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STABILITY AND NON-LINEAR APPROXIMATION FOR $y'(t) = -f(y(t-1))$

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Stability and uniform attraction of the zero solution of the equation $y'(t) = -f(y(t-1))$ and its n-th approximation $y'(t) = -\frac{1}{n!}f^{(n)}(0)\cdot(y(t-1))^n$ for $n > 1$ is considered.

1. Definitions and main results

1.1 The problem

For $n \in \mathbb{N}$ let

$$F_n := \{f:\mathbb{R} \longrightarrow \mathbb{R} \mid \text{(i) } f \text{ is } n{+}1 \text{ times continuously differentiable; (ii) for all } \nu \in \{0,1,\ldots,n-1\} \text{ we have } f^{(\nu)}(0) = 0; \text{ (iii) } f^{(n)}(0) \neq 0\}.$$

For $f \in F_n$ we consider

$$y'(t) = -f(y(t-1)). \tag{1}$$

Stability properties of this equation with $n = 1$ are investigated e.g. by Barnea [1] and Walther [7]. In the following we deal with the case $n > 1$.

Let

$$C := \{x:[-1,0] \longrightarrow \mathbb{R} \mid x \text{ is continuous}\}$$

have the topology of uniform convergence (with the usual maximum-norm $\|.\|$). Then it is well known (see [2]) that for any $x \in C$ there exists a unique solution of (1) $y_x:[-1,\infty) \longrightarrow \mathbb{R}$ with $y_x|[-1,0] = x$.

For any $x \in C$ and any positive real number δ we denote the ball with center x and radius δ by

$$B(x,\delta) := \{z \in C \mid \|z-x\| < \delta\}.$$

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As in [3] we call $0 \in C$ <u>orbitally stable</u>, if $\forall \varepsilon > 0$ $\exists \delta > 0$ $\forall x \in B(0,\delta)$ $\forall t \geq 0 : |y_x(t)| < \varepsilon$; 0 is called a <u>uniform attractor</u>, if the <u>region of uniform attraction</u>

$$A_u(f) := \{x \in C \mid \exists \delta > 0 \quad \forall \varepsilon > 0 \quad \exists T \geq 0 \quad \forall z \in B(x,\delta) \quad \forall t \geq T : |y_z(t)| < \varepsilon \}$$

is a neighborhood of 0; and 0 is called <u>uniformly asymptotically stable</u>, if 0 is orbitally stable and a uniform attractor.

Together with equation (1) we consider its <u>variational equation of n-th order</u>

$$(2) \qquad y'(t) = -\frac{1}{n!} f^{(n)}(0) \cdot (y(t-1))^n .$$

Defining $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$u \longmapsto f_n(u) := \frac{1}{n!} f^{(n)}(0) \cdot u^n ,$$

the variational equation of n-th order of (1) has the form

$$(2) \qquad y'(t) = -f_n(y(t-1)).$$

In the case $n = 1$ we have the well known facts:

(A_1) 0 is uniformly asymptotically stable for (2) if and only if $0 < f'(0) < \frac{\pi}{2}$ (see [5]).

(A_2) If 0 is uniformly asymptotically stable for (2), then 0 is uniformly asymptotically stable for (1); the converse is false (see [4]).

In the following we are going to investigate how the assertions (A_1) and (A_2) have to be modified in the case $n > 1$. The analogous problem for ordinary differential equations has been studied e.g. by Lasota and Strauss [6].

<u>1.2 Further definitions</u>

To formulate our results we need some notations. To this purpose let $f \in F_n$ be given with $n > 1$.

The mapping $M : \mathbb{R}^+ \longrightarrow \mathbb{R}$ is defined by

$$\varepsilon \longmapsto M(\varepsilon) := \sup \left\{ \frac{1}{(n+1)!} |f^{(n+1)}(u)| \;\middle|\; |u| \leq \varepsilon \right\}.$$

Then M is continuous and monotone increasing.

Further let $\phi : \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ be given by

$$(\xi,\eta) \longmapsto \phi(\xi,\eta) := \left(\frac{1}{n!} |f^{(n)}(0)| + M(\xi)\,\eta \right) \eta^{n-1} .$$

Then ϕ is continuous; for fixed η the function $\phi(\cdot,\eta)$ is monotone increasing and for fixed ξ the function $\phi(\xi,\cdot)$ is strictly increasing. In particular the function $\xi \longmapsto \phi(\xi,\xi)$ is continuous, strictly increasing, $\phi(0,0) = 0$, and $\lim_{\xi\to\infty} \phi(\xi,\xi) = \infty$. Hence there exists exactly one $\varkappa > 0$ such that

$$\phi(\varkappa,\varkappa) = 1.$$

Furthermore we need the following mappings:

$$\varphi : \mathbb{R}^+ \longrightarrow \mathbb{R} \quad \text{with} \quad \xi \longmapsto \varphi(\xi) := \phi(\varkappa,\xi)$$

$$\psi : \mathbb{R}^+ \longrightarrow \mathbb{R} \quad \text{with} \quad \xi \longmapsto \psi(\xi) := \xi\cdot(1+\varphi(\xi)).$$

Then φ and ψ are continuous, strictly increasing, and we have $\varphi(0) = \psi(0) = 0$ and $\lim_{\xi\to\infty}\varphi(\xi) = \lim_{\xi\to\infty}\psi(\xi) = \infty$. In particular the inverse functions φ^{-1} and ψ^{-1} exist and have the same properties as stated for φ and ψ.

Let

$$N(f) := \{\, |u| \;\big|\; u\in\mathbb{R},\ u \neq 0,\ f(u) = 0\}$$

and

$$\varepsilon_0 := \begin{cases} \min\{\varkappa, \inf N(f)\} & \text{if } N(f) \neq \emptyset \\ \varkappa & \text{if } N(f) = \emptyset. \end{cases}$$

Then $\varepsilon_0 > 0$, as $\varkappa > 0$ and $f \in F_n$.

Let

$$r(f) := \min\{\varphi^{-1}(\tfrac{1}{2}),\ \psi^{-1}(\varepsilon_0)\}.$$

Then $r(f) > 0$ for all $f \in F_n$.

1.3 The results

The following two theorems give a characterization of the uniformly asymptotic stability of $0 \in C$ for the equation (1):

<u>THEOREM 1</u>: <u>Let</u> $f \in F_n$ <u>with</u> $n > 1$. <u>If</u> n <u>is odd and</u> $f^{(n)}(0) > 0$, <u>then for the equation</u> (1) 0 <u>is uniformly asymptotically stable</u>, <u>where</u> $B(0,\delta) \subseteq A_u(f)$ <u>for all</u> δ <u>such that</u> $0 < \delta < r(f)$.

<u>THEOREM 2</u>: <u>Let</u> $f \in F_n$ <u>with</u> $n > 1$. <u>If either</u> $f^{(n)}(0) < 0$ <u>or</u> n <u>even and</u> $f^{(n)}(0) > 0$, <u>then for the equation</u> (1) 0 <u>is not uniformly asymptotically stable</u>.

If $f \in F_n$, so $f_n \in F_n$ too, and we can apply these two theorems to the equation (2). Thus we get:

COROLLARY 1: Let $f \in F_n$ with $n > 1$. Then we have:
(i) For the equation (2) 0 is uniformly asymptotically stable if and only if n is odd and $f^{(n)}(0) > 0$.
(ii) Is for the equation (2) 0 uniformly asymptotically stable, then $B(0,\delta) \subseteq A_u(f_n)$ for all δ such that $0 < \delta < r(f_n)$, where $r(f_n) = \varrho_n(\frac{1}{n!} f^{(n)}(0))^{-1/n-1}$ and ϱ_n is the positive solution of $\varrho^n + \varrho - 1 = 0$.

COROLLARY 2: Let $f \in F_n$ with $n > 1$. Then we have:
0 is uniformly asymptotically stable for (1) if and only if 0 is uniformly asymptotically stable for (2).

The Corollary 1 (i) corresponds to the assertion (A_1) for $n = 1$ whereas the Corollary 2 corresponds to the assertion (A_2) for $n = 1$. This shows that non-linear approximation leads to simpler conditions as was to be expected.

2. Proofs

2.1 Some fundamental lemmata

To prove Theorem 1 we first show some lemmata.

The meaning of the function ϕ (defined in 1.2) follows from our

LEMMA 1: Let $f \in F_n$ with $n > 1$. Then we have:
For all $\varepsilon > 0$ and for all $u \in \mathbb{R}$ such that $|u| < \varepsilon$ we get $|f(u)| \leq \phi(\varepsilon, |u|)|u|$, and if furthermore $\varepsilon < \varkappa$, then $|f(u)| \leq \varphi(\varepsilon)\varepsilon$.

Proof: From $f \in F_n$ we get: for all $u \in \mathbb{R}$ there exists a $v \in \mathbb{R}$, v between 0 and u, such that

$$f(u) = \frac{1}{n!} f^{(n)}(0)u^n + \frac{1}{(n+1)!} f^{(n+1)}(v)u^{n+1}.$$

Hence for $|u| < \varepsilon$: $|f(u)| \leq \phi(\varepsilon, |u|)|u|$. And if furthermore $\varepsilon < \varkappa$, then $|f(u)| < \phi(\varkappa, \varepsilon)\varepsilon = \varphi(\varepsilon)\varepsilon$.//

Let $f \in F_n$ be given with $n > 1$. For any real numbers ξ and η such that $0 < \xi < \eta < \varepsilon_0$ the set $\{u \in \mathbb{R} \mid \xi \leq |u| \leq \eta\}$ is compact, and with the definition of ε_o we derive $\min\{|f(u)| \mid \xi \le |u| \le \eta\} > 0$. Therefore we can define

$$D := \{(\xi,\eta) \in \mathbb{R}^2 \mid 0 < \xi < \eta < \varepsilon_o\}$$

and $T: D \longrightarrow \mathbb{R}$ by

$$(\xi,\eta) \longmapsto T(\xi,\eta) := \frac{\eta - \xi}{\min\{|f(u)| \mid \xi \le |u| \le \eta\}} .$$

Then $T(\xi,\eta) > 0$ for all $(\xi,\eta) \in D$. The function T gives a measure for the length of an interval, such that the condition $\xi \le |y_x| \le \eta$ is fulfilled for any solution y_x of (1). More precisely we have:

<u>LEMMA 2</u>: <u>Let</u> $f \in F_n$ <u>with</u> $n > 1$, n <u>odd</u> <u>and</u> $f^{(n)}(0) > 0$, <u>furthermore</u> <u>let</u> $x \in C$, $t_o \ge 0$ <u>and</u> $0 < \delta < \varepsilon < \varepsilon_o$, <u>then</u> <u>we</u> <u>have</u>: <u>If</u> $|y_x| < \varepsilon$ <u>in</u> $(t_o, t_o + T(\delta,\varepsilon) + 1]$ <u>and</u> $|y_x| \ge \delta$ <u>in</u> $(t_o, t_o + 1]$, <u>then</u> <u>there</u> <u>exists</u> <u>a</u> <u>real</u> <u>number</u> $t_1 \in [t_o + 1, t_o + T(\delta,\varepsilon) + 1]$ <u>such</u> <u>that</u> $|y_x(t_1)| < \delta$.

Proof: Assume that for all $t \in [t_o + 1, t_o + T(\delta,\varepsilon) + 1]$: $|y_x(t)| \ge \delta$. Using the mean value theorem there exists a $t_2 \in [t_o + 1, t_o + T(\delta,\varepsilon) + 1]$ such that

$$y_x'(t_2) = \frac{y_x(t_o + T(\delta,\varepsilon) + 1) - y_x(t_o + 1)}{T(\delta,\varepsilon)} .$$

As n is odd and $f^{(n)}(0) > 0$ we get $uf(u) > 0$ for all u such that $0 < |u| < \varepsilon_o$. If $y_x > 0$ in $[t_o, t_o + T(\delta,\varepsilon) + 1]$, then in this interval y_x is monotone decreasing and thus

$$-f(y_x(t_2 - 1)) = y_x'(t_2) > \frac{\delta - \varepsilon}{T(\delta,\varepsilon)} = -\min\{|f(u)| \mid \delta \le |u| \le \varepsilon\},$$

therefore

(3) $|f(y_x(t_2 - 1))| = f(y_x(t_2 - 1)) < \min\{|f(u)| \mid \delta \le |u| \le \varepsilon\}$.

The same is obtained, if $y_x < 0$ in $[t_o, t_o + T(\delta,\varepsilon) + 1]$. As $t_2 - 1 \in [t_o, t_o + T(\delta,\varepsilon)]$ we have from our assumption: $\delta \le |y_x(t_2 - 1)| \le \varepsilon$ which leads to a contradiction to (3).//

For the present we concentrate our investigations on two types of $x \in C$: first we consider those x, for which y_x has no zeros greater than a certain t*; secondly we deal with those x, for which y_x has zeros with the property that the distance of each two consecutive ones is at least 1. This motivates the following definitions:

For $\varepsilon > 0$ and $x \in C$ let

$$S_1(x, \varepsilon) := \{t^* \geq 0 \mid \text{(i) } \forall t > t^* : y_x(t) \neq 0 \;\; \text{(ii) } \forall t \in [t^*, t^* + 1] : |y_x(t)| < \varepsilon\}$$

$$S_2(x, \varepsilon) := \{t^* \geq 0 \mid \text{(i) } \forall t \in (t^*, t^* + 1] : y_x(t) \neq 0 \;\; \text{(ii) } \forall t \in [-1, t^* + 1] : |y_x(t)| < \varepsilon \;\; \text{(iii) } \exists \tau > t^* + 1 : y_x(\tau) = 0\}.$$

As the set of all zeros of a continuous function is closed, and taking (i) and (iii) in the definition of $S_2(x, \varepsilon)$ into consideration we define for $\varepsilon > 0$, $x \in C$ and $t^* \in S_2(x, \varepsilon)$
$\tau(x, \varepsilon, t^*) := \min\{\tau \in \mathbb{R} \mid \tau > t^* + 1, \ y_x(\tau) = 0\}$.
With these sets $S_1(x, \varepsilon)$ and $S_2(x, \varepsilon)$ we are going to formulate two lemmata which are fundamental for all that follows.

<u>LEMMA 3</u>: <u>Let</u> $f \in F_n$ <u>with</u> $n > 1$, n <u>odd</u> <u>and</u> $f^{(n)}(0) > 0$.
<u>Then</u> $\forall \varepsilon^* \in (0, \varepsilon_0)$ $\forall x \in C$ $\forall t^* \in S_1(x, \varepsilon^*)$:
(a) $\forall t > t^* + 1 : y_x(t) y_x'(t) < 0$ <u>and</u> $|y_x(t)| < \varepsilon^*$
(b) $\forall \varepsilon \in (0, \varepsilon^*)$ $\forall t \geq t^* + T(\varepsilon, \varepsilon^*) + 2 : |y_x(t)| < \varepsilon$.

This lemma implies that for those x, for which y_x has no zeros greater than a certain t^*, y_x converges monotonously to zero, whereas the part (b) gives an estimate of the convergence using the function T.

Proof of Lemma 3: (a) It is easy to show by induction that for all $k \in \mathbb{N}$ the following holds : in $(t^* + k, t^* + k + 1]$ we have $y_x y_x' < 0$ and $|y_x| < \varepsilon^*$. This proves (a).
(b) From (a) we have $|y_x(t)| < \varepsilon^*$ for all $t > t^* + 1$.
Now let $\varepsilon \in (0, \varepsilon^*)$. If for a certain $t_1 \in [t^*+1, t^*+2]$: $|y_x(t_1)| < \varepsilon$, then $|y_x(t)| < \varepsilon$ for all $t \geq t^* + T(\varepsilon, \varepsilon^*) + 2$, as y_x is monotonic. If on the other hand $|y_x(t)| \geq \varepsilon$ for all $t \in [t^*+1, t^*+2]$, then Lemma 2 gives the existence of a $t_1 \in [t^*+2, t^*+T(\varepsilon, \varepsilon^*)+2]$ such that $|y_x(t_1)| < \varepsilon$, and thus $|y_x(t)| < \varepsilon$ for all $t \geq t^*+T(\varepsilon, \varepsilon^*)+2$, as y_x is monotonic.//

<u>LEMMA 4</u>: <u>Let</u> $f \in F_n$ <u>with</u> $n > 1$, n <u>odd</u> <u>and</u> $f^{(n)}(0) > 0$.
<u>Then</u> $\forall \varepsilon \in (0, \varepsilon_0)$ $\forall x \in C$ $\forall t^* \in S_2(x, \varepsilon)$:
(a) y_x <u>has a local extremum at</u> $\tau(x, \varepsilon, t^*)+1$ <u>such that</u>

$|y_x(\tau(x,\varepsilon,t^*)+1)| < \varphi(\varepsilon)\max\{|y_x(t)| \mid t^* \le t \le t^*+1\}$.

(b) In $[-1, \tau(x,\varepsilon,t^*)+1]$ we have $|y_x| < \varepsilon$ and in $(\tau(x,\varepsilon,t^*), \tau(x,\varepsilon,t^*)+1]$ y_x is monotonic and there are no zeros; more exactly, the following is true: if $y_x > 0$ (resp. < 0) in $(t^*,t^*+1]$, then $y_x < 0$ (resp. > 0) in $(\tau(x,\varepsilon,t^*), \tau(x,\varepsilon,t^*)+1]$ and monotone decreasing (resp. increasing).

(c) For all $\delta \in (0,\varepsilon)$ we have: if $\tau(x,\varepsilon,t^*) > t^*+T(\delta,\varepsilon)+3$ then $|y_x| < \delta$ in $[t^*+T(\delta,\varepsilon)+2, \tau(x,\varepsilon,t^*)]$.

This lemma describes properties of y_x in a neighborhood of a zero τ. If y_x has several zeros with the property that two consecutive ones have at least the distance 1, then the part (a) supplies: the absolute values of the extrema lying between these zeros diminish in each case with the factor $\varphi(\varepsilon) < 1$, and the part (c) again yields an estimate of $|y_x|$ using the function T.

Proof of Lemma 4: Let $\varepsilon \in (0,\varepsilon_0)$, $x \in C$ and $t^* \in S_2(x,\varepsilon)$, then $y_x(t) \neq 0$ for all $t \in (t^*,t^*+1]$. Here we assume $y_x > 0$ in $(t^*,t^*+1]$; the proof for the case $y_x < 0$ runs completely analogous.

(a) We first show:

(4) $\forall t \in (t^*+1, \tau(x,\varepsilon,t^*)] : y_x'(t) < 0$.

As $t^* \in S_2(x,\varepsilon)$ we get $|y_x| < \varepsilon < \varepsilon_0$ in $(t^*,t^*+1]$ and $y_x' < 0$ in $(t^*+1,t^*+2]$. If $\tau(x,\varepsilon,t^*) \le t^*+2$, nothing remains to show. Therefore we consider $\tau(x,\varepsilon,t^*) > t^*+2$. Assume now that there exists a $t_0 \in (t^*+2, \tau(x,\varepsilon,t^*)]$ such that $y_x'(t_0) = 0$. However then $y_x(t_0-1) = 0$ which gives a contradiction to the definition of $\tau(x,\varepsilon,t^*)$. This proves (4).

In particular $y_x'(\tau(x,\varepsilon,t^*)) < 0$ holds and thus $\tau(x,\varepsilon,t^*)$ is an isolated zero of y_x, and y_x changes its sign there; therefore $\tau(x,\varepsilon,t^*)+1$ is an isolated zero of y_x', and y_x' changes its sign there. So y_x has a local extremum at $\tau(x,\varepsilon,t^*)+1$.

Using (4) we produce for all $t \in [\tau(x,\varepsilon,t^*)-1, \tau(x,\varepsilon,t^*)]$ $|y_x(t)| \le \varepsilon < \varepsilon_0 \le \varkappa$ and thus with Lemma 1 we get:

$|y_x(\tau(x,\varepsilon,t^*)+1)| \le \max\{|y_x'(t)| \mid \tau(x,\varepsilon,t^*) \le t \le \tau(x,\varepsilon,t^*)+1\}$

$= \max\{|f(y_x(t-1))| \mid \tau(x,\varepsilon,t^*) \le t \le \tau(x,\varepsilon,t^*)+1\}$
$\le \phi(\varkappa,\varepsilon)\max\{|y_x(t-1)| \mid \tau(x,\varepsilon,t^*)\le t\le \tau(x,\varepsilon,t^*)+1\}$
$< \varphi(\varepsilon)\max\{|y_x(t)| \mid t^* \le t \le t^*+1\}$.
Thus we have proved (a) and at the same time (b), as is easy to see.
(c) Let $\delta \in (0,\varepsilon)$ and $\tau(x,\varepsilon,t^*) > t^*+T(\delta,\varepsilon)+3$.
Assume that $y_x(t^*+T(\delta,\varepsilon)+2) \ge \delta$. Then as $\tau(x,\varepsilon,t^*) > t^*+T(\delta,\varepsilon)+2$ and the monotony of y_x in $(t^*+1, \tau(x,\varepsilon,t^*)]$ we have

(5) $\quad y_x \ge \delta$ in $[t^*+1, t^*+T(\delta,\varepsilon)+2]$

and in particular $y_x \ge \delta$ in $[t^*+1,t^*+2]$. From (b) we get furthermore $|y_x| < \varepsilon$ in $(t^*+1,t^*+T(\delta,\varepsilon)+2]$. And therefore Lemma 2 yields: there exists a $t_1 \in [t^*+2, t^*+T(\delta,\varepsilon)+2]$ such that $|y_x(t_1)| < \delta$, contradicting (5).
Thus $y_x(t^*+T(\delta,\varepsilon)+2) < \delta$ and because of the monotony of y_x we get $y_x(t) < \delta$ for all $t \in [t^*+T(\delta,\varepsilon)+2, \tau(x,\varepsilon,t^*)]$. And this proves (c).//

2.2 Proofs of the theorems of 1.3

After these preliminary lemmata we come to the Proof of Theorem 1: It has to be shown: 0 is orbitally stable and 0 is a uniform attractor such that $B(0,\delta) \subseteq A_u(f)$ for all $\delta \in (0, r(f))$.

I. 0 is orbitally stable.
This means:

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in B(0,\delta) \quad \forall t \ge 0 : |y_x(t)| < \varepsilon .$$

Let $\gamma : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be defined by

$$\delta \longmapsto \gamma(\delta) := \varphi(\delta)\delta .$$

Then we have

I.1: $\quad \forall \varepsilon \in (0,\varepsilon_0) \quad \forall \delta \in (0, \min\{\varphi^{-1}(\tfrac{1}{2}), \psi^{-1}(\varepsilon)\})$:

$$0 < \gamma(\delta) < \tfrac{1}{2}\delta \quad \text{and} \quad \gamma(\delta) < \varepsilon - \delta .$$

Proof: From $0 < \delta < \varphi^{-1}(\frac{1}{2})$ we get $0 < \gamma(\delta) < \frac{1}{2}\delta$ and $\delta < \psi^{-1}(\varepsilon)$ applies $\gamma(\delta) = \varphi(\delta)\delta = \delta\cdot(1+\varphi(\delta)) - \delta$ $= \psi(\delta) - \delta < \varepsilon - \delta$.

In particular for $\delta < \psi^{-1}(\varepsilon)$: $\delta < \varepsilon$ holds.

The simple proof of the orbital stability is based on the following notation:

Let $\varepsilon > 0$ and $x \in C$, then call $t_o \in \mathbb{R}^+$ $\underline{\varepsilon\text{-good for } x}$, if
(i) $\forall t \in [-1, t_o+1] : |y_x(t)| < \varepsilon$
(ii) $\forall t \in (t_o, t_o+1] : y_x(t) \neq 0$.
The crucial property of ε-good t-points is the following:
I.2: $\forall \varepsilon \in (0, \varepsilon_o)$ $\forall x \in C$ we have: the set of all ε-good points for x is either empty or unbounded.
Proof: Let $\varepsilon \in (0, \varepsilon_o)$ and $x \in C$. We are going to show: If t_o is ε-good for x, then there exists a $t_1 > t_o+1$ such that t_1 is ε-good for x. Obviously this proves our assertion. Now let t_o be ε-good for x. If there exists no zero of y_x in the interval (t_o+1, ∞), then $t_o \in S_1(x, \varepsilon)$ and thus, applying Lemma 3 (a), $|y_x(t)| < \varepsilon$ for all $t > t_o+1$; so all $t > t_o+1$ are ε-good for x. If on the other hand there exists a zero of y_x in (t_o+1, ∞), then $t_o \in S_2(x, \varepsilon)$. Define $t_1 := \tau(x, \varepsilon, t_o)$, then $t_1 > t_o+1$ and Lemma 4 (b) implies that t_1 is ε-good for x.

Now we are ready to show:
I.3: $\forall \varepsilon \in (0, \varepsilon_o)$ $\forall \delta \in (0, \min\{\varphi^{-1}(\frac{1}{2}), \psi^{-1}(\varepsilon)\})$
$\forall x \in B(0, \delta)$ $\forall t \geq 0 : |y_x(t)| < \varepsilon$.
Proof: Let $\varepsilon \in (0, \varepsilon_o)$, $\delta \in (0, \min\{\varphi^{-1}(\frac{1}{2}), \psi^{-1}(\varepsilon)\})$ and $x \in B(0, \delta)$. Consider the set $P := \{t \geq 0 \mid |y_x(t)| > \gamma(\delta)\}$. If $P = \emptyset$, then for all $t \geq 0$ we get $|y_x(t)| \leq \gamma(\delta) < \delta < \varepsilon$ (see I.1). Now let $P \neq \emptyset$, then we can show: there exists a t_o, which is ε-good for x. To this purpose let $t^* := \inf P$, then $|y_x(t^*)| = \gamma(\delta)$, and because of the continuity of y_x there exists a $t_o > t^*$ such that $|y_x(t_o)| > \gamma(\delta)$ and for all $t \in [-1, t_o] : |y_x(t)| < \delta$. This t_o is indeed ε-good for x, as can be seen as follows: From I.1 and Lemma 1 we derive for all $t \in (t_o, t_o+1] :$ $|y_x(t)| - \delta < |y_x(t)| - |y_x(t_o)| \leq$

$$\leq |y_x(t) - y_x(t_o)| \leq \left|\frac{y_x(t) - y_x(t_o)}{t - t_o}\right| \leq$$

$\max\{|y_x'(s)| \mid t_o \leq s \leq t_o+1\} =$
$= \max\{|f(y_x(s-1))| \mid t_o \leq s \leq t_o+1\} < \varphi(\delta)\delta = \gamma(\delta) <$
$< \varepsilon - \delta$, thus $|y_x(t)| < \varepsilon$; and this is also true for

$t = t_o$. Hence for all $t \in [-1, t_o+1] : |y_x(t)| < \varepsilon$, which is the property (i) of the definition of an ε-good point for x. Furthermore for all $t \in (t_o, t_o+1]$ we have:
$|y_x(t)| = |y_x(t) - y_x(t_o) + y_x(t_o)| \geq$
$\geq |y_x(t_o)| - |y_x(t) - y_x(t_o)| > \gamma(\delta) - \gamma(\delta) = 0$, which means $y_x(t) \neq 0$ for all $t \in (t_o, t_o+1]$; and this is the property (ii) of the definition of an ε-good point for x. Thus t_o is ε-good for x. On account of I.2 the set of all ε-good points for x is unbounded, and this applies $|y_x(t)| < \varepsilon$ for all $t \geq 0$.

In particular from I.3 we have: for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x \in B(0, \delta)$ and for all $t \geq 0$ we have $|y_x(t)| < \varepsilon$, which means that 0 is orbitally stable.

II. 0 is a uniform attractor such that $B(0, \delta) \subseteq A_u(f)$ for all $\delta \in (0, r(f))$.

First of all we prove: $0 \in A_u(f)$.

To this purpose we choose $\delta_0 \in (0, r(f))$ and consider δ_0 as fixed. Then we are going to show:

(6) $\forall \varepsilon > 0 \ \exists\, T \geq 0 \ \forall x \in B(0, \delta_0) \ \forall t \geq T : |y_x(t)| < \varepsilon$.

To that end we need some preliminary remarks:

Let

$$\varepsilon_1 := \tfrac{1}{2}(\psi(\delta_0) + \varepsilon_0),$$

then

II.1: $\forall x \in B(0, \delta_0) \ \forall t \geq 0 : |y_x(t)| < \varepsilon_1$.

Proof: From $\delta_0 < r(f) \leq \psi^{-1}(\varepsilon_0)$ we get $\psi(\delta_0) < \varepsilon_0$ and therefore, using the definition of ε_1, $\psi(\delta_0) < \varepsilon_1 < \varepsilon_0$ and thus $\delta_0 < \psi^{-1}(\varepsilon_1)$. Because of $\delta_0 < r(f) < \varphi^{-1}(\frac{1}{2})$ we finally have $\delta_0 < \min\{\varphi^{-1}(\frac{1}{2}), \psi^{-1}(\varepsilon_1)\}$; so I.3 supplies II.1.

Now let $\alpha : \mathbb{R}^+ \longrightarrow \mathbb{R}$ be defined by

$$\delta \longmapsto \alpha(\delta) := 2\varphi(\delta),$$

and let

$$\delta_n := (\alpha(\delta_0))^n \quad \text{for } n \in \mathbb{N} \text{ and}$$

$$\gamma_n := \gamma(\delta_n) \quad \text{for } n \in \mathbb{N} \cup \{0\}.$$

Then we derive immediately from the definitions:

II.2: (i) If $0 < \delta < \varphi^{-1}(\frac{1}{2})$, then $0 < \alpha(\delta) < 1$.

(ii) (δ_n) and (γ_n) are converging monotonously to zero.

(iii) $\forall n \in \mathbb{N} \cup \{0\} : 2\gamma_n \le \delta_{n+1}$.

To prove (6) we have to find for any ε a certain T; obviously it is sufficient to restrict to the case $0 < \varepsilon < \varepsilon_0$. Let now $\varepsilon \in (0, \varepsilon_0)$ be given. Then we define

$$\delta^* := \tfrac{1}{2}\min\{ \varphi^{-1}(\tfrac{1}{2}), \psi^{-1}(\varepsilon), \varepsilon_1 \}.$$

According to I.3 we have

(7) $\forall x \in B(0, \delta^*) \quad \forall t \ge 0 : |y_x(t)| < \varepsilon$.

Therefore it is sufficient to find a T with the following property: in the interval $[-1,T]$ there exists for any $x \in B(0, \delta_0)$ an interval J, having length 1, and such that $|y_x| < \delta^*$ in J. To this purpose choose $k_0 \in \mathbb{N}$ such that

(i) $\delta_{k_0} < \delta^*$ and

(ii) $(\varphi(\varepsilon_1))^{k_0} \varepsilon_1 < \delta^*$.

Such a k_0 exists as from II.2 (ii) we know that (δ_n) is a null sequence, and because of $\varepsilon_1 < \varepsilon_0 \le \varkappa$ we get $\varphi(\varepsilon_1) < \varphi(\varkappa) = \phi(\varkappa, \varkappa) = 1$ (see 1.2).

Now we define

$T^* := T(\delta^*, \varepsilon_1)$ and

$T := (k_0+1)(T^*+4)$.

Thus for any $\varepsilon \in (0, \varepsilon_0)$ we have found a $T > 0$; and it remains to show:

$$\forall x \in B(0, \delta_0) \quad \forall t \ge T : |y_x(t)| < \varepsilon .$$

In the following we distinguish two different types of $x \in B(0, \delta_0)$:

$X_1 := \{x \in B(0, \delta_0) \mid \forall n \in \{0,1,..,k_0\} : |y_x(n)| \le \gamma_n\}$

$X_2 := \{x \in B(0, \delta_0) \mid \exists n \in \{0,1,..,k_0\} : |y_x(n)| > \gamma_n\}$.

Obviously $X_1 \cap X_2 = \emptyset$ and $X_1 \cup X_2 = B(0, \delta_0)$ hold.

II.3: $\forall x \in X_1 \quad \forall t \ge T : |y_x(t)| < \varepsilon$.

We prove this assertion in several steps.

II.3.1: $\forall m \in \mathbb{N} \cup \{0\}$ the following is true:

If $|y_x(m)| \le \gamma_m$ and $\forall t \in [m-1,m] : |y_x(t)| < \delta_m$,
then $\forall t \in [m,m+1] : |y_x(t)| < \delta_{m+1}$.

Proof: Lemma 1 and II.2 (iii) yield: for all $t \in (m,m+1]$:
$|y_x(t)| - \gamma_m \le |y_x(t)| - |y_x(m)| \le |y_x(t) - y_x(m)| \le$

$\leq \left|\frac{y_x(t) - y_x(m)}{t - m}\right| \leq \max\{|f(y_x(s-1))| \mid m \leq s \leq m+1\} <$
$< \varphi(\delta_m)\delta_m = \gamma_m \leq \delta_{m+1} - \gamma_m$. Thus for all $t \in (m,m+1]$: $|y_x(t)| < \delta_{m+1}$. And for $t = m$: $|y_x(m)| \leq \gamma_m < \delta_{m+1}$, proving II.3.1.

II.3.2: $\forall m \in \mathbb{N} \cup \{0\}$ we have the following:

If $\forall j \in \{0,1,\dots,m\}$: $|y_x(j)| \leq \gamma_j$, then for all $t \in [m,m+1]$: $|y_x(t)| < \delta_{m+1}$.

Proof: This follows immediately from II.3.1 by induction.

II.3.3: $\forall x \in X_1 \quad \forall t \geq T$: $|y_x(t)| < \varepsilon$.

Proof: For $x \in X_1$ we have from the definition $|y_x(n)| \leq \gamma_n$ for all $n \in \{0,1,\dots,k_o\}$. Thus using II.3.2 we derive for all $t \in [k_o,k_o+1]$: $|y_x(t)| < \delta_{k_o+1} < \delta_{k_o} < \delta^*$ and (7) implies for all $t \geq k_o+1$: $|y_x(t)| < \varepsilon$. As $k_o+1 <$ $< (k_o+1)(T^*+4) = T$, we have for all $t \geq T$: $|y_x(t)| < \varepsilon$, which proves II.3.3 and II.3.

II.4: $\forall x \in X_2 \quad \forall t \geq T$: $|y_x(t)| < \varepsilon$.

Again we divide the proof into several steps.

Let $x \in X_2$ be given, then there exists by definition an integer $n \in \{0,1,\dots,k_o\}$ such that $|y_x(n)| > \gamma_n$, where we can assume n to be minimal.

Define $x^*: [-1,0] \longrightarrow \mathbb{R}$ by

$x^*(t) := y_x(t+n)$ for all $t \in [-1,0]$.

Then $x^* \in C$ and we are going to treat x^* first. Notice that for all $t \geq 0$ we have $y_{x^*}(t) = y_x(t+n)$. The following notion is of importance:

$x \in C$ is called <u>simply oscillating</u>, if y_x only has simple zeros in the interval $(0,\infty)$ and the distance of two consecutive zeros is greater than 1.

The decisive property of x^* is the following:

II.4.1: $\forall x \in X_2$: x^* is simply oscillating.

Proof: Let $N := \{t > 0 \mid y_x(t) = 0\}$. If $N = \emptyset$ there is nothing to prove. Let now $N \neq \emptyset$, then we define:

$t_o := 0$, $t_1 := \inf N$ and for $n \in \mathbb{N}$:

$$t_{n+1} := \begin{cases} \inf N \cap (t_n, \infty) & \text{if } N \cap (t_n, \infty) \neq \emptyset \\ t_n & \text{if } N \cap (t_n, \infty) = \emptyset . \end{cases}$$

We realize at once that (t_n) is a monotone increasing sequence, which has the following property: if for a certain $n_o \in \mathbb{N}$: $t_{n_o+1} = t_{n_o}$, then $t_n = t_{n_o}$ for all $n \geq n_o$.
Moreover we can prove:
(i) $\forall\, n \in \mathbb{N}$: $(t_n, t_n+1] \cap N = \emptyset$
(ii) $N = \{t_n |\ n \in \mathbb{N}\}$.
Proof of (i): This is easyly proved by induction, if one firstly derives $t_{n-1} \in S_2(x^*, \varepsilon_1)$ from II.1 and II.2 and then using Lemma 4 (b).
Proof of (ii): As N is closed we have $\{t_n |\ n \in \mathbb{N}\} \subseteq N$. To show $N \subseteq \{t_n |\ n \in \mathbb{N}\}$, let $\tau \in N$ and let the set B be defined by $B := \{t_n |\ n \in \mathbb{N}, t_n \leq \tau\}$. Then $B \neq \emptyset$, as $t_1 \in B$, and B is finite because of (i). Let $t_m :=$ max B , then by definition of B we have $t_m \leq \tau$. Assume now $t_m < \tau$. Then the definition of (t_n) yields $t_{m+1} \leq \tau$ and according to (i) : $t_m < t_{m+1} \leq \tau$ which contradicts the definition of t_m. Thus $t_m = \tau$ and $N \subseteq \{t_n |\ n \in \mathbb{N}\}$. This proves (ii).
From (i) and (ii) we derive that the distance of two consecutive zeros of y_x is greater than 1, moreover for all $n \in \mathbb{N}$ we have $y'_{x*}(t_n) = - f(y_{x*}(t_n-1)) \neq 0$ as $0 < |y_{x*}(t_n-1)| < \varepsilon_o$. Thus all zeros of y_{x*} are simple. This proves II.4.1.
A simple consideration shows that for the proof of II.4 it is not sufficient to know x^* to be simply oscillating; in addition we need an upper bound of the difference of two consecutive zeros. To this purpose we define:

$$k(x^*) := \begin{cases} 0 & \text{if either } N = \emptyset \text{ or } t_1 - t_o > T^*+3 \\ k & \text{if } [\forall\, j \in \{1,\dots,k\}: 0 < t_j - t_{j-1} \leq T^*+3\ ] \\ & \text{and } [\ |N| = k \text{ or } t_{k+1} - t_k > T^*+3\ ] \\ \infty & \text{if } |N| = \infty \text{ and } \forall\, j \in \mathbb{N}: t_j - t_{j-1} \leq T^*+3. \end{cases}$$

So $k(x^*)$ has the following meaning: concerning the first $k(x^*)$ zeros of y_x , the difference of two consecutive ones is not greater than T^*+3, and moreover we have: either

there is no further zero of y_x or the next zero has from the preceding one a distance greater than T^*+3.
In the following we distinguish the two cases $k_o \leq k(x^*)$ and $k_o > k(x^*)$.
II.4.2: If $k_o \leq k(x^*)$, then for all $t \geq T$: $|y_x(t)| < \varepsilon$.
Proof: Let $I := \{0,1,\ldots,|N|-1\}$ if $|N| < \infty$ and $I := \mathbb{N}$ if $|N| = \infty$. Because of $t_j \in S_2(x^*, \varepsilon_1)$ for all $j \in I$, we obtain by induction and Lemma 4 (a): for all $j \in I$: $|y_{x^*}(t_{j+1}+1)| < (\varphi(\varepsilon_1))^{j+1}\varepsilon_1$. From $1 \leq k_o \leq k(x^*) \leq |N|$, we derive $|y_{x^*}(t_{k_o}+1)| < (\varphi(\varepsilon_1))^{k_o}\varepsilon_1 < \delta^*$. Thus using Lemma 4 (b): $|y_{x^*}(t)| < \delta^*$ for all $t \in [t_{k_o}, t_{k_o}+1]$ and so (7) yields for all $t \geq t_{k_o}+1$: $|y_{x^*}(t)| < \varepsilon$. This means for all $t \geq n+t_{k_o}+1$: $|y_x(t)| < \varepsilon$. However $n+t_{k_o}+1 < k_o+k_o(T^*+3)+1 < (k_o+1)(T^*+4) = T$ and hence for all $t \geq T$: $|y_x(t)| < \varepsilon$, which proves II.4.2.
II.4.3: If $0 \leq k(x^*) < k_o$, then for all $t \geq T$: $|y_x(t)| < \varepsilon$.
Proof: First consider the case $|N| = k(x^*)$. Then we see at once $t_{k(x^*)} \in S_1(x^*, \varepsilon_1)$ and in consequence of Lemma 3 part (b) we have for all $t \geq t_{k(x^*)}+T(\delta^*, \varepsilon_1)+2$: $|y_{x^*}(t)| < \delta^*$. Thus (7) yields for all $t \geq t_{k(x^*)}+T(\delta^*, \varepsilon_1)+3$: $|y_{x^*}(t)| < \varepsilon$ and therefore for all $t \geq n+t_{k(x^*)}+T(\delta^*, \varepsilon_1)+3$: $|y_x(t)| < \varepsilon$. As $n+t_{k(x^*)}+T(\delta^*, \varepsilon_1)+3 < k_o+k_o(T^*+3)+T^*+3 < (k_o+1)(T^*+4) = T$, we have for all $t \geq T$: $|y_x(t)| < \varepsilon$.
Now consider the case $|N| > k(x^*)$. Then we have $t_{k(x^*)+1} - t_{k(x^*)} > T^*+3$, which gives $t_{k(x^*)+1} > t_{k(x^*)}+T^*+3$ and $t_{k(x^*)} \in S_2(x^*, \varepsilon_1)$. According to Lemma 4 (c) we get $|y_{x^*}(t)| < \delta^*$ for all $t \in [t_{k(x^*)}+T^*+2, t_{k(x^*)+1}]$ and in particular $|y_{x^*}(t)| < \delta^*$ for all t lying in the interval $[t_{k(x^*)}+T^*+2, t_{k(x^*)}+T^*+3]$. Again (7) yields for all $t \geq t_{k(x^*)}+T^*+3$: $|y_{x^*}(t)| < \varepsilon$, and so as above for all $t \geq T$: $|y_x(t)| < \varepsilon$. This proves II.4.3.
The assertions II.4.2 and II.4.3 prove II.4, and to-

gether with II.3 we have shown (6), which means $0 \in A_u(f)$.

Now for any $z \in B(0, \delta_0)$ there exists a $\delta > 0$ such that $B(z, \delta) \subseteq B(0, \delta_0)$. This obtains $B(0, \delta_0) \subseteq A_u(f)$, and this holds indeed for all $\delta_0 \in (0, r(f))$. This proves II, completing the proof of Theorem 1.//

Proof of Theorem 2: Assume that 0 is uniformly asymptotically stable, then especially there exists an $\varepsilon_1 > 0$ such that for all $x \in B(0, \varepsilon_1)$ we have $\lim_{t\to\infty} y_x(t) = 0$, where obviously $\varepsilon_1 < \varepsilon_0$ may be supposed (the definition of ε_0 see 1.2). For this ε_1 there exists a $\delta_1 > 0$ such that for all $x \in B(0, \delta_1)$ and all $t \geq 0$ we have $|y_x(t)| < \varepsilon_1$; and here we also suppose $\delta_1 < \varepsilon_1$. According to the requirements of Theorem 2 we consider the following two cases:

(a) $f^{(n)}(0) < 0$

(b) n is even and $f^{(n)}(0) > 0$.

Now we define $x: [-1, 0] \longrightarrow \mathbb{R}$ by $x(t) := \frac{\delta_1}{2}$ in the case (a) and $x(t) := -\frac{\delta_1}{2}$ in the case (b) for all $t \in [-1, 0]$. Then $x \in B(0, \delta_1)$ and hence, as noticed above: for all $t \geq 0$ we have $|y_x(t)| < \varepsilon_1$ and $\lim_{t\to\infty} y_x(t) = 0$. Now

$$y_x'(0) = -f(y_x(-1)) = -f(x(-1)) = \begin{cases} -f(\frac{\delta_1}{2}) > 0 & \text{in case (a)} \\ -f(-\frac{\delta_1}{2}) < 0 & \text{in case (b),} \end{cases}$$

thus there exists a right hand neighborhood of $t = 0$, such that $|y_x| > \frac{\delta_1}{2}$ holds there. We are going to show:

For all $t > 0$: $|y_x(t)| > \frac{\delta_1}{2}$.

To this purpose assume that there exists a $t_1 > 0$ such that $|y_x(t_1)| = \frac{\delta_1}{2}$, where t_1 may be supposed to be minimal. Then there exists a $\tau \in (0, t_1)$ such that $y_x'(\tau) = 0$, but on the other hand $y_x'(\tau) = -f(y_x(\tau - 1)) \neq 0$ as $\frac{\delta_1}{2} \leq |y_x(\tau - 1)| < \varepsilon_1 < \varepsilon_0$. This proves $|y_x| > \frac{\delta_1}{2}$ in $(0, \infty)$, leading to a contradiction to $\lim_{t\to\infty} y_x(t) = 0$. Thus 0 is not uniformly asymptotically stable.//

Proof of Corollary 1: (i) This is an immediate consequence of the Theorems 1 and 2.

(ii) The uniformly asymptotic stability of 0 is clear from Theorem 1. It only remains to show:

$$r(f_n) = \varrho_n(\tfrac{1}{n!}f^{(n)}(0))^{-1/n-1},$$

where ϱ_n is the positive solution of $\varrho^n + \varrho - 1 = 0$. For abbreviation we set $a := \frac{1}{n!}f^{(n)}(0)$ and obtain $f_n(u) = \frac{1}{n!}f^{(n)}(0)u^n = au^n$, where n is an odd integer and a is a positive real number. Hence we derive for all $\varepsilon \in \mathbb{R}^+$ $M(\varepsilon) = 0$ and thus for all $(\xi, \eta) \in \mathbb{R}^+ \times \mathbb{R}^+ : \phi(\xi,\eta) = a\eta^{n-1}$. The condition $\phi(\varkappa,\varkappa) = 1$ yields $\varkappa = a^{-1/n-1}$. As $N(f) = \emptyset$, we get $\varepsilon_0 = \varkappa$. Moreover for all $\xi \in \mathbb{R}^+$ we have $\varphi(\xi) = a\xi^{n-1}$ and $\psi(\xi) = \xi\cdot(1+a\xi^{n-1})$. Set $r_1 := \varphi^{-1}(\frac{1}{2})$ and $r_2 := \psi^{-1}(\varepsilon_0)$. Then $r_1 = (2a)^{-1/n-1}$, and for all $n \geq 3$ $(n \in \mathbb{N})$ we have $\psi(r_1) = \frac{3}{2}(2a)^{-1/n-1} >$ $> a^{-1/n-1} = \varkappa = \varepsilon_0 = \psi(r_2)$, which gives $r_1 > r_2$. Thus we obtain $r(f_n) = \min\{\varphi^{-1}(\frac{1}{2}), \psi^{-1}(\varepsilon_0)\} = r_2$. If ϱ_n is the positive solution of $\varrho^n + \varrho - 1 = 0$, then we get

$$\psi(\varrho_n a^{-1/n-1}) = a^{-1/n-1}(\varrho_n + \varrho_n^n) = a^{-1/n-1} = \varkappa = \varepsilon_0,$$

which means $r_2 = \varrho_n a^{-1/n-1}$, or $r(f_n) = r_2 =$ $= \varrho_n(\frac{1}{n!}f^{(n)}(0))^{-1/n-1}$.//

Proof of Corollary 2: This is an immediate consequence of the Theorems 1 and 2 as well as the Corollary 1.//

3. Remarks and examples

3.1 In the Theorem 1 the requirement "n is odd and $f^{(n)}(0) > 0$" is equivalent to "there exists a neighborhood U of 0, such that $uf(u) > 0$ for all $u \in U \setminus \{0\}$". This is an easy result of $f \in F_n$ and Taylor's formula of f.

3.2 It is easy to see that the equation $\varrho^n + \varrho - 1 = 0$ (see Corollary 1) has one and only one positive solution ϱ_n for each n, and moreover we have $\frac{1}{2} \leq \varrho_n < \varrho_{n+1} < 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \varrho_n = 1$.

3.3 Barnea stated in [1] the following result:
For the equation

$$x'(t) = b\,|x(t-r)|^{\alpha}\operatorname{sign}(x(t-r))$$

the zero solution is orbitally stable if $\alpha > 1$ and $b<0$.

3.4 Let $q \geq 0$, $C^q := \{\phi:[-q,0]\longrightarrow\mathbb{R} \mid \phi \text{ continuous}\}$, and for $\phi \in C^q$: $\|\phi\| := \sup\{|\phi(s)| \mid s \in [-q,0]\}$. Let $C^q_\beta := \{\phi \in C^q \mid \|\phi\| < \beta\}$. If $x:[t-q,t]\longrightarrow\mathbb{R}$ is continuous, write x_t for the function $x_t(s) = x(s+t)$ for all $s \in [-q,0]$, hence $x_t \in C^q$. Consider

$$\dot{x}(t) = F(t,x_t(.))$$

and define for $\phi \in C^q$: $M(\phi) := \sup\{0,\sup\{\phi(s) \mid s\in[-q,0]\}\}$.
Then Yorke in [8] has obtained the following result:
Theorem: Let $\beta > 0$ and $q > 0$. Let $F:[0,\infty)\times C^q_\beta \longrightarrow \mathbb{R}$ be continuous. Assume for some $\alpha \geq 0$
$\alpha M(\phi) \geq -F(t,\phi) \geq -\alpha M(-\phi)$ for all $\phi \in C^q_\beta$.
(i) Assume $\alpha q \leq \frac{3}{2}$. Then $x(t) \equiv 0$ is a solution and is uniformly stable.
(ii) Assume $0 < \alpha q < \frac{3}{2}$ and for all sequences $t_n \longrightarrow \infty$ and $\phi_n \in C^q_\beta$ converging to a constant nonzero function in C^q_β, $F(t_n, \phi_n)$ does not converge to 0. Then 0 is uniformly asymptotically stable, and if $\|x_{t_0}\| < \frac{2}{5}\beta$ for any $t_0 \geq 0$, then $x(t) \longrightarrow 0$ as $t \longrightarrow \infty$.

It can be shown that the uniformly asymptotic stability of 0 in Theorem 1 is a consequence of the Theorem of Yorke. However the proof given here is completely elementary using geometrical ideas, and it also seems that the estimate of the size of the region of uniform attraction - which is important for applications - is better than that of Yorke, as is indicated by the following examples.

3.5 We consider the equation (1):

$$y'(t) = -f(y(t-1))$$

for special function f. The notation $r(f)$ is defined in 1.2, whereas $r^*(f) := \sup\{\frac{2}{5}\beta\}$, β ranging over all values, which are possible according to the Theorem of Yorke with $q = 1$ and $F(t,y_x(.)) := -f(y(t-1))$.

Example 1: For $m \in \mathbb{N}$ let $g_m:\mathbb{R}\longrightarrow\mathbb{R}$ be defined by

$$u \longmapsto g_m(u) := u^{2m+1},$$

then we obtain for the equation (1) with $f = g_m$:

m	$r(g_m)$	$r^*(g_m)$
1	0.68	0.49
2	0.75	0.44
3	0.80	0.43
4	0.82	0.42
5	0.84	0.42
10	0.90	0.41
100	0.98	0.40

As can be shown, we have $\lim_{m\to\infty} r(g_m) = 1$ and $\lim_{m\to\infty} r^*(g_m) = 0.4$.

Example 2: For $m \in \mathbb{N}$ let $g_m : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$u \longmapsto g_m(u) := u^{2m+1} - u^{2m+2},$$

then we obtain for the equation (1) with $f = g_m$:

m	$r(g_m)$	$r^*(g_m)$
1	0.53	0.36
2	0.66	0.38
3	0.72	0.38
4	0.76	0.39
5	0.79	0.39
10	0.87	0.39
100	0.98	0.40

As can be shown, we have $\lim_{m\to\infty} r(g_m) = 1$ and $\lim_{m\to\infty} r^*(g_m) = 0.4$.

Example 3: For $m \in \mathbb{N}$ let $g_m : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$u \longmapsto g_m(u) := \sum_{k=2m+1}^{\infty} \frac{1}{k!} u^k ,$$

Then we obtain for the equation (1) with $f = g_m$:

m	$r(g_m)$	$r^*(g_m)$
1	1.06	0.87
2	1.75	1.25
3	2.38	1.60
4	2.98	1.93
5	3.57	2.26
10	6.44	3.84
100	56.81	30.75

It can be shown that $\lim_{m\to\infty} r(g_m) = \lim_{m\to\infty} r^*(g_m) = \infty$ holds.

Notation: $\mathbb{N}, \mathbb{R}, \mathbb{R}^+$ denotes the set of integers, real numbers, and non-negative real numbers respectively; for any set N $|N|$ denotes the cardinal number of N. "//" means the end of a proof.

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