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Non-Equilibrium Potentials and Stationary Probability Distributions of Some Dissipative Models Without Manifest Detailed Balance

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Practical methods are developed to analyze the steady state probability distributions of dynamical Fokker-Planck models without manifest detailed balance in the limit of small noise. We consider two different models, whose deterministic attractor is a fixed point at the origin, and construct the leading terms of systematic expansions in various small parameters of the "non-equilibrium potential" and the stationary probability distribution of the models.

1. Introduction

The dynamical behaviour of macroscopic systems can often be successfully modelled by a Fokker-Planck process [1, 2, 3]. Such a description is useful for systems in thermodynamic equilibrium, where the dynamics is constrained by the requirement of detailed balance [4] against the microscopically defined transformation of time reversal [2]. The Fokker-Planck description is also useful in non-equilibrium systems where detailed balance, in general, only holds with respect to some more complicated transformation of time reversal, which does not restrict the form of the dynamics [5] but usually is not known explicitely. Sometimes, even in non-equilibrium systems the explicit form of the time reversal transformation, with respect to which the dynamics is in detailed balance, is apparent from the Fokker-Planck equation of the system [4], which is the situation that has been called "manifest detailed balance" in an earlier paper by one of us [5]. However, usually a non-equilibrium system does not have this special property, and its detailed balance is "hidden", since the associated time reversal transformation is not known explicitely. In this case, it is in general not possible to solve the Fokker-Planck equation exactly, even for the time independent probability distribution. The development of approximation methods for this case is therefore of great practical interest.

An effective numerical method developed recently makes use of matrix continued fractions to solve the Fokker-Planck equation using a convenient basis in the function space [6]. However, this method - apart from giving little analytical insight - is not well suited for the case of weak noise, which is often the relevant limit in practical applications, due to the macroscopic nature of the systems considered.

In the present paper, we therefore present an approximate but systematic analytical method for finding the stationary distribution of a Fokker-Planck process with hidden detailed balance in the limit of weak noise. For clarity the method is presented in the framework of two elementary examples of dissipative systems having a single attracting fixed point in the deterministic limit. An application of our method to optical bistability has already been given in [7, 8]. An extension of the method to dissipative systems with limit cycles [9] and an application to mode-locking in multi-mode lasers [10] will be published elsewhere.

The remainder of the paper is organized as follows. In Sect. 2 the general method is explained and the basic equations are derived. In Sects. 3 and 4 we derive approximate expressions for the non-equilibrium potentials of the two specific examples. In Sect. 5 we obtain approximate analytical results for the probability distribution in the limit of weak noise. Section 6 contains the discussion of our results and our final conclusions.

2. The Method

We start from a general Fokker-Planck model. For simplicity we assume that the diffusion matrix of the model is constant

$$\frac{\partial}{\partial t} W(q,t) = \left[-\frac{\partial}{\partial q^\nu} K^\nu(q) + \frac{\varepsilon}{2} \frac{\partial^2}{\partial q^\nu \partial q^\mu} Q^{\nu\mu} \right] W(q,t). \tag{2.1}$$

The summation convention is implied. $q=\{q^\nu\}$, $(\nu=1,2,\ldots,n)$ are the variables of the model, $K^\nu(q)$ is the deterministic drift, $\varepsilon Q^{\nu\mu}$ is the symmetric diffusion matrix, $W(q,t)$ is the probability density of the variables q at time t. We assume natural boundary conditions for $W(q,t)$ at infinity. We are interested in the time independent solution $W_\infty(q)$ of (2.1) for the case that ε is a small parameter, and assume that the following expansion in ε exists.

$$W_\infty(q) = N \cdot \exp\left[-\left(\frac{1}{\varepsilon} \Phi_0(q) + \Phi_1(q) + O(\varepsilon) \right) \right]. \tag{2.2}$$

An analysis of this assumption is given elsewhere [11].

Introducing (2.2) into (2.1) and collecting terms of equal orders in ε we obtain in order ε^{-1}:

$$K^\nu(q) \frac{\partial \Phi_0}{\partial q^\nu} + \frac{1}{2} Q^{\nu\mu} \frac{\partial \Phi_0}{\partial q^\nu} \frac{\partial \Phi_0}{\partial q^\mu} = 0 \tag{2.3}$$

while the terms in order ε^0 lead to

$$\left(K^\nu(q) + Q^{\nu\mu} \frac{\partial \Phi_0}{\partial q^\mu} \right) \frac{\partial \Phi_1}{\partial q^\nu} = \frac{\partial}{\partial q^\nu} \left(K^\nu(q) + \frac{1}{2} Q^{\nu\mu} \frac{\partial \Phi_0}{\partial q^\mu} \right). \tag{2.4}$$

In the limit of weak noise it is sufficient to solve (2.3), (2.4) which are differential equations of first order in the variables q. Because the order of the differential equation (2.1) has been lowered by one there may occur problems in (2.3) and (2.4) when we try to impose the boundary conditions, which have to be satisfied by Φ_0 and Φ_1 in order to qualify as solutions of (2.1) via (2.2). In the examples we will discuss below, these problems can be overcome in a rather simple way. However, in general one must be aware of the fact that the expansion in ε in (2.2) is a singular perturbation theory and it may be necessary to use boundary layer methods in order to satisfy all boundary conditions for $W_\infty(q)$.

The analytical solution of equation (2.3) which is of the form of a Hamilton Jacobi equation, in general, is only possible if the equations contain an additional small parameter in which the solution can be expanded. This small parameter then can also be used for solving (2.4). This completes the outline of the basic method, which is exemplified for two special cases in the following sections. The solution of (2.3) is the non-equilibrium potential of the system.

The deterministic dynamics

$$\dot{q}^\nu = K^\nu(q) \tag{2.5}$$

is the superposition of a gradient flow of this potential

$$-\frac{1}{2} Q^{\nu\mu} \frac{\partial \Phi_0}{\partial q^\mu}$$

and a flow $r^\nu(q)$ on equi-potential surfaces

$$K^\nu(q) = -\frac{1}{2} Q^{\nu\mu} \frac{\partial \Phi_0}{\partial q^\mu} + r^\nu(q) \tag{2.6}$$

$$r^\nu(q) \cdot \frac{\partial \Phi_0}{\partial q^\nu} = 0. \tag{2.7}$$

Equations (2.6) and (2.7), clearly, are equivalent to (2.3). The non-equilibrium potential can not increase in time along the deterministic trajectories.

$$\frac{d\Phi_0}{dt} = -\frac{1}{2} Q^{\nu\mu} \cdot \frac{\partial \Phi_0}{\partial q^\nu} \cdot \frac{\partial \Phi_0}{\partial q^\mu} \leqq 0 \tag{2.8}$$

and gives the leading singular term of the probability distribution $W_\infty(q)$ in the weak noise limit, according to (2.2). However, the next order correction Φ_1 which solves (2.4) is also needed in order to obtain a complete expression for $W_\infty(q)$ in the weak noise limit, because only the higher order contributions become negligibly small when the strength of the noise approaches zero.

3. Non-Equilibrium Potential of the First Model

The first model we want to discuss is defined on the two-dimensional phase space of the variables x, y with the drift

$$K_x = -x + \delta y^3$$
$$K_y = -y - \delta x^3 \tag{3.1}$$

and the diagonal constant diffusion matrix

$$\mathbf{Q} = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}. \tag{3.2}$$

The deterministic drift contains an attracting fixed point – a node – at the origin with the entire (x,y)

plane as its stable manifold. In the infinitesimal neighbourhood of the origin the deterministic dynamics is on straight lines towards the origin. At finite distances there is also a non-linear part to the dynamics which encircles the origin. The model defined by (3.1) and (3.2) does not have manifest detailed balance. The exact steady state distribution of its Fokker-Planck equation is not known. However, by a simple rescaling of the variables of the model it is easily shown that $W_\infty(x,y)$ is a function of the three quantities $Q\delta$, $x\sqrt{\delta}$, $y\sqrt{\delta}$ only. For small Q the method of Sect. 2 is applicable. The formal small parameter ε of (2.1) is put equal to one, for convenience. Equation (2.3) for this case reads

$$(-x+\delta y^3)\frac{\partial \Phi_0}{\partial x}+(-y-\delta x^3)\frac{\partial \Phi_0}{\partial y}+\frac{1}{2}Q\left(\left(\frac{\partial \Phi_0}{\partial x}\right)^2+\left(\frac{\partial \Phi_0}{\partial y}\right)^2\right)=0. \tag{3.3}$$

We will now construct approximate solutions of (3.3) for two cases:

i) Expansion in δ

This expansion is applicable if the frequency δ can be assumed to be small. Since δ measures the strength of the nonlinearity, the expansion represents Φ_0 only for small x, y

$$\Phi_0=\sum_{n=0}^{\infty}\delta^n\Phi_{0n}, \quad (x^2, y^2 \ll \delta^{-1}). \tag{3.4}$$

For $\delta=0$ we obtain from (3.3) the solution

$$\Phi_{00}(x,y)=\frac{1}{Q}(x^2+y^2). \tag{3.5}$$

We note that for $\delta=0$ the full Fokker-Planck equation of the model has manifest detailed balance, and its exact solution is given by (2.2) with $\Phi_0=\Phi_{00}$ from (3.5), Φ_1 and all corrections of higher order in ε vanish. This remark will be useful for the construction of $\Phi_1(q)$ for $\delta \neq 0$ in Sect. 5.

We proceed to calculate the first order correction of Φ_0 in δ, which satisfies

$$x\frac{\partial \Phi_{01}}{\partial x}+y\frac{\partial \Phi_{01}}{\partial y}=\frac{2}{Q}xy(x^2-y^2). \tag{3.6}$$

Equation (3.6) can be solved by the methods of characteristics and we obtain the general solution

$$\Phi_{01}(x,y)=-\frac{\delta xy}{2Q}(x^2-y^2)+F\left(\frac{y}{x}\right) \tag{3.7}$$

where $F(u)$ is an arbitrary function of u. Φ_{01} has the correct symmetry if $F(u)=F(-1/u)$. In order to determine $F(y/x)$ we note that it must be proportional to $1/Q$ and to δ in order to contribute in (3.7) in the correct orders of the small parameters. Hence

$$F(x,y)=\frac{\delta}{Q}f\left(\frac{y}{x}\right).$$

However, this form of $F(y/x)$ is not compatible with the general scaling property $W_\infty(x,y,Q)=G(\sqrt{\delta}x, \sqrt{\delta}y, \delta Q)$ of the model. Therefore we have to take $F(y/x)$ equal to zero. The results in first order in δ then reads

$$\Phi_0(x,y)=\frac{1}{Q}\left[x^2+y^2+\frac{\delta}{2}xy(x^2-y^2)\right]. \tag{3.8}$$

The calculation can easily be carried on to higher orders in δ, but no new points of principle arise. We also note that (3.8) is, indeed, the beginning of an expansion in powers of x, y, as expected, and the result is reliable only in regions of phase space where the term proportional to $\sim\delta$ is a small correction to the leading term. Equation (3.8) makes explicit the destruction of rotational invariance of Φ_0 by the nonlinear parts of the drift in (3.1).

ii) Expansion in $1/\delta$

This expansion is complementary to the δ-expansion and provides information for large x, y; $(x^2, y^2 \gg \delta^{-1})$. Putting

$$\Phi_0=\sum_{n=0}^{\infty}\Phi_{0n}\left(\frac{1}{\delta}\right)^n \tag{3.9}$$

we obtain in lowest order

$$\delta y^3\frac{\partial \Phi_{00}}{\partial x}-\delta x^3\frac{\partial \Phi_{00}}{\partial y}=0 \tag{3.10}$$

with the general solution

$$\Phi_{00}(x,y)=F(x^4+y^4) \tag{3.11}$$

where F remains an arbitrary function. In order to determine F uniquely we have to proceed to the next order in $1/\delta$ and have to consider

$$y^3\frac{\partial \Phi_{01}}{\partial x}-x^3\frac{\partial \Phi_{01}}{\partial y}=4(x^4+y^4)\cdot F'(x^4+y^4)-8Q(x^6+y^6)(F'(x^4+y^4))^2 \tag{3.12}$$

where F' is the first derivative of F. A solution for Φ_{01} exists if and only if the right hand side of (3.12) is orthogonal on all solutions of the associated adjoint homogeneous problem

$$\left(-y^3\frac{\partial}{\partial x}+x^3\frac{\partial}{\partial y}\right)G(x,y)=0 \tag{3.13}$$

which is solved again by the arbitrary function

$$G(x,y)=G(x^4+y^4). \tag{3.14}$$

The orthogonality condition then reads explicitly

$$\iint dx\,dy\,G(x^4+y^4)\cdot F'(x^4+y^4)\cdot[4(x^2+y^2)-8Q(x^6+y^6)\cdot F'(x^4+y^4)]. \tag{3.15}$$

Changing the variables of integration into x, $u=\sqrt[4]{x^4+y^4}$ and noting that the integration over u can be dropped because $G(u^4)$ is arbitrary, we obtain a condition of the form

$$\int_0^4 dx(u^4-x^4)^{-\frac{3}{4}}\cdot(u^4-2Qx^6F'(u^4)-2Q(u^4-x^4)^{\frac{3}{2}}\cdot F'(u^4)) \tag{3.16}$$

which can be reduced to

$$F'(u^4)=\frac{h(u^4)}{g(u^4)} \tag{3.17}$$

with

$$h(u^4)=u^2\int_0^1 d\xi\frac{1}{(1-\xi^4)^{\frac{3}{4}}}\equiv\beta u^2 \tag{3.18}$$

and

$$g(u^4)=2Qu^4\int_0^1 d\xi\frac{\xi^6+(1-\xi^4)^{\frac{3}{2}}}{(1-\xi^4)^{\frac{3}{4}}}\equiv 2Q\alpha u^4.$$

We finally obtain

$$F(x^4+y^4)=\frac{\beta}{\alpha Q}(x^4+y^4)^{\frac{1}{2}}. \tag{3.19}$$

The numerical constants α, β are defined by the definite integrals of (3.18). It is now possible to continue and solve (3.12) explicitly for Φ_{01}. This function again is determined only up to an arbitrary function of the form $F_1(x^4+y^4)$ which has to be fixed by an equivalent orthogonality relation, arising in second order in $1/\delta$. In this way the expansion can in principle be carried to increasingly higher orders in $1/\delta$.

We remark, parenthetically, that an orthogonality condition analogous to (3.15) must also be satisfied for (3.6) where the function G is of the form $G(x,y)=1/x^2\,\tilde{G}(y/x)$ with $\tilde{G}$ arbitrary. It turns out that this orthogonality condition is automatically satisfied for (3.6) due to symmetry, which is the reason why the solution (3.6) could be found without explicitly imposing the orthogonality condition. Symmetry also guarantees that the solubility conditions are satisfied in higher orders; it is therefore consistent to determine the function F in (3.7) by an independent argument.

It is interesting to note that our results for the non-equilibrium potential to lowest order in $1/\delta$ has the symmetry

$$\Phi_0(x,y)=\frac{\beta}{\alpha Q}(x^4+y^4)^{\frac{1}{2}}+O(1/\delta) \tag{3.20}$$

imposed by the form of the reversible part of the drift (3.1) but grows only like x^2, y^2. The growth of Φ_0 at large amplitudes, however, is not at all determined by the reversible part of the drift – which in this approximation is a drift on equipotential surfaces – but by the linear irreversible part alone. Thus, equation (3.20) may be understood as arising from an optimized compromise between nonlinear drift, which imposes the symmetry, and the linear drift, which imposes the overall growth rate. We also note that the result (3.20) ensures the normalizability of the probability density (2.2) at large amplitudes for $\varepsilon\to 0$.

4. Non-Equilibrium Potential of the Second Model

Some additional points arise from the following model

$$\begin{aligned} K_x &= -x^3+\delta y\\ K_y &= -y^3-\delta x \end{aligned} \tag{4.1}$$

and

$$\mathbf{Q}=Q\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{4.2}$$

The origin of the (x,y) plane is still the only attractor of the drift (4.1), but the attractor arises only from the nonlinear terms, while the linear part of the drift describes a conservative flow which encircles the origin. Simple scaling arguments show that the steady state distribution can only depend on $x/\sqrt{\delta}$, $y/\sqrt{\delta}$, Q/δ^2. Again, we consider the solution of

$$(\delta y-x^3)\frac{\partial\Phi_0}{\partial x}+(-\partial x-y^3)\frac{\partial\Phi_0}{\partial y}+\frac{Q}{2}\left(\left(\frac{\partial\Phi_0}{\partial x}\right)^2+\left(\frac{\partial\Phi_0}{\partial y}\right)^2\right)=0 \tag{4.3}$$

as an expansion in powers of δ and δ^{-1}.

i) Expansion in δ

This expansion is justified only for large x, y; $(x^2, y^2\gg\delta)$. For $\delta=0$, the model (4.1), (4.2) has manifest detailed balance and we obtain in the notation used in the previous chapter

$$\Phi_{00}=\frac{1}{2Q}(x^4+y^4). \tag{4.4}$$

The correction to first order in δ is easily obtained along the lines of the first expansion in the preceding section and we obtain

$$\Phi_{01}=-\frac{2}{Q}\delta x y\frac{x^2+y^2}{x^2-y^2}+F\left(\frac{x^2y^2}{x^2-y^2}\right) \tag{4.5}$$

where the function F remains unspecified. We now try to determine F in such a way that Φ_{01} remains finite in the limit $x^2\to y^2$ and is invariant under the symmetry transformation $x\to -x$, $y\to -y$ and $x\to y$, $y\to -x$. We first consider the solution (4.5) in the region $x>0$, $y>0$ and use

$$x^2-y^2=z. \tag{4.6}$$

Then we obtain

$$\Phi_{01}=F\left(\frac{x^4-zx^2}{z}\right)-\frac{2\delta}{Q}\frac{x^4}{z}\sqrt{1-\frac{z}{x^2}}\left(2-\frac{z}{x^2}\right). \tag{4.7}$$

In order to avoid a pole for $z=0$ we must put F into the form

$$F(u)=\frac{4\delta}{Q}u+\tilde{F}(u) \tag{4.8}$$

where $\tilde{F}(u)$ is a function which is assumed finite for all u including $u\to\infty$. From (4.8) we obtain

$$\Phi_{01}=-\frac{2\delta}{Q}x\cdot y(x-y)+\tilde{F}\left(\frac{x^2y^2}{x^2-y^2}\right). \tag{4.9}$$

In order to determine $\tilde{F}$ uniquely we first note that it must be proportional to δ/Q in order to contribute to (4.10). The scaling property of $W_\infty(x,y)$ then implies

$$\tilde{F}\left(\frac{x^2y^2}{x^2-y^2}\right)=\lambda\frac{\delta}{Q}\frac{x^2y^2}{x^2-y^2}$$

where λ is a numerical constant. However, for $\lambda\neq 0$ this form of $F(u)$ is not compatible with the requirement that it remains finite for $u\to\infty$. Hence we set $\lambda=0$.

The same analysis can be carried through in the regions $x<0$, $y>0$; $x<0$, $y<0$; and $x>0$, $y<0$ with the final result

$$\Phi_{01}=-\frac{2\delta}{Q}xy\frac{|x|-|y|}{|x|+|y|}. \tag{4.10}$$

We note that (4.10) is free from singularities and has the correct symmetries $x\to -x$, $y\to -y$ and $x\to y$, $y\to -x$.

ii) Expansion in δ^{-1}

This expansion is adequate for small x,y; $x^2,y^2\ll\delta$. The lowest order in δ^{-1} the function Φ_{00}, in the notation of (3.9) is obtained as

$$\Phi_{00}(x,y)=F(x^2+y^2) \tag{4.11}$$

where the function F is still arbitrary. The solubility condition of the corrections in next order in δ^{-1} is obtained along the lines of the previous example, (3.12)–(3.17) and determines the functional dependence of F in the form

$$F'(u)=\frac{h(u)}{g(u)} \tag{4.12}$$

with

$$g(u)=u\cdot Q\int_0^1 d\xi\frac{1}{\sqrt{1-\xi^2}}=\frac{\pi}{2}Qu$$
$$h(u)=2\cdot u^2\int_0^1 d\xi\frac{(\xi^4-\xi^2+\frac{1}{2})}{\sqrt{1-\xi^2}}=\frac{3}{8}\pi u^2. \tag{4.13}$$

With these results, the non-equilibrium potential in lowest order reads

$$\Phi_{00}(x,y)=\frac{3}{8Q}(x^2+y^2)^2. \tag{4.14}$$

Again, the result appears as an optimized compromise between the conservative part of the drift which determines the symmetry of (4.4), and the non-linear, dissipative part which determines the extension of the probability density in phase space. The expansion, in principle, can be carried to any desired order in δ^{-1}.

5. Steady State Probability Density for Small Noise

For the two models we have considered we now want to determine the probability density in the steady state, by extending our results for the non-equilibrium potential to next order in Q. We consider the various cases in succession.

i) First Model in δ-Expansion

We expand Φ_1 in powers of δ:

$$\Phi_1=\Phi_{10}+\delta\Phi_{11}+\dots \tag{5.1}$$

and note that

$$\Phi_{10}=0 \tag{5.2}$$

since the model (3.1), (3.2) for $\delta=0$ has manifest detailed balance. In first order in δ it is therefore sufficient to replace Φ_0 by Φ_{00} on the left hand side of (2.4). The inhomogeneity on the right hand side of (2.4) turns out to vanish to first order in δ. We then obtain

$$x\frac{\partial \Phi_{11}}{\partial x}+y\frac{\partial \Phi_{11}}{\partial y}=0 \tag{5.3}$$

with the general solution

$$\Phi_{11}(x,y)=F_1(y/x). \tag{5.4}$$

As in the previous solution for Φ_0 we must take $F(y/x)$ equal to zero, since, in order to contribute to (5.1) it should be proportional to δ, which violates the scaling property of $W_\infty(x,y)$. Therefore the probability density (2.2) to first order in δ is given by

$$W_\infty(x,y)=N\cdot\exp\left[-\frac{1}{Q}\left(x^2+y^2+\frac{\delta}{2}xy(x^2-y^2)+O\left(\frac{\delta^2}{Q},Q\right)\right)\right]. \tag{5.5}$$

It should be recalled that this result is only valid in the region $x^2, y^2 \ll 1/\delta$. In fact, (5.5) cannot be correct for large x, y since $W_\infty(x,y)$ can not be normalized due to the divergence at infinity. In this region the $1/\delta$-expansion can characterize the probability density.

ii) First Model in the $1/\delta$-Expansion

The $1/\delta$-expansion of Φ_0 and Φ_1 is easily introduced in (2.4). To lowest order in $1/\delta$ we obtain

$$\Phi_{10}=F_1(x^4+y^4). \tag{5.6}$$

The real problem is to determine F_1 from the solubility condition for Φ_{11}, which satisfies

$$y^3\frac{\partial \Phi_{11}}{\partial x}-x^3\frac{\partial \Phi_{11}}{\partial y}=\left[4(x^4+y^4)-2\frac{\beta}{\alpha}\frac{x^6+y^6}{\sqrt{x^4+y^4}}\right]F_1' -2+\frac{\beta}{\alpha}\left(3\frac{x^2+y^2}{\sqrt{x^4+y^4}}-2\frac{x^6+y^6}{(x^4+y^4)^{\frac{3}{2}}}\right). \tag{5.7}$$

We now proceed in the same way as in the section following (3.12) and obtain

$$F_{10}'(u^4)=\frac{1}{u^4}\cdot\frac{3\gamma/4-2(\alpha+\beta)}{\beta} \tag{5.8}$$

where α and β are defined in (3.17), (3.18) and γ is defined by

$$\gamma=\int_0^1 dx\,\frac{x^2+(1-x^4)^{\frac{1}{2}}}{(1-x^4)^{\frac{3}{4}}}. \tag{5.9}$$

From (5.8) we obtain

$$F_{10}'(u^4)=\frac{1}{\beta}\left(\tfrac{3}{4}\gamma-2(\alpha+\beta)\right)\cdot\ln u^4. \tag{5.10}$$

The probability density (2.2) to lowest order in δ^{-1} is therefore given by

$$W_\infty(x,y)=N\cdot(x^4+y^4)^{2+\frac{8\alpha-3\gamma}{4\beta}}\cdot\exp\left(-\frac{\beta}{\alpha Q}\sqrt{x^4+y^4}+O\left(\frac{1}{Q\delta},Q\right)\right). \tag{5.11}$$

This expression is complementary to our result (5.5) since it is valid only for large x, y. It shows that the normalization integral converges for large x, y.

iii) Second Model in δ-Expansion

We expand Φ_1 in the form (5.1). Since for $\delta=0$ the model (4.1), (4.2) has manifest detailed balance, we obtain

$$\Phi_{10}(x,y)=0. \tag{5.12}$$

In first order in δ we have to solve the following equation

$$x^3\frac{\partial \Phi_{11}}{\partial x}+y^3\frac{\partial \Phi_{11}}{\partial y}=6\delta\frac{x^3-y^3}{(x+y)^3} \tag{5.13}$$

for $x>0$, $y>0$. The form of the corresponding equations in the other quadrants of the (x,y)-plane follows from the repeated application of the symmetry transformation $x\to y$, $y\to -x$ to (5.13). The general solution of (5.13) for $x>0$, $y>0$ is given by

$$\Phi_{11}(x,y)=3\delta\frac{x^2-y^2}{x^2y^2}\,\Phi\left(\frac{x^2}{y^2}\right)+F\left(\frac{x^2-y^2}{x^2y^2}\right) \tag{5.14}$$

with

$$\Phi(z)=\int_0^z du(u^{\frac{3}{2}}-1)(u^{\frac{1}{2}}-1)^3(u-1)^{-5} \tag{5.15}$$

and arbitrary F. Note the symmetry $\Phi(z)=\Phi(1/z)$ which follows from (5.15). The integral defining $\Phi(z)$ is elementary but lengthy, and is not written out explicitly. We determine the function F by compensating the singularity of Φ_{11} at $x^2-y^2\to 0$. For

$|x^2-y^2| \ll x^2$ we obtain from (5.14), (5.15)

$$\Phi_{11}(x,y)=\frac{9}{16}\delta\frac{x^2-y^2}{x^4}\left(\ln\left|\frac{x^2-y^2}{x^4}\right|+\ln x^2\right) +F\left(\frac{x^2-y^2}{x^4}\right). \tag{5.16}$$

The logarithmic term in Φ_{11} leads to a logarithmic divergence of the first derivatives of Φ_{11} at $x^2=y^2$ which we consider as unphysical. It can be compensated by an appropriate choice of F

$$F(u)=-\tfrac{9}{16}\delta\cdot u\cdot\ln|u|+\tilde{F}(u) \tag{5.17}$$

where $\tilde{F}(u)$ is differentiable at $u=0$. Hence, we obtain

$$\Phi_{11}(x,y)=3\delta\frac{x^2-y^2}{x^2y^2} \cdot\left(\Phi\left(\frac{x^2}{y^2}\right)-\frac{3}{16}\ln\left|\frac{x^2-y^2}{x^2y^2}\right|\right)+\tilde{F}\left(\frac{x^2-y^2}{x^2y^2}\right). \tag{5.18}$$

In order to fix also $\tilde{F}$ we invoke the scaling properties of the steady state distribution function. The correct scaling of the logarithmic term in (5.18) requires that $\tilde{F}$ contains a term

$$-\frac{9}{16}\delta\frac{x^2-y^2}{x^2y^2}\ln\delta.$$

The appearance of such contributions shows that the expansion in δ contains logarithmic terms beyond the first order in δ and cannot be extended to higher orders without including such terms. In addition to the terms already mentioned F can contain also terms proportional to δ. Thus

$$\tilde{F}(u)=-\tfrac{9}{16}(\delta\ln\delta)\,u+\tfrac{3}{2}\lambda\cdot\delta\cdot u \tag{5.19}$$

where λ is a numerical constant, which must be determined by the solubility condition which arises in second order in δ. Leaving λ undetermined we write our final result in order δ in symmetric form as:

$$W_\infty(x,y)=N\cdot\exp\left\{-\frac{1}{2Q}\left(x^4+y^4-4\delta xy\frac{|x|-|y|}{|x|+|y|}\right)\right. -\frac{3}{2}\delta\frac{x^2-y^2}{xy|xy|}\left(\Phi\left(\frac{x^2}{y^2}\right)+\Phi\left(\frac{y^2}{x^2}\right)\right. \left.\left.-\frac{3}{8}\ln\left|\delta\frac{x^2-y^2}{x^2y^2}\right|+\lambda\right)\right\}. \tag{5.20}$$

iv) Second Model in the δ^{-1}-Expansion

Expanding Φ_1 in δ^{-1} we find in lowest order in δ^{-1}:

$$\Phi_{10}(x,y)=F_1(x^2+y^2) \tag{5.21}$$

with arbitrary F_1. In first order in δ^{-1} we must solve

$$y\frac{\partial\Phi_{11}}{\partial x}-x\frac{\partial\Phi_{11}}{\partial y}=2\cdot F'\left(x^4+y^4+Q\left(x\frac{\partial\Phi_0}{\partial x}+y\frac{\partial\Phi_0}{\partial y}\right)\right) -3(x^2+y^2)+\frac{Q}{2}\left(\frac{\partial^2\Phi_0}{\partial x^2}+\frac{\partial^2\Phi_0}{\partial y^2}\right). \tag{5.22}$$

The solubility condition requires that the right hand side is orthogonal on an arbitrary function of (x^2+y^2), when integrated over x, y. Introducing $u=x^2+y^2$ as a new variable of integration we find the condition

$$\int_0^{\sqrt{u}} dx(4x^4-4ux^2-u^2)\frac{1}{\sqrt{u-x^2}}\cdot F_1'(u)=0 \tag{5.23}$$

with the solution

$$F_1(u)=\text{const.} \tag{5.24}$$

The probability density to lowest order in δ^{-1} is therefore given by

$$W_\infty(x,y)=N\cdot\exp\left(-\frac{3}{8Q}(x^2+y^2)^2+O\left(\frac{1}{Q\delta},\frac{1}{\delta},Q\right)\right). \tag{5.25}$$

The results (5.18) and (5.25) are complementary in the sense that they apply to large and to small values of x, y respectively. In our results we explicitly see the change of symmetry of the distribution from small values of x, y to large ones. It is caused by the fact that the conservative dynamics with rotational symmetry dominates for small amplitudes, while the dissipation with a different symmetry dominates at large amplitudes.

6. Conclusions

We have presented in this paper a practical scheme, which allows us to compute the steady state distribution function of a Fokker-Planck process in the absence of manifest detailed balance. This method has been derived under the assumption of weak noise and the existence of an additional small parameter in which the solution can be expanded. The latter conditions of course, impose limits on the applicability of our method, but the weak noise limit is often realized in physical systems and additional small parameters can be found by considering limiting cases.

Let us briefly compare the advantages and disadvantages of a number of available alternative, analytical methods with the method we have presented here.

i) If a small parameter is available besides the noise intensity one may try to solve directly the Fokker-Planck equation by perturbative methods. The advantage would be that the weak noise limit is not required. However, the disadvantage of this more ambitious procedure is that closed form expressions can rarely be obtained even in first order. Rather, the perturbation theory of the Fokker-Planck equation requires the full knowledge of the eigenvalues and of the left and right eigenvectors of the unperturbed Fokker-Planck operator. In addition, the matrixelements of the perturbation must be calculated, and infinite sums over the spectrum of the unperturbed operator have to be evaluated. For this reason, the weak noise limit provides a very essential simplification if the physical problem allows it.

ii) The method we described combines features of perturbation theory and WKB-methods familiar from the solution of Schrödinger equations, which are separable into problems with a single degree of freedom. Of course, a full-fledged WKB approach would be preferable to our procedure, since an additional small parameter besides the weak noise would not be required. However, such an approach is severely hampered by the fact that the equations we want to solve are not separable, and it seems that practicable and effective WKB-methods for such cases are not available. Therefore this ambitious program, in practice, could hardly succeed without another small parameter at hand.

iii) In the weak noise limit, the steady state distribution is dominated by the attractors of the system in the deterministic limit. A very reasonable approximation in this case is the linearisation around the attractors, which reduces the problem locally to a linear Gaussian process. Approximations of this type, also for the time dependent Fokker-Planck equation, have been advocated by van Kampen [3]. Our remarks, however, are only concerned with the steady state distribution. Indeed, if this procedure works, which is not always the case, however, it corresponds to the lowest order in the expansion we presented in Sects. 3–5. In particular in the first model the δ-expansion was of this type. However, in the second model the full process cannot be reduced to a linear Gaussian process in the vicinity of the attractor, and the linearisation procedure does not work, whereas our procedure still can be applied.
There are other disadvantages of the linearisation method which are avoided by our approach. If the system has several attractors it is very difficult to match the different linear Gaussian processes in their vicinity. Hence, in the linearisation method it is not known how to compare the relative probabilities of the different attractors in steady state. The method presented allows such a comparison to be made, provided a small parameter exists common to all attractors which are to be compared. An example of this kind has been discussed in great detail in [7] in the context of optical bistability. After emphasizing the advantages, we also want to mention a principle disadvantage of our method. It rests squarely on the assumption that an expansion of $W_\infty(q)$ of the form (2.2) exists, and that $\Phi_0(q)$, $\Phi_1(q)$ are single valued continuous and differentiable functions. However, in general, there may be also non-analytical terms in this expansion, and we must assume that they do not contribute, at least to order $(\varepsilon)^0$. This assumption conceivably may break down, in particular if the boundary conditions on the process (2.1) are more complicated than the natural boundary conditions assumed in this work. A similar problem is associated with the expansion in the second small parameter. In the δ-expansion of the second model we encountered an example where logarithmic terms in the small parameter appeared explicitly, when we went beyond first order. For these reasons the method, for all its advantages, has to be applied with care in order to yield meaningful results.

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