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ESTIMATES FROM BELOW FOR LEBESGUE CONSTANTS OF FOURIER SERIES ON COMPACT LIE GROUPS

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In this paper the Lebesgue constants $(L_R^K(G))_{R>0}$ of Fourier series on compact Lie groups G corresponding to general one-dimensional groupings on the dual object G^ are estimated from below by the associated (abelian) Lebesgue constants $(L_R^K(T))_{R>0}$ on a maximal torus T in G. For spherical groupings this leads to the estimate $L_R^{\Theta}(G) \geq \text{const.} R^{(l-1)/2}$, $l=\dim T \geq 2$.

1. Introduction

Let G be a n-dimensional compact connected Lie group and T be a maximal torus in G. Denote by G^ and T^ the dual objects of G and T. The formal Fourier series of a function f in the Lebesgue space $L_1(G)$ has the form $f \sim \sum_{\lambda\in G^\wedge} d_\lambda \chi_\lambda * f$ where χ_λ is the character and $d_\lambda = \chi_\lambda(e)$ the dimension of λ. A function $\phi: G^\wedge \to \mathbb{C}$ is said to be a (central) L_p Fourier multiplier on G if

$$(1.1) \qquad \|\phi\|_{m_p(G)} = \sup_{0 \neq f \in L_p} \|\phi^\vee * f\|_{L_p} / \|f\|_{L_p} < \infty.$$

Here $\phi^\vee$ denotes the inverse Fourier transform of ϕ, i.e.

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$\phi^{\vee}(g) \sim \sum_{\lambda \in G^{\wedge}} d_{\lambda}\phi(\lambda)\chi_{\lambda}(g)$, $g \in G$. In the case suppϕ finite $\phi^{\vee}$ is a central trigonometric polynomial on G and we call the number $\|\phi\|_{m_1(G)} = \|\phi^{\vee}\|_1$ Lebesgue constant for ϕ on G. In [4;Lemma 3.2] we proved a general transference result for estimates of Lebesgue constants which allows to get estimates from above of $\|\phi(\cdot+\rho)\|_{m_1(G)}$ (ρ is the half sum of the positive roots of (G,T)) from estimates of the associated Euclidean Lebesgue constants $\|\phi\|_{m_1(\mathbb{T}^l)}$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. In the following note we prove for characteristic functions ϕ of certain finite subsets of $G^{\wedge}$ the estimate $\|\phi\|_{m_1(G)} \geq M\|\phi\|_{m_1(T)}$, $M > 0$. With this result one can transfer estimates from below of the Euclidean Lebesgue constant $\|\phi\|_{m_1(\mathbb{T}^l)}$ to estimates from below of the non Euclidean Lebesgue constant $\|\phi\|_{m_1(G)}$.

2. Estimates from below of Lebesgue constants on G

We formulate and prove our main result for semisimple simply connected Lie groups G. By standard techniques it easily extends to arbitrary compact connected Lie groups.

Denote by $\underline{g}$ and $\underline{t}$ the Lie algebras of G and T and let $\underline{g}_C$ and $\underline{t}_C$ be their complexifications. Let Δ be the set of roots of $(\underline{g}_C, \underline{t}_C)$, P be a system of positive roots in Δ and W be the corresponding Weyl group. The exponential function exp on $\underline{g}$ is a homomorphism of $\underline{t}$ onto T. If $\xi \in T^{\wedge}$, $\xi \circ \exp$ is a character of $\underline{t}$. Thus there is a $\lambda \in \underline{t}_C^*$, which takes pure imaginary values on $\underline{t}$, such that $\xi \circ \exp = e^{\lambda}$. Let Λ^* be the set of all integral linear forms on $\underline{t}$. If Λ is the kernel of exp and $i\underline{t}^*$ denotes the set of all pure imaginary valued linear forms on $\underline{t}$ one has $\Lambda^* = \{\lambda \in i\underline{t}^* : \lambda(H) \in 2\pi i\mathbb{Z} \text{ for all } H \in \Lambda\}$. Finally let $\Lambda(G)$ be the set of all dominant integral forms on $\underline{t}_C$. Then $\Lambda(G)$ is a semilattice in Λ^* which is in one-to-one correspondence with $G^{\wedge}$.

Let $\mathbb{K}$ be the set of all $K \subset i\underline{t}^*$ which are compact, convex, Weyl group invariant, and with the point zero as inner point. For $K \in \mathbb{K}$ denote by 1_K the indicator function of K. The numbers $L_R^K(G) = \| 1_K(\frac{\cdot}{R}) \|_{m_1(G)}$, $R > 0$, are called the K Lebesgue constants on G.

<u>THEOREM</u> <u>Let</u> $K \in \mathbb{K}$. <u>Then there exists a constant</u> $C > 0$ <u>which only depends on</u> G <u>such that</u>

$$(2.1) \qquad L_R^K(T) \leq C|W|L_R^K(G) \qquad (R > 0).$$

<u>PROOF</u> Let $Z \subset i\underline{t}^*$ be Weyl group invariant. Denote by $T_Z(G)$ the (algebraic) span of all characters χ_λ with $\lambda \in Z \cap \Lambda^*$. Then $T_Z(T)$ is the span of all exponentials e^λ with $\lambda \in Z \cap \Lambda^*$, Z is W-invariant, and the space $T_Z(T)$ is W-invariant. Denote by $T_Z^W(T)$ the subspace of all W-invariant trigonometric polynomials in $T_Z(T)$. As characters are class functions, it follows from the conjugation theorem that the character χ_π of some $\pi \in G^\wedge$ is completely determined by its restriction to T:

$$\chi_\pi(\exp X) = \sum_{i=1}^{p} m_\pi(\lambda_i) e^{\lambda_i(X)} \qquad (X \in \underline{t}),$$

where $\lambda_1, \ldots, \lambda_p \in \Lambda^*$ are the weights of the representation π and $m_\pi(\lambda_i) \in \mathbb{N}$, $i=1,\ldots,p$. Therefore restriction to T defines an isomorphism i^* of $T_{\Lambda^*}(G)$ onto a subspace of $T_{\Lambda^*}(T)$. In the following we prove that

$$(2.2) \qquad i^*(T_Z(G)) = T_Z^W(T)$$

for all $Z \in \mathbb{K}$. For $\lambda, \mu \in i\underline{t}^*$ we say $\mu \leq \lambda$ if μ lies in the convex hull of the orbit of λ under the Weyl group W. We say $\mu < \lambda$ if $\mu \leq \lambda$ and $\mu \neq \lambda$ (cf. [1]). For $\mu \in i\underline{t}^*$ let $W(\mu)$ be the orbit of μ under the Weyl group W. Then

$S(\mu) = \sum_{\lambda\in W(\mu)} e^{\lambda}$ is a W-invariant function on $\underline{t}$, $S(\mu) = S(\nu)$ if and only if $W(\mu) = W(\nu)$ and $S(\mu)$, $S(\nu)$ are orthogonal with respect to any Lebesgue measure on $\underline{t}$ if and only if $W(\mu) \neq W(\nu)$. Let d_Z be the number $|\{W(\mu): \mu\in Z\}|$. We construct a basis of $T_Z^W(T)$ of length d_Z. This implies $\dim T_Z^W(T) = d_Z$. Thus the functions $(S(\mu))_{\mu\in Z\cap\Lambda^*}$ generate $T_Z^W(T)$. For $\lambda\in\Lambda^*$ define $f_\lambda = 1/|W|\cdot\sum_{\gamma\in W} e^{\gamma(\lambda)}$. Then $f_\lambda\in T_Z^W(T)$ if $\lambda\in Z\cap\Lambda^*$ and $(f_\lambda)_{\lambda\in Z\cap\Lambda^*}$ contains a basis of $T_Z^W(T)$ which has length d_Z. On the other hand we have for $\lambda\in\Lambda(G)$ (cf. [1])

$$\chi_\lambda \circ \exp = S(\lambda) + \sum m_j S(\lambda_j)$$

with $m_j\in\mathbb{Z}$ and $\lambda_j < \lambda$. Since Z is convex and W-invariant it follows $i^*(T_Z(G)) \subset T_Z^W(T)$ and $W(Z\cap\Lambda(G)) = Z\cap\Lambda^*$. Thus $\dim(i^*(T_Z(G)) = d_Z$ and (2.2) is proved.

Let $C^c(G)$ denote the subspace of all central functions in $C(G)$ and let $P_Z: C^c(G) \to T_Z(G)$ be the Fourier projection, i.e. P_Z is of the convolution type $P_Z f = D_Z * f$, $f\in C^c(G)$, where $D_Z = \sum_{\lambda\in Z\cap\Lambda(G)} d_\lambda\chi_\lambda$ is the Dirichlet kernel for Z. The operator norm $\|P_Z\|$ is equal to $\|D_Z\|_1$ and P_Z induces a projection $P_Z^W: C^W(T) \to T_Z^W(T)$ with the same norm. Here $C^W(T)$ denotes the subspace of all W-invariant functions in $C(T)$. Denote by $S_Z = \sum_{\lambda\in Z\cap\Lambda^*} e^{\lambda}$ the (abelian) Fourier projection of the space $T_Z(T)$. For $T = \mathbb{T}^1$ we proved in [5; Cor. 1(ii)] the inequality $\|S_Z\| \leq C|W|\,\|P_Z^W\|$ by a symmetrization technique. This result can be proved by the same method for arbitrary tori T. Set $Z = RK$, $K\in\mathbb{K}$, $R > 0$. Since $\|S_{RK}\| = L_R^K(T)$ and $\|P_{RK}^W\| = L_R^K(G)$ our assumption (2.1) is proved.-

<u>COROLLARY</u> <u>Let</u> $\Theta \subset it^*$ <u>be the unit ball with center zero and let</u> l <u>be the dimension of</u> T. <u>Then</u>

$$L_R^{\Theta}(G) \geq \begin{cases} \text{const.}\log R & \text{for } l = 1 \\ \text{const.}R^{(l-1)/2} & \text{for } l \geq 2 \end{cases} \quad (R > 0). \tag{2.3}$$

<u>PROOF</u> For $l = 1$ the classical estimate $L_R^{\Theta}(T) \geq \text{const.}\log R$ holds. For $l \geq 2$ the estimate $L_R^{\Theta}(T) \geq \text{const.}R^{(l-1)/2}$ follows from a result of Il'in in [7]. Thus (2.3) follows from (2.1).-

The recent preprints [6] and [11] contain a general Cohen type inequality for compact Lie groups which can be used to prove estimates from below for $L_R^K(G)$. Lets restrict for simplicity to simple compact Lie groups G. The number of points in $R\Theta \cap \Lambda^*$ increases asymptotically as the volume of $R\Theta$, i.e. as $\text{const.}R^l$ $(R \to \infty)$. From the inequality in [6], [11] one gets $L_R^{\Theta}(G) \geq \text{const.}R$. Tis result is equal to (2.3) for $l = 3$, better than (2.3) for $l = 1,2$ and worse than (2.3) for $l > 3$. In [8] it has been proved that $L_R^K(T) \geq \text{const.}\log R$, $K \in \mathbb{K}$. Combined with (2.1) this implies $L_R^K(G) \geq \text{const.}\log R$. But it follows from [6], [11] that $L_R^K(G) \geq \text{const.}R$.

Our method of prove and the methods in [6], [11] and [8] are totally different. In [11] a theorem of R.S. Cahn [2] is used and in [6] an estimate of A.H. Dooley [3] for norms of characters is essential. A reduction to a general theorem of Olevskiĭ on orthonormal systems is the idea in [8].

We finish with a remark on a further method of estimating $L_R^K(G)$ from below. Let H be a compact subgroup of G. Then $X = G/H$ is a G-homogeneous space and C(X) can be considered as the subspace of all right H-invariant functions in C(G). C(X) is G-invariant. Let $P_{RK}^{\flat}$ denote the restriction of the Fourier projection P_{RK} to the space C(X). Then $\| P_{RK} \| \geq \| P_{RK}^{\flat} \|$ if the second norm is taken with respect to the subspace $C(X) \subset C(G)$. In various special cases it is easier to estimate $\| P^{b}_{RK} \|$ instead of $\| P_{RK} \|$ because P^{b}_{RK} is explicitly known. Consider for example $G = SO(2n)$, $n \geq 2$, and $H = SO(2n-1)$. Then we have from (2.3) that $L^{\odot}_{R}(SO(2n)) \geq \text{const.} R^{(n-1)/2}$. On the other hand we know that X is the real unit sphere S_{2n-1} in $\mathbb{R}^{2n}$ of dimension $2n-1$. The projection P^{b}_{RK} can be expressed by Gegenbauer polynomials. Thus one can deduce from [10] or [9] that $\| P^{b}_{RK} \| \sim R^{n-1}$ $(R \to \infty)$. This leads to the estimate $L^{\odot}_{R}(SO(2n)) \geq \text{const.} R^{n-1}$.

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