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SMALL CYCLE DOUBLE COVERS OF 4-CONNECTED PLANAR GRAPHS

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A *cycle double cover* of a graph, G, is a collection of cycles, $\mathscr{C}$, such that every edge of G lies in precisely two cycles of $\mathscr{C}$. The *Small Cycle Double Cover Conjecture*, proposed by J. A. Bondy, asserts that every simple bridgeless graph on n vertices has a cycle double cover with at most $n-1$ cycles, and is a strengthening of the well-known *Cycle Double Cover Conjecture*. In this paper, we prove Bondy's conjecture for 4-connected planar graphs.

1. Introduction

A *cycle double cover* (CDC) of a graph, G, is a collection of cycles $\mathscr{C}$, such that every edge of G lies in precisely two cycles of $\mathscr{C}$. An obvious necessary condition for a graph to have a CDC is that the graph be bridgeless. P. D. Seymour [9] conjectures that this condition is also sufficient.

Conjecture 1 (CDC conjecture). *Every bridgeless graph has a cycle double cover.*

Though Seymour is most often credited with making this conjecture, Szekeres [10] conjectures that every cubic bridgeless graph has a CDC, which turns out to be an equivalent conjecture.

Let G be a simple graph on n vertices. A CDC of G consisting of at most $n-1$ cycles is called a *small cycle double cover* (SCDC) of G. Notice that if G contains a vertex v of degree $n-1$, at least $n-1$ cycles are required in a CDC of G, in order to doubly cover the edges incident with v. This provides us with some motivation for the following strengthening of the CDC Conjecture, due to J. A. Bondy [1].

Conjecture 2 (SCDC Conjecture). *Every simple bridgeless graph has a small cycle double cover.*

We note that this conjecture is clearly false if not restricted to simple graphs, since any graph on n vertices, containing a vertex of degree at least n has no SCDC.

The class of bridgeless planar graphs is one for which the CDC Conjecture is easily verified: we simply take the collection of cycles which bound the faces of the graph. However, this approach does not always produce an SCDC. If G is a simple

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2-connected planar graph on n vertices containing more than $2n-3$ edges, then it follows from Euler's formula that G has at least n faces, and so the collection of facial cycles does not constitute an SCDC of G.

The SCDC Conjecture has been verified for certain classes of simple planar graphs. Bondy and Seyffarth (see [1]) have shown that maximal planar graphs, that is, simple planar graphs in which each face is a triangle, have SCDC's; in fact, this result extends to triangulations of arbitrary surfaces. Simple planar graphs on n vertices, containing a vertex of degree $n-1$ can also be easily shown to have SCDC's [7].

In this paper, we present a proof of the SCDC Conjecture for simple 4-connected planar graphs. There are two main results about planar graphs on which our proof relies. The first of these is the fact that any 4-connected planar graph has a Hamilton cycle, a result due to Tutte [12]. The other fact we require concerns the partition of the edge set of an even graph into cycles.

2. Cycle decompositions and Hajós' Conjecture

Let G be a simple even graph; that is, a graph in which every vertex has even degree. It is well-known that the edge set of such a graph can be partitioned into cycles, and we call such a partition a *cycle decomposition* of G. We denote by $c(G)$ the minimum number of cycles required in a cycle decomposition of G. The following conjecture about the size of $c(G)$ was made by G. Hajós (see [6]).

Conjecture 3 (Hajós' Conjecture). *If G is a simple even graph on n vertices, then $c(G) \leq \lfloor (n-1)/2 \rfloor$.*

The bound in Hajós' Conjecture is actually $\lfloor n/2 \rfloor$ rather than $\lfloor (n-1)/2 \rfloor$, but Dean [2] has shown that the two conjectures are equivalent.

Hajós' Conjecture has been proved for graphs with maximum degree at most four by Granville and Moisiadis [4], and independently by Favaron and Koudier [3].

Suppose G is an even graph on n vertices and m edges, and suppose that the underlying simple graph of G contains m' edges. Then Hajós' Conjecture can be reformulated as follows: $c(G) \leq \lfloor (n+m-m'-1)/2 \rfloor$. To prove this, we first delete 2-cycles as long as the underlying simple graph is not affected. In the remaining graph, every edge has multiplicity at most two. We subdivide one of the edges in each pair of parallel edges with a new vertex, and apply Hajós' Conjecture to the resulting simple graph.

Of particular interest to us is the fact that Hajós' conjecture has been verified for planar graphs. This result is due to Tao [11]; an alternative proof is given in [8]. It plays an important role in our proof of the SCDC Conjecture for 4-connected planar graphs.

Theorem 1 (Tao). *If G is a simple planar even graph on n vertices, then $c(G) \leq \lfloor (n-1)/2 \rfloor$.*

3. The main theorem

Our proof of the SCDC Conjecture for 4-connected planar graphs makes use of the fact that any such graph contains a Hamilton cycle. Tutte proves this in [12], and in fact proves a stronger result.

Theorem 2 (Tutte). *Let G be a 4-connected planar graph, and let e and f be two edges of G that are incident with a common face. Then G has a Hamilton cycle containing both e and f.*

One additional lemma is needed for the proof of the SCDC Conjecture for 4-connected planar graphs. The proof is straight-forward and is left to the reader.

Lemma 3. *Let G be a bridgeless plane graph (that is, a planar graph together with a fixed embedding in the plane) and let*

$$f \ : \ F(G) \to \{1,2,3,4\}$$

be a proper 4-colouring of the set $F(G)$ of faces of G. For $2 \le j \le 4$, let G_{1j} denote the subgraph of G consisting of the edges incident with a face of colour 1 or a face of colour j, but not both. Then G_{1j} is an even subgraph of G, and furthermore, every edge of G lies in exactly two of G_{12}, G_{13}, G_{14}.

We are now ready to prove the main result of this paper.

Theorem 4. *Every simple 4-connected planar graph has a small cycle double cover.*

Proof. Let G be a simple 4-connected planar graph on n vertices, and assume that it is embedded in the plane. From Theorem 2, we know that G has a Hamilton cycle, H.

We 4-colour the faces of G by properly 2-colouring the faces in the interior of H with colours 1 and 2, and properly 2-colouring the faces in the exterior of H with colours 3 and 4. This gives us a proper colouring of the faces of G, and we call this colouring the *4-face colouring of G induced by H.* We now apply Lemma 3.

For $j=2,3,4$, it is easy to see that G_{1j} spans G, and so $v(G_{1j})=n$. Since each G_{1j} is even, its edge set can be partitioned into cycles, and since each edge of G lies in precisely two of G_{12}, G_{13}, G_{14}, the collection of cycles obtained by taking cycle decompositions of the G_{1j}'s constitutes a CDC of G. If q denotes the minimum number of cycles in a CDC obtained in this manner, then $q=c(G_{12})+c(G_{13})+c(G_{14})$, where we recall that $c(G_{1j})$ denotes the minimum number of cycles in a cycle decomposition of G_{1j}.

Clearly, $G_{12} = H$, so we have $c(G_{12}) = 1$. Also, by Theorem 1 $c(G_{13}) \le \lfloor (n-1)/2 \rfloor$ and $c(G_{14}) \le \lfloor (n-1)/2 \rfloor$. Therefore,

$$q \le 2\left\lfloor \frac{n-1}{2} \right\rfloor + 1 = \begin{cases} n-1 & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd.} \end{cases}$$

Thus if n is even, we have shown that G has as SCDC.

Now suppose that n is odd, $n = 2k+1$. Since G is 4-connected and planar, G has minimum degree four or five. Let v be a vertex of minimum degree in G, with neighbours $v_0, v_1, \ldots, v_l$ in cyclic order, where $l=3$ or $l=4$ according as G has minimum degree four or five. By Theorem 2, G has a Hamilton cycle, H, containing edges vv_0 and vv_1.

Let G' denote the graph obtained from G by duplicating edge vv_3; clearly, H is a Hamilton cycle in G'. We now consider the proper 4-face colouring of G' induced by H. If v has degree four in G, then, without loss of generality, the faces of G' incident with v are coloured as shown in Figure 1. If v has degree five in G, then, without loss of generality, the faces of G' incident with v are coloured as shown in Figure 2.

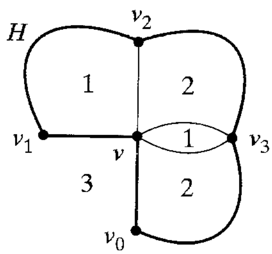

Fig. 1

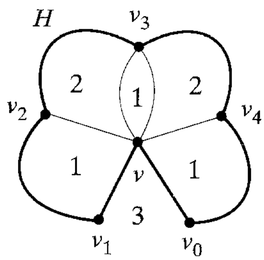

Fig. 2

We consider the even subgraphs G'_{13} and G'_{14} of G'. In both cases, v has degree four in G'_{13}, and the 2-cycle vv_3v is in both G'_{13} and G'_{14}. Let G''_{13} denote the graph obtained from G'_{13} by deleting the 2-cycle vv_3v. Since G is 4-connected, every triangle of G bounds a face. Therefore, in the case where v has degree four in G, v_0v_2 is not an edge of G, and hence not an edge of G'. Similarly, in the case where v has degree five in G, v_2v_4 is not an edge of G, and hence not an edge of G'. Thus, in both cases, v has degree two in G''_{13}, and the neighbours of v are non-adjacent in G''_{13}, so $c(G''_{13}) \leq \lfloor((n-1)-1)/2\rfloor = \lfloor n/2\rfloor - 1 = k-1$.

If m denotes the number of edges in G'_{14}, then the underlying simple graph of G'_{14} has $m-1$ edges. By first subdividing one of the edges in the 2-cycle vv_3v and then applying Theorem 1 we have $c(G'_{14}) \leq \lfloor((n+1)-1)/2\rfloor = \lfloor n/2\rfloor - 1 = k$.

Let $\mathbf{S}_1$ denote a partition of the edge set of G''_{13} into $c(G''_{13})$ cycles. Then $\mathbf{S}_1$ is a partition of the edges of $G_{13} - \{vv_3\}$ into cycles. Let $\mathbf{S}_2$ denote a partition of the edge set of G'_{14} into $c(G'_{14})$ cycles.

If the 2-cycle vv_3v is not a member of $\mathbf{S}_2$, then $\mathbf{S}_2$ gives a rise to a cycle cover, $\mathbf{S}'_2$, of G_{14}, where vv_3 is covered twice and all other edges are covered once. Thus $\mathscr{C} = \mathbf{S}_1 \cup \mathbf{S}'_2 \cup \{H\}$ is a CDC of G, and $|\mathscr{C}| = |\mathbf{S}_1| + |\mathbf{S}'_2| + 1 = c(G''_{13}) + c(G'_{14}) + 1 \leq (k-1)+k+1 = 2k = n-1$.

If the 2-cycle vv_3v is a member of $\mathbf{S}_2$, we let $\mathbf{S}'_2$ denote the collection of cycles obtained from $\mathbf{S}_2$ by deleting the 2-cycle vv_3v. Then $\mathbf{S}'_2$ is a partition of the edges of $G_{14} - \{v, v_3\}$ into cycles, and $|\mathbf{S}'_2| \leq k-1$. We now write $H = P_1 \cup P_2$, where $P_1 \cap P_2 = \{v, v_3\}$, and let $C_i = P_i \cup \{vv_3\}$, $i = 1,2$. Then $\mathscr{C} = \mathbf{S}_1 \cup \mathbf{S}'_2 \cup \{C_1, C_2\}$ is a CDC of G, and $|\mathscr{C}| = |\mathbf{S}_1| + |\mathbf{S}'_2| + 2 = c(G''_{13}) + (c(G'_{14}) - 1) + 2 \leq (k-1)+(k-1)+2 =$

$2k = n-1$. In both cases, $\mathscr{C}$ is an SCDC of G. This establishes the case when n is odd, and completes the proof of the theorem. ∎

4. Summary and related results

When n is even, the proof of Theorem 4 can be extended to simple planar hamiltonian graphs in general. If G is a simple planar hamiltonian graph on n vertices, where n is odd, then the proof shows that such a graph has a CDC with at most n cycles.

The proofs of Hajós' Conjecture for even planar graphs and for even graphs with maximum degree at most four imply that these graphs have SCDC's. Jaeger [5] has shown that a 4-edge-connected graph on n vertices is the union of two even subgraphs with the property that each edge lies in precisely two of them. By decomposing each of the even subgraphs into cycles, we obtain a CDC. The truth of Hajós' Conjecture in general would imply that a simple 4-edge-connected graph on n vertices has a CDC with at most $3\lfloor (n-1)/2 \rfloor$ cycles. It follows from the Four Colour Theorem, Lemma 3, and the validity of Hajós' Conjecture for planar graphs that every simple planar graph on n vertices has a CDC with at most $3\lfloor (n-1)/2 \rfloor$ cycles; moreover, this bound is valid for any 4-face-colourable planar graph.

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