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# ON THE FRACTIONAL MATCHING POLYTOPE OF A HYPERGRAPH

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For a hypergraph $\mathcal{H}$ and $b:\mathcal{H}\to\mathbb{R}^+$ define

$$\nu_b^* = \max\left\{ \sum_{A\in\mathcal{H}} b(A)w(A) : w \text{ a fractional matching of } \mathcal{H} \right\}.$$

Conjecture. *There is a matching* $\mathcal{M}$ *of* $\mathcal{H}$ *such that*

$$\sum_{A\in\mathcal{M}} (|A|-1+1/|A|)b(A) \geq \nu_b^*.$$

For uniform $\mathcal{H}$ and $b$ constant this is the main theorem of [4]. Here we prove the conjecture if $\mathcal{H}$ is uniform or intersecting, or $b$ is constant.

## 1. Introduction, results

As usual, a hypergraph $\mathcal{H}$ is a pair $(V(\mathcal{H},E(\mathcal{H}))$, where $V(H)$ is a finite set, the set of *vertices*, and $E(\mathcal{H})$, the *edge set*, is a multiset of subsets of $V(\mathcal{H})$. Where no confusion will result we abbreviate $V(\mathcal{H})$ and $E(\mathcal{H})$ to $V$ and $\mathcal{H}$. Note that $\mathcal{H}$ may contain the same set more than once. We say that $\mathcal{H}$ is *k-uniform* if all its edges are of size $k$. The *degree* of a vertex $v$, denoted $\deg_{\mathcal{H}}(v)$, or simply $\deg(v)$, is $|\{E : v\in E\in\mathcal{H}\}|$. $\mathcal{H}$ is *d-regular* if $\deg_{\mathcal{H}}(v)=d$ for all $v\in V$. A subset of edges $\mathcal{S}\subseteq\mathcal{H}$ is called a *subhypergraph*. A subhypergraph $\mathcal{M}\subseteq\mathcal{H}$ is called a *matching* if every two of its members are disjoint. We write $|\ (\mathcal{H})$, or just $|$ , for the set of matchings of $\mathcal{H}$. The largest cardinality of a matching in $\mathcal{H}$ is the matching number $\nu(\mathcal{H})$. If $\nu(\mathcal{H})=1$, then $\mathcal{H}$ is called *intersecting.*

With each $\mathcal{S}\subseteq\mathcal{H}$ we associate its characteristic vector $\chi(\mathcal{S})\in {}^{\mathcal{H}}$, namely $(\chi(\mathcal{S}))_E$ is 1 if $E\in\mathcal{S}$ and 0 otherwise. The convex hull of the vectors $\{\chi(\mathcal{M}) : \mathcal{M}\in |\ \}$ is called the *matching polytope*, $MP(\mathcal{H})$.

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A *fractional matching* $w$ of $\mathcal{H}$ is the real relaxation of a matching, that is, a function $w:\mathcal{H}\to\mathbb{R}^+$ such that

$$\sum_{E\ni v} w(E)\le 1$$

holds for each $v\in V$. The *fractional matching number* of $\mathcal{H}$ is

$$\nu^*(\mathcal{H}) := \max\left\{\sum_{E\in\mathcal{H}} w(E) : w \text{ is a fractional matching of } \mathcal{H}\right\}.$$

The set of all fractional matchings forms a polytope in the positive orthant of $\mathbb{R}^{\mathcal{H}}$, called the *fractional matching polytope*, and denoted by $FMP(\mathcal{H})$. Obviously,

$$MP(\mathcal{H})\subseteq FMP(\mathcal{H})\subset\mathbb{R}^{\mathcal{H}}.$$

On the other hand,

$$FMP(\mathcal{H})\subseteq A\times MP(\mathcal{H}), \tag{1.1}$$

where $A$ is an $\mathcal{H}\times\mathcal{H}$ diagonal matrix with $(A)_{E,E}=|E|$. This means that the polytope obtained by blowing up the matching polytope in the direction $x_E$ by the factor $|E|$ contains $FMP(\mathcal{H})$. See e.g. [1], [5], [6], [7] for more backround.

To reformulate (1.1), let us introduce the following weighted versions of the matching and fractional matching numbers. For any non-negative vector $b\in\mathbb{R}^{\mathcal{H}}$ (i.e. a non-negative function on the edges $b:\mathcal{H}\to\mathbb{R}^+$), let

$$\nu_b = \max\left\{\sum_{E\in\mathcal{M}} b(E) : \mathcal{M}\in |\ \right\},$$

$$\nu_b^* = \max\left\{\sum_{E\in\mathcal{H}} b(E)w(e) : w \text{ is a fractional matching of } \mathcal{H}\right\}.$$

(So $\nu$ and $\nu^*$ correspond to $b\equiv 1$.) For $k$-uniform hypergraphs, Lovász (see [5]) realized that the trivial inequality $\nu^*\le k\nu$ never holds with equality. His conjecture concerning $\nu^*/\nu$ was proved in [4] in the following form.

**Theorem 1.1.** *If $\mathcal{H}$ is a $k$-uniform hypergraph, then*

$$\nu^*(\mathcal{H})\le\left(k-1+\frac{1}{k}\right)\nu(\mathcal{H}). \tag{1.2}$$

*Moreover, if there is no finite projective plane among the subhyphergraphs of $\mathcal{H}$, and $k>2$, then $\nu^*(\mathcal{H})\le(k-1)\nu(\mathcal{H})$.*

Our work is motivated by

**Conjecture 1.1.** *For any hypergraph $\mathcal{H}$ and $b:\mathcal{H}\to\mathbb{R}^+$ there exists a matching $\mathcal{M}$ such that*

$$\sum_{E\in\mathcal{M}}\left(|E|-1+\frac{1}{|E|}\right)b(E)\geq\nu_b^*.$$

This conjecture is equivalent to the statement

$$FMP(\mathcal{H})\subseteq\hat{A}\times MP(\mathcal{H}),$$

where $\hat{A}$ is the $\mathcal{H}\times\mathcal{H}$ diagonal matrix with $(\hat{A})_{E,E}=|E|-1+1/|E|$.

When $\mathcal{H}$ is $k$-uniform and $b\equiv 1$, Conjecture 1.1 is just Theorem 1.1. So the conjecture generalizes Theorem 1.1 in two ways, namely by allowing nonuniform $\mathcal{H}$, and by considering weights, rather than just sizes, of matchings. In fact, we can prove Conjecture 1.1 if either one, but not both, of these relaxations is in force, i.e.

**Theorem 1.2.** *Any hypergraph $\mathcal{H}$ has a matching $\mathcal{M}$ with*

$$\sum_{E\in\mathcal{M}}\left(|E|-1+\frac{1}{|E|}\right)\geq\nu^*(\mathcal{H}).$$

Note this is sharp for any disjoint union of projective planes.

**Theorem 1.3.** *For any $k$-uniform hypergraph $\mathcal{H}$ and $b:\mathcal{H}\to\mathbb{R}^+$*

$$\left(k-1+\frac{1}{k}\right)\nu_b\geq\nu_b^*.$$

Conjecture 1.1 and Theorems 1.2–3 were announced in [5].

We also prove Conjecture 1.1 for intersecting $\mathcal{H}$. In this case $MP(\mathcal{H})$ is an $|\mathcal{H}|$-dimensional simplex, so the statement reduces to

**Theorem 1.4.** *If $w$ is a fractional matching of an intersecting hypergraph $\mathcal{H}$, then*

$$\sum_{E\in\mathcal{H}}w(E)\frac{1}{|E|-1+1/|E|}\leq 1$$

Theorem 1.4 depends mainly on establishing the following extremal property of projective planes, which is thought to be of independent interest.

**Theorem 1.5.** *If $\mathcal{H}$ is $k$-uniform and intersecting, and $\bigcap_{E\in\mathcal{H}}E=\emptyset$, then*

$$\frac{1}{|\mathcal{H}|^2}\sum_{A\in\mathcal{H}}\sum_{B\in\mathcal{H}}|A\cap B|\geq\frac{k^2}{k^2-k+1},$$

*with equality iff $\mathcal{H}$ is (the line set of) a projective plane with each edge multiplied the same number of times.*

There is another form of our main Conjecture 1.1, which is rather pretty although not so amenable to linear programing, as follows. Let $\mathcal{H}$ be a hypergraph.

We denote by $\chi'(\mathcal{H})$ the *edge-chromatic number* of $\mathcal{H}$, that is, the minimum number of matchings of $\mathcal{H}$ with union $\mathcal{H}$. $\chi'^{*}(\mathcal{H})$ is a fractional relaxation of this, the minimum of $\sum_{\mathcal{M}\in\mathfrak{M}} q(M)$ over all $q : | \to \mathbb{R}^+$ satisfying $\sum_{\mathcal{M}\ni E} q(\mathcal{M}) \geq 1$ for all $E\in\mathcal{H}$; or equivalently, $\chi'^{*}(\mathcal{H})$ is the minimum $d>0$ such that the constant vector $(1/d,\dots,1/d)$ belongs to the matching polytope. We define

$$\Omega(\mathcal{H}) = \max_{p\in V(\mathcal{H})} \sum_{E\ni p} \left(|E| - 1 + \frac{1}{|E|}\right).$$

The following is equivalent to Conjecture 1.1.

**Conjecture 1.6.** *For any hypergraph* $\mathcal{H}$, $\chi'^{*}(\mathcal{H})\leq\Omega(\mathcal{H})$.

Indeed, as far as we can see, the stronger conjecture that $\chi'(\mathcal{H})\leq\Omega(\mathcal{H})$ may also be true. For graphs $\mathcal{G}$ this is just Shannon's theorem ([8], also see in [2]), that $\chi'(\mathcal{G})\leq\frac{3}{2}\Delta(\mathcal{G})$, where $\Delta(\mathcal{G})$ is the maximum degree of $\mathcal{G}$; while for intersecting $\mathcal{H}$ it is Theorem 1.4, since for such $\mathcal{H}$ $\chi'^{*}$ and $\chi'$ agree. The problem of bounding $\chi'(\mathcal{H})$ for uniform $\mathcal{H}$ in terms of the edge size and maximum degree was raised by Faber and Lovász in [3] more than 20 years ago.

To see the equivalence of Conjecture 1.6 and 1.1 we proceed as follows. Conjecture 1.1 asserts that for any $b:\mathcal{H}\to\mathbb{R}^+$, if $w:\mathcal{H}\to\mathbb{R}^+$ satisfies $\sum_{E\ni p} w(E)\leq 1$ for all $p\in V(\mathcal{H})$ then

$$\sum_{E} w(E)b(E) \leq \max_{\mathcal{M}\in\mathfrak{M}} \sum_{E\in\mathcal{M}} \left(|E| - 1 + \frac{1}{|E|}\right) b(E).$$

Equivalently, for any $b:\mathcal{H}\to\mathbb{R}^+$ and $w:\mathcal{H}\to\mathbb{R}^+$,

$$\sum_{E} w(E)b(E) \leq \left(\max_{p\in V(\mathcal{H})} \sum_{E\ni p} w(E)\right) \max_{\mathcal{M}\in\mathfrak{M}} \left(\sum_{E\in\mathcal{M}} \left(|E| - 1 + \frac{1}{|E|}\right) b(E)\right).$$

Substituting $\ell(E) = \left(|E|-1+\frac{1}{|E|}\right) b(E)$ and $m(E) = \frac{w(E)}{|E|-1+1/|E|}$, we see that an equivalent conjecture is: for any $\ell:\mathcal{H}\to\mathbb{R}^+$ and $m:\mathcal{H}\to\mathbb{R}^+$,

$$\sum_{E} \ell(E)m(E) \leq \left(\max_{p\in v(\mathcal{H})} \sum_{E\ni p} m(E) \left(|E| - 1 + \frac{1}{|E|}\right)\right) (\max_{\mathcal{M}\in\mathfrak{M}} \sum_{E\in\mathcal{M}} \ell(E)).$$

This is true if and only if it is true for integral $m$; and (by replacing every edge $E$ by $m(E)$ copies) if and only if it is true all $\mathcal{H}$ when $m\equiv 1$. Thus, an equivalent conjecture is: for all $\mathcal{H}$ and all $\ell:\mathcal{H}\to\mathbb{R}^+$,

$$\sum_{E} \ell(E) \leq \Omega(\mathcal{H}) \max_{\mathcal{M}\in\mathfrak{M}} \left(\sum_{E\in\mathcal{M}} \ell(E)\right).$$

By Farkas' lemma, this is equivalent to 1.6. ∎

## 2. Proof of Theorem 1.2

For a fractional matching $w$ of $\mathcal{H}$ let $S(w)$ be the set of $w$-saturated vertices,

$$S(w) := \left\{ v \in V : \sum_{E \ni v} w(E) = 1 \right\}.$$

Suppose $\mathcal{H}$ is a minimal counterexample to Theorem 1.2. Then $\mathcal{H}$ is $\nu^*$-*critical*, that is, $\nu^*(\mathcal{H}') < \nu^*(\mathcal{H})$ for all $\mathcal{H}' \subset \mathcal{H}$, so in particular

**(2.1)** *if $w$ is an optimal fractional matching of $\mathcal{H}$, then $w(E) > 0$ for every $E \in \mathcal{H}$.*

The key observation here is similar to that in [4]:

**(2.2)** *There exists an optimal fractional matching $w$ of $\mathcal{H}$ such that $|S(w)| \geq |\mathcal{H}|$.*

**Proof of (2.2)** The linear program defining $\nu^*(\mathcal{H})$, that is,

$$\max \sum w(E)$$

subject to $w \in \mathbb{R}^{\mathcal{H}}$

$$w(E) \geq 0 \text{ for } E \in \mathcal{H} \tag{2.3}$$

$$\sum_{E \ni v} w(E) \leq 1 \text{ for } v \in V \tag{2.4}$$

has an optimal solution $w$ for which at least $|\mathcal{H}|$ of the inequalities (2.3), (2.4) are equalities. Since the inequalities (2.3) are strict (by (2.1)), we have $|S(w)| \geq |\mathcal{H}|$. ∎

To prove Theorem 1.2 it is enough to show that for some $E \in \mathcal{H}$,

$$|E| - 1 + \frac{1}{|E|} \geq \sum_{F \cap E \neq \emptyset} w(F). \tag{2.5}$$

For then, setting $\mathcal{H}' = \{F \in \mathcal{H} : F \cap E = \emptyset\}$, we have (inducting)

$$\max_{\mathcal{M} \in \mathfrak{M}(\mathcal{H})} \sum_{F \in \mathcal{M}} \left( |F| - 1 + \frac{1}{|F|} \right) \geq |E| - 1 + \frac{1}{|E|} + \max_{\mathcal{M} \in \mathfrak{M}(\mathcal{H}')} \sum_{F \in \mathcal{M}} \left( |F| - 1 + \frac{1}{|F|} \right)$$

$$\geq \sum_{F \cap E \neq \emptyset} w(F) + \nu^*(\mathcal{H}') \geq \nu^*(\mathcal{H}).$$

But for $w$ as in (2.2),

$$|\mathcal{H}| \leq |S(w)| \leq \sum_{v \in V} \sum_{E \ni v} w(E) = \sum_{E \in \mathcal{H}} |E| w(E).$$

In particular there is some $E \in \mathcal{H}$ with $w(E) \geq 1/|E|$. But for such an $E$ we have (2.5), since

$$\sum_{F \cap E \neq \emptyset} w(F) \leq w(E) + |E|(1 - w(E))$$

$$= |E| - (|E| - 1) w(E) \leq |E| - 1 + 1/|E|.$$ ∎

## 3. Proof of Theorem 1.3

We must show for $k$-uniform $\mathcal{H}$ that

$$\sum_{E\in\mathcal{H}} b(E)w(E) \le k-1+1/k, \tag{3.1}$$

whenever

$$w \textit{ is a fractional matching of } \mathcal{H} \tag{3.2}$$

and $b:\mathcal{H}\to\mathbb{R}^+$ satisfies

$$\sum_{E\in\mathcal{M}} b(E) \le 1 \quad \text{for all } \mathcal{M}\in\mathfrak{M}(\mathcal{H}). \tag{3.3}$$

Suppose $\mathcal{H}$ is a minimal counterexample. Then for any optimal pair, $b$, $w$ (that is, $b$, $w$ maximizing $\sum b(E)w(E)$ subject to (3.2), (3.3)) we have

$$b(E), w(E) > 0 \quad \text{for all } E\in\mathcal{H}. \tag{3.4}$$

Moreover, as in (2.2) we may choose, for any particular $b$, an optimal $w$ for which

$$|S(w)| \ge |\mathcal{H}| \tag{3.5}$$

(where as before $S(w)=\{v\in V: \sum_{E\ni v} w(E)=1\}$).

Fix $b$, $w$ optimal satisfying (3.2), (3.3) and let $\mathfrak{M}_0$ be the set of $b$-saturated matchings:

$$\mathfrak{M}_0 := \{\mathcal{M}\in\mathfrak{M} : \sum_{E\in\mathcal{M}} b(E)=1\}.$$

By linear programming duality, applied to the program

$$\max_b \sum_{E\in\mathcal{H}} b(E)w(E)$$

subject to

$$b(E)\ge 0 \quad \text{for } E\in\mathcal{H},$$

$$\sum_{E\in\mathcal{H}} b(E) \le 1 \quad \text{for } \mathcal{M}\in\mathfrak{M},$$

there exists $\alpha:\mathfrak{M}\to\mathbb{R}^+$ such that

$$\sum_{E\in\mathcal{M}\in\mathfrak{M}} \alpha(\mathcal{M}) \ge w(E) \quad \text{for all } E\in\mathcal{H}, \tag{3.6}$$

$$\sum_{\mathcal{M}\in\mathfrak{M}} \alpha(\mathcal{M}) = \sum_{E\in\mathcal{H}} b(E)w(E). \tag{3.7}$$

By complementary slackness (using (3.4)) equality holds in (3.6) for each $E\in\mathcal{H}$, and $\mathrm{supp}(\alpha)\subseteq\mathcal{M}_0$.

Fix $E\in\mathcal{H}$. Since each $\mathcal{M}\in\mathcal{M}_0$ contains a set meeting $E$,

$$\sum_{\mathcal{M}\in\mathfrak{M}_0}\alpha(\mathcal{M})\le\sum_{F\cap E\neq\emptyset}\sum_{\mathcal{M}\ni F}\alpha(\mathcal{M})=\sum_{F\cap E\neq\emptyset}w(F)\le w(E)+k(1-w(E))=k-(k-1)w(E). \tag{3.8}$$

We now finish as in the proof of Theorem 1.2. It follows from (3.5) that there exists $E$ with $w(E)\ge 1/k$. Inserting such an $E$ in (3.8) we have (summarizing):

$$\sum_{E\in\mathcal{H}}b(E)w(E)=\sum_{\mathcal{M}\in\mathfrak{M}_0}\alpha(\mathcal{M})\le\sum_{F\cap E\neq\emptyset}w(F)\le k-(k-1)w(E)\le k-1+1/k. \qquad \blacksquare$$

As in Theorem 1.1 we have the following sharpening.

**Theorem 3.1.** *If $\mathcal{H}$ is a $k$-uniform hypergraph, $k\ge 3$, and there is no projective plane among the subhypergraphs of $\mathcal{H}$, then for any $b:\mathcal{H}\to\mathbb{R}^+$*

$$(k-1)\nu_b(\mathcal{H})\ge\nu_b^*(\mathcal{H})$$

**Proof.** First, one can prove the following form of (2.2).

**(3.9).** *For any particular $b$, there exists an optimal fractional matching $w$ such that the characteristic vectors of the edges $E\in\mathcal{H}$ with $w(E)>0$ restricted to $S(w)$ are linearly independent.* $\blacksquare$

For the proof of Theorem 3.1 follow the preceding proof as far as the inequality (3.8). To finish from this point it suffices to show the existence of an edge $E\in\mathcal{H}$ with $w(E)\ge 1/(k-1)$.

Suppose instead that for every $E, w(E)<1/(k-1)$ holds. Then for every $v\in S(w)$ we have $\deg(v)\ge k$. This implies that $\mathcal{H}$ is $k$-regular that $|S(w)|=|\mathcal{H}|$. Thus by (3.9), there is only one optimal fractional matching, namely $w(E)\equiv 1/k$.

Since every edge is intersected by at most $k^2-k$ others, and $\mathcal{H}$ does not contain $k^2-k+1$ pairwise intersecting edges (they would form a projective plane), Brooks' theorem (see [2]) implies that $\mathcal{H}$ can be decomposed into $k^2-k$ matching, $\mathcal{M}_i$, $1\le i\le k^2-k$, Thus

$$\sum_{E\in\mathcal{H}}b(E)w(E)=\frac{1}{k}\sum_{E\in\mathcal{H}}b(E)=\frac{1}{k}\sum_{i=1}^{k^2-k}\sum_{E\in\mathcal{M}_i}b(E)\le\frac{1}{k}(k^2-k). \qquad \blacksquare$$

## 4. Proof of Theorem 1.4

In this section we reduce Theorem 1.4 to Theorem 1.5, which we use in the form

**(4.1).** *If $\mathcal{H}$ is $k$-uniform and intersecting, and $\bigcap_{E\in\mathcal{H}} E=\emptyset$, then for any $x:\mathcal{H}\to\mathbb{R}^+$ with $\sum x(E)=1$,*

$$\sum_{E\in\mathcal{H}}\sum_{F\in\mathcal{H}} |E\cap F|x(E)x(F) \geq \frac{k^2}{k^2-k+1}. \tag{4.2}$$

(This follows from Theorem 1.5 by rational approximation and clearing of denominators in (4.2).)

By linear programming duality Theorem 1.4 is equivalent to

**Theorem 1.4'.** *For any intersecting $\mathcal{H}$ there exists $t:V\to\mathbb{R}^+$ such that*
(a) $\sum_{v\in E} t(v)\geq \frac{|E|}{|E|^2-|E|+1}$ *for all $E\in\mathcal{H}$,*
(b) $\sum_{v\in V} t(v)\leq 1$.

**Proof of Theorem 1.4'.** We show that an appropriate $t$ may be obtained as follows. Let $k=\min\{|B|:B\in\mathcal{H}\}$, $\mathcal{H}_k=\{B\in\mathcal{H}:|B|=k\}$.

Suppose first that there exists $p\in\cap\{E:E\in\mathcal{H}_k\}$. Let $E_1\setminus\{p\},\dots,E_l\setminus\{p\}$ be a maximal matching in the family $\{E\setminus\{p\}:E\in\mathcal{H}_k\}$. If $l>k+1$, then $|F|>k+1$ for $p\notin F\in\mathcal{H}$, and we may define $t$ as follows.

$$t(v)=\begin{cases}\frac{k}{k^2-k+1} & \text{for } v=p\\ \frac{k-1}{k^2-k+1} & \text{for } v\in E_1\setminus\{p\}\\ 0 & \text{for } v\notin E_1.\end{cases}$$

If $l\leq k+1$, then let

$$t(v)=\begin{cases}\frac{k+2}{k^2+k+1} & \text{for } v=p\\ \frac{k+1}{l(k^2+k+1)} & \text{for } v\in (E_1\cup\cdots\cup E_l)\setminus\{p\}\\ 0 & \text{for } v\notin (E_1\cup\cdots\cup E_l).\end{cases}$$

Suppose now that $\cap\mathcal{H}_k=\emptyset$. We construct $t$ using another function $f$. Suppose $f:\mathcal{H}_k\to\mathbb{R}^+$ satisfies $\sum f(B)=1/k$. Define $t=t_f:V\to\mathbb{R}^+$ by $t(v)=\sum_{B\ni v} f(B)$. Then for $A\in\mathcal{H}$,

$$\sum_{v\in A} t(v)=\sum_{B\in\mathcal{H}_k} |A\cap B|f(B).$$

Now

$$\sum_{v\in V} t(v)=k\sum_{B\in\mathcal{H}_k} f(B)=1$$

gives (b); and for $|A|>k$, (a) holds automatically since $\nu(\mathcal{H})=1$ implies

$$\sum_{B\in\mathcal{H}_k} |A\cap B| f(B) \geq \sum_{B\in\mathcal{H}_k} f(B) = \frac{1}{k} > \frac{|A|}{|A|^2-|A|+1}.$$

It is sufficient to show

**(4.3).** *If $\mathcal{H}$ is an intersecting $k$-uniform hypergraph with $\bigcap_{E\in\mathcal{H}} E=\emptyset$, then there exists $f:\mathcal{H}\to\mathbb{R}^+$ such that*

(a) *For all $A\in\mathcal{H}$ one has $\sum_B |A\cap B| f(B) \geq \frac{k}{k^2-k+1}$,*

(b) $\sum_{B\in\mathcal{H}} f(B) \leq \frac{1}{k}$.

**Proof of (4.3)** Consider the quadratic programming problem

$$(4.4)\qquad \begin{cases} \text{minimize} & \sum_{A\in\mathcal{H}}\sum_{B\in\mathcal{H}} |A\cap B| x(A)x(B) \\ \text{subject to} & x:\mathcal{H}\to\mathbb{R}^+ \\ & \sum_{A\in\mathcal{H}} x(A)=1. \end{cases}$$

Let $x$ be an optimal solution to (4.4) and suppose there are $A$, $A'\in\mathcal{H}$ such that

$$(4.5)\qquad \sum_{B\in\mathcal{H}} |A\cap B| x(B) > \sum_{B\in\mathcal{H}} |A'\cap B| x(B) \quad \text{and} \quad x(A)>0.$$

Define $y\in(\mathbb{R}^+)^{\mathcal{H}}$ by

$$\begin{aligned} y(A) &= x(A)-\varepsilon \\ y(A') &= x(A')+\varepsilon \\ y(B) &= x(B) \quad \text{if } B\neq A, A'. \end{aligned}$$

Then an easy calculation gives

$$\begin{aligned} &\sum_{B\in\mathcal{H}}\sum_{C\in\mathcal{H}} |B\cap C|\,(x(B)x(C)-y(B)y(C)) \\ &= 2\varepsilon \sum_{B\in\mathcal{H}} (|A\cap B|-|A'\cap B|)x(B) - O(\varepsilon^2) > 0. \end{aligned}$$

for small enough positive $\varepsilon$, contrary to our choice of $x$. We conclude that (4.5) does not occur, that is, there is a number $s$ such that

$$\sum_{B\in\mathcal{H}} |A\cap B| x(B) \begin{cases} = s & \text{if } x(A)>0 \\ \geq s & \text{if } x(A)=0. \end{cases}$$

By (4.1)

$$s = \sum_{A \in \mathcal{H}} x(A)s = \sum_{A \in \mathcal{H}} \sum_{B \in \mathcal{H}} |A \cap B| x(A) x(B) \geq \frac{k^2}{k^2 - k + 1}.$$

Now set $f(A) = x(A)/k$ for $A \in \mathcal{H}$.

Then (b) of Theorem 1.4' is automatic, and we have just shown (a), since for any $A \in \mathcal{H}$,

$$\sum_{B \in \mathcal{H}} |A \cap B| f(B) = \frac{1}{k} \sum_{B \in \mathcal{H}} |A \cap B| x(B) \geq \frac{s}{k} \geq \frac{k}{k^2 - k + 1}.$$ ∎

## 5. Proof of Theorem 1.5.

A pair $\mathcal{A}$, $\mathcal{B}$ of families of subsets of $V$ is said to be *cross-intersecting* if $A \cap B \neq \emptyset$ for all $A \in \mathcal{A}$, $B \in \mathcal{B}$. The key to Theorem 1.5 is the following simple observation.

**Lemma 5.1.** *If $\mathcal{A}$, $\mathcal{B}$ are cross-intersecting then*

$$\sum_{A \in \mathcal{A}} \sum_{A' \in \mathcal{A}} |A \cap A'| \sum_{B \in \mathcal{B}} \sum_{B' \in \mathcal{B}} |B \cap B'| \geq |\mathcal{A}|^2 |\mathcal{B}|^2.$$

**Proof.** Define vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^V$ by

$$\mathbf{a}(v) = \deg_{\mathcal{A}}(v) ;= |\{A \in \mathcal{A} : v \in A\}|,$$
$$\mathbf{b}(v) = \deg_{\mathcal{B}}(v) := |\{B \in \mathcal{B} : v \in B\}|.$$

Then, with $\langle \mathbf{a}, \mathbf{b} \rangle$ the usual inner product on $\mathbb{R}^V$,

$$\sum_{A \in \mathcal{A}} \sum_{A' \in \mathcal{A}} |A \cap A'| \sum_{B \in \mathcal{B}} \sum_{B' \in \mathcal{C}} |B \cap B'| = \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 \geq \langle \mathbf{a}, \mathbf{b} \rangle^2$$
$$= \left( \sum_{A \in \mathcal{A}} \sum_{B \in \mathcal{B}} |A \cap B| \right)^2 \geq |\mathcal{A}|^2 |\mathcal{B}|^2$$

(the second inequality holding because $\mathcal{A}$, $\mathcal{B}$ are cross-intersecting). ∎

Now let $\mathcal{H}$ be as in Theorem 1.5, say with $|\mathcal{H}| = m$, and suppose

(5.1)
$$\frac{k^2 m^2}{k^2 - k + 1} \geq \sum_{E \in \mathcal{H}} \sum_{F \in \mathcal{H}} |E \cap F|.$$

In case of $k=2$ the hypergraph $\mathcal{H}$ should be a triangle, $\mathcal{H} = \{E_1, E_2, E_3\}$ with edge multiplicities $\mu_1$, $\mu_2$, $\mu_3$, and an easy calculation shows that (5.1) implies $\mu_1 = \mu_2 = \mu_3$, and thus equality holds. From now on we suppose that $k \geq 3$.

Fix $v \in V$ and set

$$\mathcal{A} = \{A \in \mathcal{H} : v \in A\}$$
$$\mathcal{B} = \mathcal{H} \setminus \mathcal{A}$$
$$d = |\mathcal{A}| = \deg_{\mathcal{H}}(v).$$

Apply Lemma 5.1 to $\mathcal{A} \setminus \{p\}$ and $\mathcal{B}$.

$$\sum_{A \in \mathcal{A}} \sum_{A' \in \mathcal{A}} (|A \cap A'| - 1) \sum_{B \in \mathcal{B}} \sum_{B' \in \mathcal{B}} |B \cap B'| \geq |\mathcal{A}|^2 |\mathcal{B}|^2 = d^2(m-d)^2. \tag{5.2}$$

Set $a = (m-d)^{-2} \sum_{B \in \mathcal{B}} \sum_{B' \in \mathcal{B}} |B \cap B'|$. Then

$$1 < a \leq k. \tag{5.3}$$

(Notice that the hypothesis $\bigcap_{A \in \mathcal{H}} A = \emptyset$ gives $d < m$.) In terms of $a$, (5.2) becomes

$$\sum_{A \in \mathcal{A}} \sum_{A' \in \mathcal{A}} (|A \cap A'| - 1) \geq d^2/a,$$

or, equivalently,

$$\sum_{A \in \mathcal{A}} \sum_{A' \in \mathcal{A}} |A \cap A'| \geq \left(1 + \frac{1}{a}\right) d^2. \tag{5.4}$$

It follows that

$$\begin{aligned} \sum_{E \in \mathcal{H}} \sum_{F \in \mathcal{H}} |E \cap F| &\geq \sum_{A \in \mathcal{A}} \sum_{A' \in \mathcal{A}} |A \cap A'| + \sum_{B \in \mathcal{B}} \sum_{B' \in \mathcal{B}} |B \cap B'| + 2d(m-d) \\ &\geq \left(1 + \frac{1}{a}\right) d^2 + a(m-d)^2 + 2d(m-d) \\ &= \frac{1}{a}[(a^2 - a + 1)d^2 - 2(a^2 - a)md + a^2 m^2]. \end{aligned} \tag{5.5}$$

For given $m$, $a$ the right hand side of (5.5) is minimized at $d = \frac{(a^2-a)m}{a^2-a+1}$, where it takes the value $\frac{a^2 m^2}{a^2-a+1}$. Combining this with (5.1) gives

$$\frac{k^2 m^2}{k^2 - k + 1} \geq \sum_{E \in \mathcal{H}} \sum_{F \in \mathcal{H}} |E \cap F| \geq \frac{a^2 m^2}{a^2 - a + 1}. \tag{5.6}$$

For the function $g(x); = \frac{x^2}{x^2 - x + 1}$ one has

$$g(x) \begin{cases} > g(k) & \text{if } \frac{k}{k-1} < x < k \\ = g(k) & \text{if } x \in \left\{\frac{k}{k-1}, k\right\} \\ < g(k) & \text{otherwise.} \end{cases}$$

Thus by (5.3) and (5.6),

$$a \in \left(1, \frac{k}{k-1}\right] \cup \{k\}.$$

But if $a=k$ the $\mathscr{B}$ consists of copies of some fixed edge $B$ and we may sharpen (5.4) via

$$\sum_{A\in\mathscr{A}} \sum_{\substack{A'\in\mathscr{A}\\ |A\cap A'|>1}} (|A\cap A'|-1) \geq \left(\sum_{A\in\mathscr{A}} \sum_{\substack{A'\in\mathscr{A}\\ |A\cap A'|>1}} 1\right) + (k-2)d \geq d^2/k + (k-2)d.$$

In particular, for $k \geq 3$, this gives strict inequality in (5.4) and in the second inequality of (5.6), which is impossible since the left and the right hand sides of (5.6) are equal.

It follows that $1 < a \leq \frac{k}{k-1}$. Inserting this in (5.4) and letting $v\in V$ vary yields the basic inequality

$$\sum_{E\ni v} \sum_{F\ni v} |E\cap F| \geq \left(2-\frac{1}{k}\right)(\deg(v))^2 \quad \text{for all } v\in V. \tag{5.7}$$

Summing on $v$ gives

$$\sum_{E\in\mathscr{H}} \sum_{F\in\mathscr{H}} |E\cap F|^2 = \sum_{v\in V} \sum_{E\ni v} \sum_{F\ni v} |E\cap F| \geq \left(2-\frac{1}{k}\right) \sum_{v\in V} (\deg(v))^2. \tag{5.8}$$

Furthermore,

$$-2 \sum_{E\in\mathscr{H}} \sum_{F\in\mathscr{H}} |E\cap F| = -2 \sum_{v\in V} (\deg(v))^2. \tag{5.9}$$

$$\sum_{E\in\mathscr{H}} \sum_{F\in\mathscr{H}} 1 = m^2. \tag{5.10}$$

Summing (5.8)-(5.10) gives

$$\sum_{E\in\mathscr{H}} \sum_{F\in\mathscr{H}} (|E\cap F|-1)^2 \geq m^2 - \frac{1}{k} \sum_{v\in V} (\deg(v))^2. \tag{5.11}$$

Thus, noting that

$$|E\cap F| - 1 \geq (|E\cap F|-1)^2/(k-1), \tag{5.12}$$

we have

$$\sum_{v\in V} (\deg(v))^2 - m^2 = \sum_{E\in\mathscr{H}} \sum_{F\in\mathscr{H}} (|E\cap F|-1)$$

$$\geq \sum_{E\in\mathscr{H}} \sum_{F\in\mathscr{H}} (|E\cap F|-1)^2/(k-1) \geq \left(m^2 - \frac{1}{k} \sum_{v\in V} (\deg(v))^2\right) /(k-1),$$

and a little rearranging gives the inequality of Theorem 1.5:

$$\sum_{E\in\mathcal{H}}\sum_{F\in\mathcal{H}}|E\cap F| = \sum_{v\in V}(\deg(v))^2 \geq \frac{k^2m^2}{k^2-k+1}. \tag{5.13}$$

Suppose now that equality holds in (5.13). This requires that equality always hold in (5.12), in other words that

$$|E\cap F| \in \{1,k\} \quad \text{for all } E,F\in\mathcal{H}. \tag{5.14}$$

Now for $v\in V$ let $A_1,\dots,A_p$ be the distinct sets on $v$ which appear as edges of $\mathcal{H}$, (note that $p\leq k$), and let $\mu_i$ be the multiplicity of $A_i$. Since (5.7) must hold with equality we have

$$\sum_{A\ni v}\sum_{A'\ni v}(|A\cap A'|-1) = (k-1)\sum_{i=1}^{p}\mu_i^2 \geq \frac{k-1}{p}(\deg(v))^2$$
$$\geq \frac{k-1}{k}(\deg(v))^2 = \sum_{A\ni v}\sum_{A'\ni v}(|A\cap A'|-1).$$

Thus $p=k$ and all $\mu_i$ are equal to some fixed $\mu$, which (by connectedness of $\mathcal{H}$, say) does not depend on $v$. That is, $\mathcal{H}$ consists of $\mu$ copies of some $k$-regular, $k$-uniform 1-intersecting hypergraph $\mathcal{H}_0$, and such a hypergraph is easily seen to be a projective plane. ∎

**Remark 5.1** The following example shows, that one cannot easily sharpen Theorem 1.5 for hypergraphs with no projective planes.

Let $T=\{v_0,v_1,\dots,v_k\}$ and let $\mathcal{H}$ consists of $t$ copies of $\{v_1,\dots,v_k\}$ together with $(k-1)t$ edges containing $\{v_0,v_1\}$, $1\leq i\leq k$, all edges being disjoint apart from the intersections forced by these specifications. Then $|\mathcal{H}|=t(k^2-k+1)$, and we have

$$\frac{1}{|\mathcal{H}|^2}\sum_{E\in\mathcal{H}}\sum_{F\in\mathcal{H}}|E\cap F| = \frac{k^2}{k^2-k+1}+O\left(\frac{1}{kt}\right),$$

which can be arbitrarily close to the minimum ratio as $t\to\infty$.

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