+ \int_{t_1}^{t_2}\{[2EI\hat{v}_{,xxx} + \hat{N}\hat{v}_{,x} - \varkappa(\hat{v}_{,x} - 2\tilde{u}/h)]\,\delta\hat{v} - [2EI\hat{v}_{,xx}]\,\delta\hat{v}_{,x} - [2EA\tilde{u}_{,x}]\,\delta\tilde{u}\}_{x_1}^{x_2} dt$$
$$+ \int_{x_1}^{x_2}\{[m\hat{v}_{,t}]\,\delta\hat{v}\}_{t_1}^{t_2} dx = 0. \qquad (2.4)$$

A standard condition imposed on Eq. (2.4) is that $\hat{v}(x, t)$ is specified at $t = t_1$ and $t = t_2$. Thus

$$\delta\hat{v}|_{t_1} = \delta\hat{v}|_{t_2} = 0 \qquad (2.5)$$

and the last integral in Eq. (2.4) vanishes. This condition is equivalent to prescribing the initial conditions.

The natural boundary conditions for the problem are obtained by equating to zero the boundary integrals. They are

$$\left.\begin{aligned}
&\{[2EI\hat{v}_{,xxx} + \hat{N}\hat{v}_{,x} - \varkappa(\hat{v}_{,x} - 2\tilde{u}/h)]\,\delta\hat{v}\}_{x_1}^{x_2} = 0\\
&\{[2EI\hat{v}_{,xx}]\,\delta\hat{v}_{,x}\}_{x_1}^{x_2} = 0\\
&\{[2EA\tilde{u}_{,x}]\,\delta\tilde{u}\}_{x_1}^{x_2} = 0.
\end{aligned}\right\} \qquad (2.6)$$

Because of Eqs. (1.5) to (1.7), the above conditions may be written as

$$\left.\begin{aligned} &\{\hat{V}\delta\hat{v}\}_{x_1}^{x_2} = 0 \\ &\{M_b\delta\hat{v}_{,x}\}_{x_1}^{x_2} = \\ &\{2\tilde{N}\delta\tilde{u}\}_{x_1}^{x_2} = \left\{\tilde{M}\delta\left(\frac{2\tilde{u}}{h}\right)\right\}_{x_1}^{x_2} = 0. \end{aligned}\right\} \tag{2.6.1}$$

This form gives the boundary terms an obvious physical meaning. Note that although the bending moment in the structure is $\hat{M} = M_b + \tilde{M}$, as shown in Eq. (1.6), M_b and $\tilde{M}$ or their corresponding rotations $\hat{v}_{,x}$ and $2\tilde{u}/h$ must be prescribed *separately* at the boundaries.

For any case in which the natural boundary conditions are satisfied, the variational problem (2.4) reduces to the form

$$\boxed{\delta\int_{t_1}^{t_2}\int_{x_1}^{x_2}\left\{\frac{(2EI)\,\hat{v}_{,xx}^2}{2} + 2\left(\frac{EA\tilde{u}_{,x}^2}{2}\right) + \frac{\varkappa(\hat{v}_{,x} - 2\tilde{u}/h)^2}{2} - q\hat{v} - \frac{\hat{N}\hat{v}_{,x}^2}{2} - \frac{m\hat{v}_{,t}^2}{2}\right\} dx\,dt = 0}\,. \tag{2.7}$$

This is Hamilton's variational principle for the equations given in (2.1).

The standard form of Hamilton's principle is

$$\delta\int_{t_1}^{t_2}(\Pi - T)\,dt = 0, \tag{2.8}$$

where

$$T = \int_{x_1}^{x_2}\frac{m\hat{v}_{,t}^2}{2}\,dx \tag{2.9}$$

is the kinetic energy. Thus, the remaining terms form the total potential energy

$$\Pi = \int_{x_1}^{x_2}\left\{\frac{(2EI)\,\hat{v}_{,xx}^2}{2} + 2\left(\frac{EA\tilde{u}_{,x}^2}{2}\right) + \frac{\varkappa(\hat{v}_{,x} - 2\tilde{u}/h)^2}{2} - q\hat{v} - \frac{\hat{N}\hat{v}_{,x}^2}{2}\right\} dx. \tag{2.10}$$

For the planned generalization of Eq. (2.7) it will be helpful to establish the physical meaning of each term in Π. In this connection note that

$$\Pi = U - W, \tag{2.11}$$

where U is the stored elastic strain energy and W is the work potential of the external forces. The first term in Π

$$\int_{x_1}^{x_2}\frac{(2EI)\,\hat{v}_{,xx}^2}{2}\,dx = \int_{x_1}^{x_2}\frac{1}{2}\,\frac{M_b^2}{2EI}\,dx = U_{M_b} \tag{2.12}$$

represents the elastic strain energy in the two chords due to bending. The second term

$$\int_{x_1}^{x_2} 2\left(\frac{EA\tilde{u}_{,x}{}^2}{2}\right) dx = \int_{x_1}^{x_2} 2\left(\frac{1}{2}\,\frac{\tilde{N}^2}{EA}\right) dx = U_{\tilde{N}} \tag{2.13}$$

represents the elastic strain energy in both chords due to the axial force $\tilde{N}$ in each chord. The third term

$$\int_{x_1}^{x_2} \frac{\varkappa(\hat{v}_{,x} - 2\tilde{u}/h)^2}{2}\, dx = U_\varkappa \tag{2.14}$$

is the last strain energy term. It contains the stiffness parameters of the joint springs and the cross-bars. Thus, it obviously represents the strain energy in the joint springs due to the relative rotation of the cross-bars and chords and the strain energy in the cross-bars due to bending. The fourth and fifth terms represent the work potential

$$W = \int_{x_1}^{x_2} \left(q\hat{v} + \frac{\hat{N}\hat{v}_{,x}{}^2}{2}\right) dx \tag{2.15}$$

of the distributed lateral load q and the axial force $\hat{N}$.

All terms in Π, except for $U_\varkappa$, are known from the theory of beams. The strain energy term $U_\varkappa$, however, is new. Since it was derived formally, it is necessary to validate its physical meaning by considering the structure.

Fig. 3 shows the initial and the deformed state of a typical cross-bar and the adjoining chords. Each of the joint springs is deformed by the angle ψ. The cross-bar, in addition to experiencing the rigid body translations $\hat{u}$, $\hat{v}$ (that do not store strain energy in it), is subjected to end displacements $\tilde{u}$ and end rotations $\phi = \hat{v}_{,x} - \psi$.

The cross-bar and the joint springs act in series, therefore their strain energy has to be calculated using an equivalent rotational stiffness[3]. This is done next.

Using the slope-deflection relations ([1, Eq. (2.3)]), the end moments of a cross-bar may be expressed as

$$M_{ij} = M_{ji} = S_1K\phi + S_2K\phi - (S_1 + S_2)\, K2\tilde{u}/h. \tag{2.16}$$

Since $\phi = \hat{v}_{,x} - \psi$, and neglecting the effect of axial forces in the cross-bars on the bending moments, as done in [1] (thus $S_1 = 4$, $S_2 = 2$), above equation becomes

$$M_{ij} = M_{ji} = 6K[(\hat{v}_{,x} - 2\tilde{u}/h) - \psi], \tag{2.16.1}$$

[3] Note that when two linear springs with parameters c and k are attached *in series* and stretched by a force P, the one spring elongating by w_1 and the other by w_2, the stored strain energy is $U = k^*(w_1 + w_2)^2/2$, where $k^* = ck/(c + k)$ is the equivalent spring parameter.

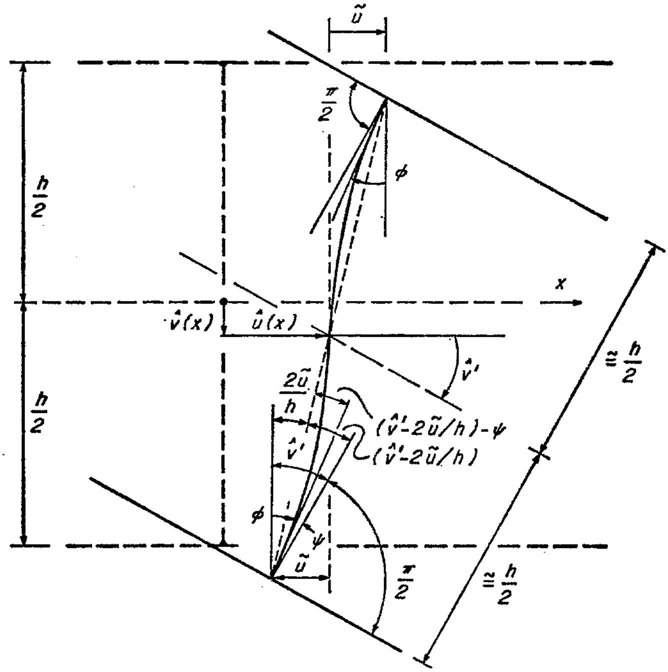

Fig. 3

where $K = E_0I_0/h$. The total moment that each cross-bar exerts on the chords is

$$M_{ij} + M_{ji} = 12K[(\hat{v}_{,x} - 2\tilde{u}/h) - \psi]. \tag{2.17}$$

Thus, the averaged torsional stiffness of the cross-bars at the joints, per unit length of axis, is

$$\frac{12K}{a} = 12K^*. \tag{2.18}$$

The averaged torsional stiffness of the joint springs, at both chords, is

$$\frac{2s}{a} = 2s^*. \tag{2.19}$$

Because the cross-bars and joint springs act in series, the "equivalent rotational stiffness" at the joint is

$$\frac{1}{s^*_{\text{eq}}} = \frac{1}{2s^*} + \frac{1}{12K^*} = \frac{6K^* + s^*}{12s^*K^*}.$$

Hence

$$s^*_{\text{eq}} = \frac{12s^*K^*}{6K^* + s^*}. \tag{2.20}$$

Comparing above expression with Eq. (1.2) it follows that

$$s^*_{\text{eq}} = \varkappa. \tag{2.21}$$

This establishes the physical meaning of the parameter $\varkappa$, which appeared in the formal derivation presented in [1].

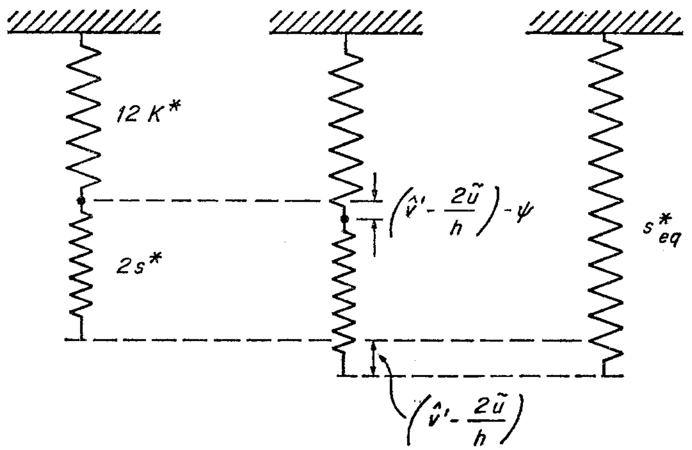

Fig. 4

In Fig. 4, the series arrangement of the cross-bars and joint springs is presented in a straight line, for the sake of clarity. Noting that the angle of rotation of the "equivalent spring" is $(\hat{v}_{,x} - 2\tilde{u}/h)$ it follows that the strain energy stored in the cross bar-joint spring system is

$$U_\varkappa = \int_{x_1}^{x_2} \frac{\varkappa(\hat{v}_{,x} - 2\tilde{u}/h)^2}{2}\, dx$$

which agrees with the corresponding expression in Eq. (2.10). This completes the interpretation of the $U_\varkappa$ term.

The strain energy due to axial extension of the cross-bars does not appear in the expression for Π, Eq. (2.10), since the effect of axial forces in the cross-bars was suumed to be negligible in [1] and this axial force does not appear in (1.1).

3. The Generalized Equations for the Frame-Type Structure

The differential equations in (1.1) for a long frame-type structure were derived under the assumption that the structural parameters, such as the bending stiffnesses EI and E_0I_0, the spring joint stiffness s, and the cross-bar spacing does not vary. Also the axial load $\hat{N}$ was assumed to be constant. In the present section the equations in (1.1) will be generalized, by dropping these assumptions, and by including the effect of a uniform temperature increase.

To obtain a formulation valid for a variable $\hat{N}(x, t)$, we include a distributed axial load $n(x, t)$, acting along the reference axis of the structure. We also include a non-linearity in the strain-displacement relations, in order to couple the lateral and axial displacements in the formulation.

Referring to Fig. 5, the Lagrange strains in the top and bottom chords are defined as

$$\left.\begin{aligned} \varepsilon_{xx}^T &= \lim_{a_n\to 0} \frac{s_n{}^T - a_n}{a_n} = s,_x{}^T - 1 \\ \varepsilon_{xx}^B &= \lim_{a_n\to 0} \frac{s_n{}^B - a_n}{a_n} = s,_x{}^B - 1. \end{aligned}\right\} \tag{3.1}$$

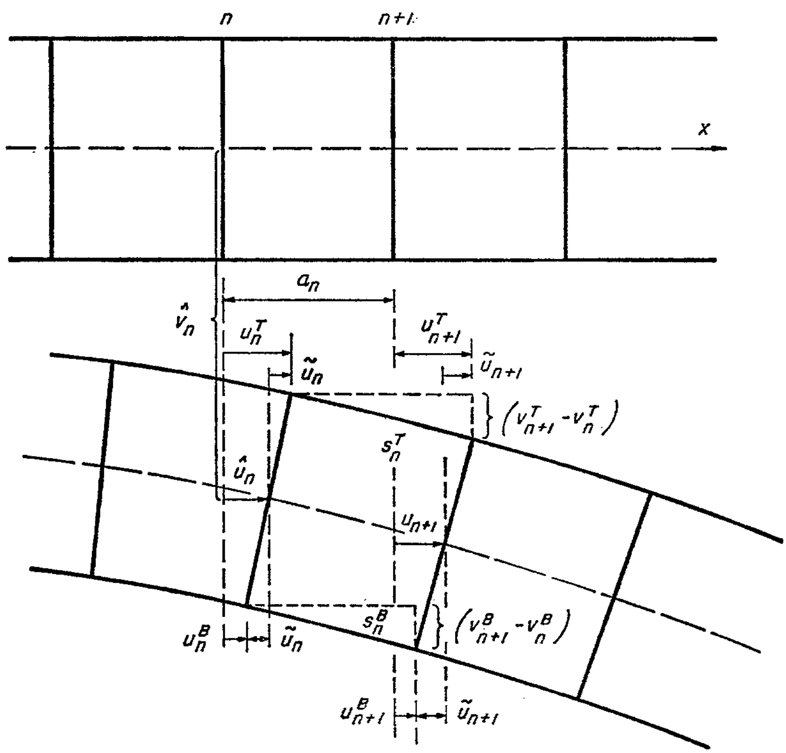

Fig. 5

The lengths of the deformed chord segments, $s_n{}^T$ and $s_n{}^B$, are expressed according to Fig. 5 (noting that $u_n{}^T = \hat{u}_n + \tilde{u}_n$, $u_n{}^B = \hat{u}_n - \tilde{u}_n$, etc.), as

$$\left.\begin{aligned} (s_n{}^T)^2 &= (a_n + u_{n+1}^T - u_n{}^T)^2 + (v_{n+1}^T - v_n{}^T)^2 \\ (s_n{}^B)^2 &= (a_n + u_{n+1}^B - u_n{}^B)^2 + (v_{n+1}^B - v_n{}^B)^2. \end{aligned}\right\} \tag{3.2}$$

Utilizing the definition of the forward difference operator ([1, Sec. 2]), the equations in (3.2) may be rewritten as

$$\left.\begin{aligned}\left(\frac{s_n{}^T}{a_n}\right)^2 &= \left(1 + \frac{\Delta_f u_n{}^T}{\Delta x}\right)^2 + \left(\frac{\Delta_f v_n{}^T}{\Delta x}\right)^2 \\ \left(\frac{s_n{}^B}{a_n}\right)^2 &= \left(1 + \frac{\Delta_f u_n{}^B}{\Delta x}\right)^2 + \left(\frac{\Delta_f v_n{}^B}{\Delta x}\right)^2.\end{aligned}\right\} \tag{3.3}$$

Performing lim $a_n \to 0$ on the above equations, we obtain

$$\left.\begin{aligned}s,_x{}^T &= \sqrt{(1 + u,_x{}^T)^2 + (v,_x{}^T)^2} \\ s,_x{}^B &= \sqrt{(1 + u,_x{}^B)^2 + (v,_x{}^B)^2}.\end{aligned}\right\} \tag{3.4}$$

Substitution of Eq. (3.4) into the definition for strains, as stated in (3.1), yields

$$\left.\begin{aligned}\varepsilon_{xx}^T &= \sqrt{1 + 2u,_x{}^T + (u,_x{}^T)^2 + (v,_x{}^T)^2} - 1 \\ \varepsilon_{xx}^B &= \sqrt{1 + 2u,_x{}^B + (u,_x{}^B)^2 + (v,_x{}^B)^2} - 1.\end{aligned}\right\} \tag{3.5}$$

The resulting strains are highly non-linear. In order to simplify them, the right hand sides are expanded in a binomial series of the form

$$\sqrt{1 + a} = 1 + \frac{a}{2} + \cdots; \qquad a \ll 1$$

and higher order terms are neglected. The resulting strains are

$$\left.\begin{aligned}\varepsilon_{xx}^T &= 1 + u,_x{}^T + (u,_x{}^T)^2/2 + (v,_x{}^T)^2/2 + \cdots - 1 \\ \varepsilon_{xx}^B &= 1 + u,_x{}^B + (u,_x{}^B)^2/2 + (v,_x{}^B)^2/2 + \cdots - 1.\end{aligned}\right\} \tag{3.5.1}$$

Based on the anticipated geometry of the deformed structure, it is assumed that $u,_x$ is of the same order as $v,_x{}^2$. The expressions in (3.5.1) then reduce to

$$\left.\begin{aligned}\varepsilon_{xx}^T &= u,_x{}^T + \frac{1}{2}(v,_x{}^T)^2 \\ \varepsilon_{xx}^B &= u,_x{}^B + \frac{1}{2}(v,_x{}^B)^2.\end{aligned}\right\} \tag{3.6}$$

Noting that, in accordance with [1] (and Fig. 5)

$$\left.\begin{aligned}u^T &= \hat{u} + \tilde{u} \\ u^B &= \hat{u} - \tilde{u} \\ v^T &= v^B = \hat{v}\end{aligned}\right\} \tag{3.7}$$

the expressions in (3.6) become

$$\left.\begin{aligned}\varepsilon_{xx}^T &= \hat{u},_x + \tilde{u},_x + \frac{1}{2}\hat{v},_x{}^2 \\ \varepsilon_{xx}^B &= \hat{u},_x - \tilde{u},_x + \frac{1}{2}\hat{v},_x{}^2.\end{aligned}\right\} \tag{3.8}$$

The axial forces in the top and bottom chords are (Fig. 1)

$$\left.\begin{aligned}\hat{N}/2 + \tilde{N} &= -EA\varepsilon_{xx}^{T} = -EA(\hat{u}_{,x} + \tilde{u}_{,x} + \hat{v}_{,x}{}^2/2)\\ \hat{N}/2 - \tilde{N} &= -EA\varepsilon_{xx}^{B} = -EA(\hat{u}_{,x} - \tilde{u}_{,x} + \hat{v}_{,x}{}^2/2)\end{aligned}\right\} \tag{3.9}$$

where $\hat{N} > 0$ is a compression force. Solving the above equations for $\hat{N}$ and $\tilde{N}$ yields

$$\left.\begin{aligned}\hat{N}(x, t) &= -2EA\left(\hat{u}_{,x} + \frac{1}{2}\,\hat{v}_{,x}{}^2\right)\\ \tilde{N}(x, t) &= -EA\tilde{u}_{,x}.\end{aligned}\right\} \tag{3.10}$$

These are the new relations for the axial forces. Comparing them with Eqs. (1.4) and (1.5) it follows that $\tilde{N}$ remains unchanged, but $\hat{N}$ contains an additional term that is non-linear.

Taking into consideration the effect of thermal *axial* forces, the relations in (3.10) become

$$\left.\begin{aligned}\hat{N}(x, t) &= -2EA\left(\hat{u}_{,x} + \frac{1}{2}\,\hat{v}_{,x}{}^2 - \alpha T\right)\\ \tilde{N}(x, t) &= -EA\tilde{u}_{,x}.\end{aligned}\right\} \tag{3.10.1}$$

These expressions will be used in the following for constructing a more general formulation, than the one given in (1.1) and (2.7).

Hamilton's principle, Eq. (2.8), for the generalized problem, is established by assuming that the elastic strain energy U consists of the following parts:

$$U_{M_b} = \int_{x_1}^{x_2} \frac{1}{2}\,\frac{M_b{}^2}{(2EI)}\,dx = \int_{x_1}^{x_2} \frac{(2EI)\hat{v}_{,xx}^{2}}{2}\,dx \tag{3.11}$$

$$U_{\hat{N}} = \int_{x_1}^{x_2} 2\left[\frac{1}{2}\,\frac{(\hat{N}/2)^2}{EA}\right] dx = \int_{x_1}^{x_2} EA\left(\hat{u}_{,x} + \frac{1}{2}\,\hat{v}_{,x}{}^2 - \alpha T\right)^2 dx \tag{3.12}$$

$$U_{\tilde{N}} = \int_{x_1}^{x_2} 2\left[\frac{1}{2}\,\frac{\tilde{N}^2}{EA}\right] dx = \int_{x_1}^{x_2} 2\left[\frac{EA\tilde{u}_{,x}{}^2}{2}\right] dx \tag{3.13}$$

$$U_{\varkappa} = \int_{x_1}^{x_2} \frac{\varkappa(\hat{v}_{,x} - 2\tilde{u}/h)^2}{2}\,dx. \tag{3.14}$$

The corresponding work potential W is

$$W = \int_{x_1}^{x_2} (q\hat{v} + n\hat{u})\,dx \tag{3.15}$$

and the kinetic energy is

$$T = \int\limits_{x_1}^{x_2} \left(\frac{m\hat{v},_t^2}{2} + \frac{m\hat{u},_t^2}{2} \right) dx. \tag{3.16}$$

Thus, the generalized Hamilton's variational equation for the frame-type structure is

$$\boxed{\delta \int\limits_{t_1}^{t_2} \int\limits_{x_1}^{x_2} \left[EI\hat{v},_{xx}^2 + EA\left(\hat{u},_x + \frac{1}{2}\hat{v},_x^2 - \alpha T\right)^2 + EA\tilde{u},_x^2 + \frac{\varkappa}{2}(\hat{v},_x - 2\tilde{u}/h)^2 - q\hat{v} - n\hat{u} - \frac{1}{2}m\hat{v},_t^2 - \frac{1}{2}m\hat{u},_t^2 \right] dx\, dt = 0.} \tag{3.17}$$

Note the difference between the above equation and the original variational equation (2.7), which corresponds to the differential equations in (1.1). Whereas Eq. (2.7) implies that the axis of the frame-type structure is inextensible, this is not the case in the generalized equation (3.17). In (3.17) the strain energy term due to the extensibility of the axis, $U_{\hat{N}}$, was added. This in turn required a modification of the work potential W, as shown in (3.15). Since in Eq. (3.17) the axial forces $\hat{N} = \hat{N}(x, t)$, the kinetic energy in the axial direction was also included.

In Eq. (3.17) the displacements $\hat{u}(x, t)$ and $\hat{v}(x, t)$ are coupled. The coupling term is of higher order than quadratic. Thus, the corresponding differential equations will be non-linear.

Performing the variations in Eq. (3.17), then integrating by parts, and using the Fundamental Lemma, we obtain the corresponding differential equation formulation. The resulting equations for $\hat{v}(x, t)$, $\hat{u}(x, t)$, and $\tilde{u}(x, t)$ are:

$$\boxed{\begin{aligned} &(2EI\hat{v},_{xx}),_{xx} - [2EA(\hat{u},_x + \hat{v},_x^2/2 - \alpha T)\,\hat{v},_x],_x - [\varkappa(\hat{v},_x - 2\tilde{u}/h)],_x + m\hat{v},_{tt} = q \\ &(EA\tilde{u},_x),_x + \varkappa(\hat{v},_x - 2\tilde{u}/h)/h = 0 \\ &[2EA(\hat{u},_x + \hat{v},_x^2/2 - \alpha T)],_x + n = m\hat{u},_{tt}. \end{aligned}} \tag{3.18}$$

The natural boundary conditions are obtained from the boundary terms

$$\left.\begin{aligned} &\{[-(2EI\hat{v},_{xx}),_x + 2EA(\hat{u},_x + \hat{v},_x^2/2 - \alpha T)\,\hat{v},_x + \varkappa(\hat{v},_x - 2\tilde{u}/h)]\,\delta\hat{v}\}_{x_1}^{x_2} = 0 \\ &\{[2EI\hat{v},_{xx}]\,\delta\hat{v},_x\}_{x_1}^{x_2} = 0; \quad \{[2EA\tilde{u},_x]\,\delta\tilde{u}\}_{x_1}^{x_2} = 0 \\ &\{[2EA(\hat{u},_x + \hat{v},_x^2/2 - \alpha T)]\,\delta\hat{u}\}_{x_1}^{x_2} = 0. \end{aligned}\right\} \tag{3.19}$$

The initial conditions are deduced from the conditions

$$\{[m\hat{v},_t]\,\delta\hat{v}\}_{t_1}^{t_2} = 0; \quad \{[m\hat{u},_t]\,\delta\hat{u}\}_{t_1}^{t_2} = 0. \tag{3.20}$$

4. Interpretation of the Generalized Differential Equation Formulation

To give the generalized equations in (3.18) a physical interpretation, note that according to Eq. (3.10)

$$\left.\begin{aligned} \hat{N}(x,t) &= -2EA\left(\hat{u}_{,x} + \frac{1}{2}\hat{v}_{,x}{}^2 - \alpha T\right) \\ \tilde{N}(x,t) &= -EA\tilde{u}_{,x}. \end{aligned}\right\} \tag{4.1}$$

Also, from the above presentation it follows that

$$\left.\begin{aligned} M_b(x,t) &= -2EI\hat{v}_{,xx} \\ \tilde{M}(x,t) &= h\tilde{N} = -hEA\tilde{u}_{,x} \end{aligned}\right\} \tag{4.2}$$

and

$$\hat{M}(x,t) = M_b(x,t) + \tilde{M}(x,t). \tag{4.2.1}$$

Furthermore, according to the presented interpretation of the strain energy expression $U_\varkappa$, Eq. (2.14), the term

$$\varkappa(\hat{v}_{,x} - 2\tilde{u}/h) = \mu \tag{4.3}$$

is the distributed moment per unit length of axis, that the cross-bars and joint springs exert on the two chords.

With this notation, the differential equations in (3.18) may be written as

$$\left.\begin{aligned} &-M_{b,xx} + (\hat{N}\hat{v}_{,x})_{,x} - \mu_{,x} + m\hat{v}_{,tt} = q \\ &-\tilde{N}_{,x} + \mu/h = 0 \\ &-\hat{N}_{,x} + n = m\hat{u}_{,tt}, \end{aligned}\right\} \tag{3.18.1}$$

where $\hat{N} > 0$ is a compression force. Eliminating μ from the first equation in (3.18.1) by using the second equation, and noting that $\hat{M} = M_b + \tilde{M}$ and $\tilde{M} = h\tilde{N}$, the equations in (3.18.1) may be rewritten as

$$\left.\begin{aligned} &-\hat{M}_{,xx} + (\hat{N}\hat{v}_{,x})_{,x} + m\hat{v}_{,tt} = q \\ &-\hat{N}_{,x} + n = m\hat{u}_{,tt} \\ &+\tilde{M}_{,x} = \mu. \end{aligned}\right\} \tag{3.18.2}$$

The general form of the first two equations are the same as in single beam theory. The third equation represents the moment equilibrium of the "continuous" filler between the chords. It is a special feature of the frame-type structure under consideration.

In accordance with the approach that led to the equations in (2.6.1), it follows, from the first condition in (3.19), that the shearing force expression, that corresponds to the generalized formulation, is

$$\hat{V}(x, t) = -(2EI\hat{v}_{,xx})_{,x} + 2EA(\hat{u}_{,x} + \hat{v}_{,x}{}^2/2 - \alpha T)\,\hat{v}_{,x} + \varkappa(\hat{v}_{,x} - 2\tilde{u}/h). \quad (4.4)$$

Thus

$$\hat{V}(x, t) = M_{b_{,x}} - \hat{N}\hat{v}_{,x} + \mu \quad (4.5)$$

or

$$\hat{V}(x, t) = \hat{M}_{,x} - \hat{N}\hat{v}_{,x}. \quad (4.6)$$

5. Specific Examples

To demonstrate the use of the derived analysis, we discuss the necessary formulations for two problems.

At first consider the dynamic response of the structure of a tall building shown in Fig. 6. The structure is subjected to its own weight $n(x)$, and the wind load $q(x, t)$.

Neglecting the effect of the axial inertia term, it follows from the third equation in (3.18), or (3.18.1), that the axial force in the structure is

$$\hat{N}(x) = \int_x^L n(\xi)\, d\xi. \quad (5.1)$$

Thus, the formulation of the problem under consideration consists of the remaining two differential equations in (3.18)

$$\left.\begin{aligned} &(2EI\hat{v}_{,xx})_{,xx} + [(\hat{N} - \varkappa)\,\hat{v}_{,x}]_{,x} + (2\varkappa\tilde{u}/h)_{,x} + m\hat{v}_{,tt} = q(x, t) \\ &(EA\tilde{u}_{,x})_{,x} + \varkappa(\hat{v}_{,x} - 2\tilde{u}/h)/h = 0 \end{aligned}\right\} \quad (5.2)$$

the six boundary conditions, chosen from (3.19),

$$\left.\begin{aligned} &\hat{v}(0, t) = 0; \qquad \hat{v}_{,xx}(L, t) = 0 \\ &\hat{v}_{,x}(0, t) = 0; \qquad \left[(2EI\hat{v}_{,xx})_{,x} + (\hat{N} - \varkappa)\,\hat{v}_{,x} + 2\varkappa\frac{\tilde{u}}{h}\right]_{L,t} = 0 \\ &\tilde{u}(0, t) = 0; \qquad \tilde{u}_{,x}(L, t) = 0 \end{aligned}\right\} \quad (5.3)$$

and the two initial conditions

$$\left.\begin{aligned} \hat{v}(x, 0) &= f(x) \\ \hat{v}_{,t}(x, 0) &= g(x). \end{aligned}\right\} \quad (5.4)$$

Note that since $\hat{N}(x)$ was determined explicitly and is given in Eq. (5.1), the obtained formulation for $\hat{v}$ and $\hat{u}$ is linear.

When the cross-beams (floors) are "rigidly" connected to the columns (for example, by welding) then $s = \infty$ and, according to Eq. (1.2),

$$(\varkappa)_\infty = 12K^* = \frac{12E_0I_0}{ah}.$$

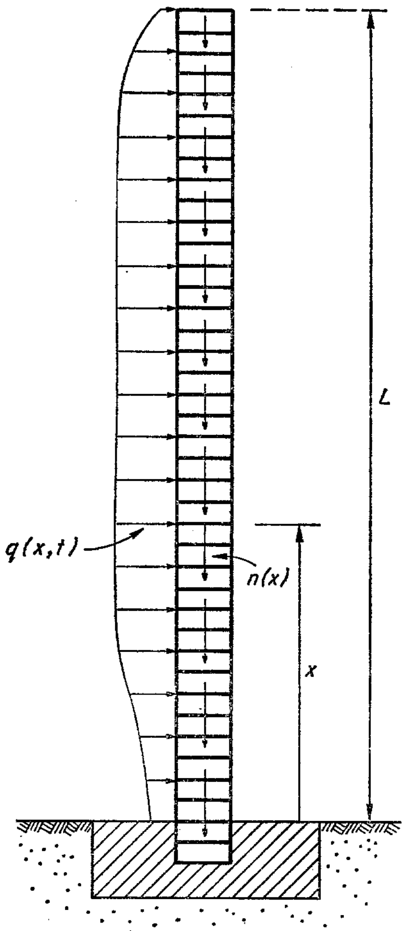

Fig. 6. Analytical model for a tall building

In the case of non-rigid elastic connections, the complete $\varkappa$ given in Eq. (1.2) should be used.

Next, consider a frame-type structure of constant geometric properties, clamped at both ends to "rigid" supports, and subjected to a distributed vertical load q, as shown in Fig. 7. At first, let us consider the general case when $q(x)$ is of such magnitude that it will generate, in addition to the $\tilde{N}$'s, also the axial force $\hat{N}$.

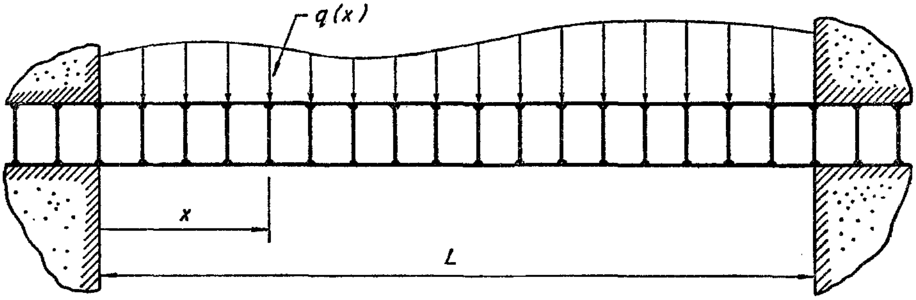

Fig. 7. Analytical model for a frame-type girder

The differential equations for this problem are obtained from (3.18) noting that $n \equiv 0$, $m\hat{v}_{,tt} \equiv 0$ and $m\hat{u}_{,tt} = 0$. They are:

$$\left.\begin{aligned} &2EI\hat{v}_{,xxxx} - 2EA[(\hat{u}_{,x} + \hat{v}_{,x}^2/2)\,\hat{v}_{,x}]_{,x} - \varkappa(\hat{v}_{,x} - 2\tilde{u}/h)_{,x} = q \\ &EA\tilde{u}_{,xx} + \varkappa(\hat{v}_{,x} - 2\tilde{u}/h)/h = 0 \\ &2EA(\hat{u}_{,x} + \hat{v}_{,x}^2/2)_{,x} = 0. \end{aligned}\right\} \tag{5.6}$$

According to (3.19), the necessary boundary conditions are

$$\left.\begin{aligned} &\hat{v}(0) = 0; &&\hat{v}(L) = 0 \\ &\hat{v}_{,x}(0) = 0; &&\hat{v}_{,x}(L) = 0 \\ &\tilde{u}(0) = 0; &&\tilde{u}(L) = 0 \\ &\hat{u}(0) = 0; &&\hat{u}(L) = 0. \end{aligned}\right\} \tag{5.7}$$

The resulting formulation is non-linear. However, because of the structure of the differential equations they may be easily solved.

If it is assumed, a priori, that the $\hat{N}$ generated by $q(x)$ is negligible, then this linearizes the formulation, because then in (5.6) $\hat{v}_{,x}^2 \ll \hat{u}$ and is neglected as small of higher order. This also uncouples the third equation in (5.6). The problem is thus reduced to the solution of the first two simultaneous differential equations with constant coefficients for $\hat{v}(x)$ and $\tilde{u}(x)$, and the uncoupled third equation for $\hat{u}(x)$.

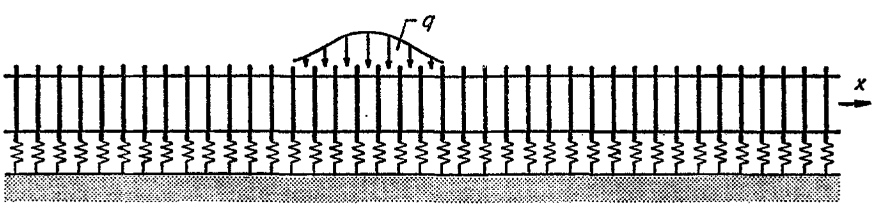

Fig. 8. Analytical model for the lateral response of a railroad track

The generalized equations, (3.18) to (3.20), may also be used for the analysis of railroad tracks subjected to lateral mechanical forces, and for the analysis of thermal track buckling in the lateral plane [7], as shown in Fig. 8.

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