Acta Mechanica 60, 67—89 (1986)

ACTA MECHANICA

A Theory of Thermoelastic Materials with Voids

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(*Received March 18, 1985; revised May 30, 1985*)

Summary

A linear theory of thermoelastic materials with voids is considered. First, some general theorems (uniqueness, reciprocal and variational theorems) are established. Then, the acceleration waves and some problems of equilibrium are studied.

1. Introduction

Nunziato and Cowin [1] have presented a nonlinear theory for the behaviour of porous solids in which the skeletal or matrix material is elastic and the interstices are void of material. In this theory the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field. This representation introduces an additional degree of kinematic freedom. The intended applications of the theory of elastic materials with voids are to geological materials like rock and soils and to manufactured porous materials.

Jarić and Golubovic [2] and Jarić and Ranković [3] have studied the nonlinear theory of thermoelastic materials with voids. The linear theory of elastic materials with voids has been established by Cowin and Nunziato [4]. Another version of the linear theory, called the dilatation theory of elasticity, was independently proposed by Markov [5]. Some applications of the linear theory were presented in [4]—[7].

In this paper we study the linear theory of thermoelastic materials with voids. In the first part of the paper we use the method given by Green and Rivlin [8] in order to obtain the basic equations from the balance of energy and the invariance requirements under superposed rigid body motions. In Section 3 we establish theorems concerning the uniqueness of solution, the reciprocity relation and variational characterization of solution in the dynamic theory. In the next section we study the acceleration waves in homogeneous and isotropic bodies. The propagation conditions and growth equations, which govern the propagation of waves, are derived. The couplings between the discontinuities are studied. In the final section of the paper, some problems of thermoelastostatics (the response to a concentrated source of heat, the deformation of a thick walled spherical shell and a hollow cylinder) are solved. In each of these applications, the change in void volume induced by the deformation is determined.

2. Basic Equations

We refer the motion of the continuum to a fixed system of rectangular Cartesian axes Ox_i $(i = 1, 2, 3)$. We shall employ the usual summation and differentiation conventions: Latin subscripts (unless otherwise specified) are understood to range over the integers (1, 2, 3) whereas Greek subscripts are confined to the range (1, 2), summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate. In what follows we present a linear theory for the behavior of porous solid in which the skeletal or matrix is a thermoelastic material and the interstices are void of material. We assume that the initial body is free from stresses. The concept of a distributed body asserts that the mass density at time t has the decomposition $\gamma\nu$ where γ is the density of the matrix material and ν is the volume fraction field [1]. We denote by u_i the components of the displacement vector. We consider an arbitrary material volume $\mathcal{V}$ in the continuum, bounded by a surface $\mathcal{A}$ at time t, and we suppose that V is the corresponding region in the initial undeformed state of the continuum, bounded by a surface A. Let the outward unit normal at A be n_i, referred to our fixed rectangular frame of reference.

We postulate an energy balance for an arbitrary material domain, in the form

$$\begin{aligned}&\int_V \varrho(v_i\dot{v}_i + \varkappa\dot{\nu}\ddot{\nu})\,dV + \int_V \varrho\dot{\varepsilon}\,dV\\ &= \int_V \varrho(f_iv_i + l\dot{\nu} + s)\,dV + \int_A (t_iv_i + h\dot{\nu} + q)\,dA,\end{aligned} \tag{2.1}$$

where $v_i = \dot{u}_i$, ϱ is the density in the reference configuration, $\varkappa$ is the equilibrated inertia, ε is the internal energy per unit mass, f_i is the body force, l is the extrinsic equilibrated body force, s is the extrinsic heat supply, t_i is the stress vector associated with the surface $\mathcal{A}$ but measured per unit area of the surface A, h is the equilibrated stress, q is the heat flux associated with the surface $\mathcal{A}$ but measured per unit area of the surface A, and a superposed dot denotes the material derivative with respect to the time. We suppose that the body has arrived at a given state at time t through some prescribed motion. Following [8], we consider a second motion which differs from the given motion only by a constant superposed rigid body translational velocity, the body occupying the same position at time t, and we assume that $\dot{\varepsilon}$, f_i, l, s, t_i, h, q are unaltered by such superposed rigid velocity. If we use the Eq. (2.1) with v_i replaced by $v_i + a_i$, where a_i is an arbitrary constant, we obtain

$$\int_V \varrho \dot{v}_i \, dV = \int_V \varrho f_i \, dV + \int_A t_i \, dA . \tag{2.2}$$

Using the well-known method, from (2.2) we get

$$t_i = t_{ji} n_j , \tag{2.3}$$

and

$$t_{ji,j} + \varrho f_i = \varrho \ddot{u}_i . \tag{2.4}$$

Taking into account (2.3) and (2.4), the Eq. (2.1) reduces to

$$\int_V \varrho(\dot{\varepsilon} + \varkappa \dot{\nu} \ddot{\nu}) \, dV = \int_V [t_{ji} v_{i,j} + \varrho(l \dot{\nu} + s)] \, dV + \int_A (\dot{\nu} h + q) \, dA . \tag{2.5}$$

With an argument similar to that used in obtaining (2.3), from (2.5) we obtain

$$(h - h_i n_i) \, \dot{\nu} + q - q_i n_i = 0 , \tag{2.6}$$

where h_i is the equilibrated stress vector [1] and q_i is the heat flux vector. With the help of (2.6), Eq. (2.5) reduces to

$$(h_{i,i} + \varrho l - \varrho \varkappa \ddot{\nu}) \, \dot{\nu} + t_{ji} v_{i,j} + q_{i,i} + \varrho s + h_i \dot{\nu}_{,i} - \varrho \dot{\varepsilon} = 0 . \tag{2.7}$$

Let us now consider a motion of the body which differs from the given motion only by a superposed uniform rigid body angular velocity, the body occupying the same position at time t, and let us assume that $\dot{\varepsilon}$, l, s, t_{ij}, h_i, q_i are unaltered by such motion. In this case, as in [8], the equation (2.7) leads to

$$t_{ij} = t_{ji} . \tag{2.8}$$

With the help of (2.8), the Eq. (2.7) reduces to

$$\varrho \dot{\varepsilon} = t_{ij} \dot{e}_{ij} - g \dot{\nu} + h_i \dot{\nu}_{,i} + q_{i,i} + \varrho s , \tag{2.9}$$

where we have used the notations

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) , \tag{2.10}$$

and

$$g = \varrho \varkappa \ddot{\nu} - h_{i,i} - \varrho l . \tag{2.11}$$

The Eq. (2.11) was obtained in [1] from the balance of equilibrated force. The function g is the intrinsic equilibrated body force [1].

The entropy production inequality is

$$\int_V \varrho \dot{\eta} \, dV \geqq \int_V \varrho \frac{s}{T} \, dV + \int_A \frac{q}{T} \, dA , \tag{2.12}$$

for an arbitrary material domain. Here η is the specific entropy and T is the absolute temperature. The free energy is

$$\psi = \varepsilon - \eta T. \tag{2.13}$$

Let us introduce the notations

$$\theta = T - T_0, \qquad \varphi = \nu - \nu_0, \tag{2.14}$$

where T_0 is the absolute temperature in the reference state and ν_0 is the volume distribution function for the reference configuration. We assume that T_0 and ν_0 are constants.

We restrict our attention to the linear theory of thermoelastic materials where the constitutive variables are e_{ij}, φ, $\varphi_{,i}$, θ, $\theta_{,i}$. It is easy to see that the constitutive variables are invariant under superposed rigid-body motions.

We assume that at each point x and for all time, ψ, t_{ij}, g, h_i, h and q_i are functions of e_{ij}, $\nu_0 + \varphi$, $\varphi_{,i}$, $T_0 + \theta$, $\theta_{,i}$ consistent with the assumption of the linear theory. Moreover, we assume that h and q are functions of e_{ij}, $\nu_0 + \varphi$, $\varphi_{,i}$, $T_0 + \theta$, $\theta_{,i}$ and n_i.

For a given deformation, $\dot{\nu}$ in (2.6) may be chosen arbitrarily so that, on the basis of the constitutive assumptions, we have

$$h = h_i n_i, \qquad q = q_i n_i. \tag{2.15}$$

Using (2.15), the inequality (2.12) reduces to

$$\varrho T \dot{\eta} - \varrho s - q_{i,i} + \frac{1}{T} q_i T_{,i} \geqq 0. \tag{2.16}$$

With the help of (2.9) and (2.13), the inequality (2.16) becomes

$$-\varrho(\dot{\psi} + \eta\dot{\theta}) + t_{ij}\dot{e}_{ij} + g\dot{\varphi} + h_i\dot{\varphi}_{,i} + \frac{1}{T}\, q_i\theta_{,i} \geqq 0. \tag{2.17}$$

If we use the constitutive equations and the standard arguments, then the inequality (2.17) implies the restrictions

$$\begin{gathered} \psi = \psi(e_{ij}, \varphi, \varphi_{,i}, \theta), \\ t_{ij} = \frac{\partial\sigma}{\partial e_{ij}}, \qquad g = -\frac{\partial\sigma}{\partial\nu}, \\ \varrho\eta = -\frac{\partial\sigma}{\partial\theta}, \qquad h_i = \frac{\partial\sigma}{\partial\varphi_{,i}}, \end{gathered} \tag{2.18}$$

and

$$q_i\theta_{,i} \geqq 0, \tag{2.19}$$

where $\sigma = \varrho\psi$.

The inequality (2.19) implies that

$$q_i = 0 \quad \text{if} \quad \theta_{,i} = 0. \tag{2.20}$$

In the linear theory, and assuming that the initial body is free from stress and has zero intrinsic equilibrated body force, we have

$$\begin{aligned}\sigma = &\frac{1}{2} C_{ijrs} e_{ij} e_{rs} - \beta_{ij} e_{ij} \theta - \frac{1}{2} a\theta^2 \\ &+ \frac{1}{2} A_{ij} \varphi_{,i} \varphi_{,j} + B_{ij} \varphi e_{ij} + D_{ijk} e_{ij} \varphi_{,k} \\ &+ d_i \varphi \varphi_{,i} + \frac{1}{2} \xi \varphi^2 - m\theta\varphi - a_i \varphi_{,i} \theta .\end{aligned} \tag{2.21}$$

Using (2.18), (2.20) and (2.21) we obtain the following constitutive equations

$$\begin{aligned} t_{ij} &= C_{ijrs} e_{rs} + B_{ij} \varphi + D_{ijk} \varphi_{,k} - \beta_{ij} \theta, \\ h_i &= A_{ij} \varphi_{,j} + D_{rsi} e_{rs} + d_i \varphi - a_i \theta, \\ g &= -B_{ij} e_{ij} - \xi \varphi - d_i \varphi_{,i} + m\theta, \\ \varrho\eta &= \beta_{ij} e_{ij} + a\theta + m\varphi + a_i \varphi_{,i}, \\ q_i &= k_{ij} \theta_{,j}. \end{aligned} \tag{2.22}$$

The coefficients in (2.22) have the following symmetries

$$\begin{aligned} &C_{ijrs} = C_{rsij} = C_{jirs}, \qquad \beta_{ij} = \beta_{ji}, \\ &D_{ijk} = D_{jik}, \qquad A_{ij} = A_{ji}, \qquad B_{ij} = B_{ji}. \end{aligned} \tag{2.23}$$

In the case of an isotropic material, the constitutive equations (2.22) become

$$\begin{aligned} t_{ij} &= \lambda e_{rr} \delta_{ij} + 2\mu e_{ij} + b\varphi \delta_{ij} - \beta\theta\delta_{ij}, \\ h_i &= \alpha \varphi_{,i}, \\ g &= -b e_{rr} - \xi\varphi + m\theta, \\ \varrho\eta &= \beta e_{rr} + a\theta + m\varphi, \\ q_i &= k\theta_{,i}, \end{aligned} \tag{2.24}$$

where δ_{ij} is Kronecker's delta and λ, μ, b, β, α, ξ, m, a, k are constitutive coefficients.

In the linear theory, the energy equation (2.9) reduces to

$$\varrho T_0 \dot{\eta} = q_{i,i} + \varrho s. \tag{2.25}$$

We assume that the body occupies a bounded regular region B of three-dimensional Euclidean space. Let us consider the subsets Σ_i $(i = \overline{1,6})$ of ∂B so that $\bar{\Sigma}_1 \cup \Sigma_2 = \bar{\Sigma}_3 \cup \Sigma_4 = \bar{\Sigma}_5 \cup \Sigma_6 = \partial B$, $\Sigma_1 \cap \Sigma_2 = \Sigma_3 \cap \Sigma_4 = \Sigma_5 \cap \Sigma_6 = 0$. We consider the following boundary conditions

$$\begin{aligned} u_i &= \tilde{u}_i \quad \text{on} \quad \bar{\Sigma}_1 \times [0, t_0), \qquad & t_i &= \tilde{t}_i \quad \text{on} \quad \Sigma_2 \times [0, t_0), \\ \varphi &= \tilde{\varphi} \quad \text{on} \quad \bar{\Sigma}_3 \times [0, t_0), \qquad & h &= \tilde{h} \quad \text{on} \quad \Sigma_4 \times [0, t_0), \\ \theta &= \tilde{\theta} \quad \text{on} \quad \bar{\Sigma}_5 \times [0, t_0), \qquad & q &= \tilde{q} \quad \text{on} \quad \Sigma_6 \times [0, t_0), \end{aligned} \tag{2.26}$$

where $\tilde{u}_i$, $\tilde{t}_i$, $\tilde{\varphi}$, $\tilde{h}$, $\tilde{\theta}$, $\tilde{q}$ are prescribed functions and t_0 is some instant that may be infinite.

The basic equations of the theory are: the equation of motion (2.4), the balance of equilibrated forces (2.11), the energy equation (2.25), the constitutive equations (2.22), the geometrical equations (2.10). To the system of field equations we adjoin the boundary conditions (2.26) and the following initial conditions

$$\begin{aligned} &u_i(x, 0) = u_i^0(x), \qquad \dot{u}_i(x, 0) = v_i^0(x), \\ &\varphi(x, 0) = \varphi^0(x), \qquad \dot{\varphi}(x, 0) = \zeta^0(x), \\ &\theta(x, 0) = \theta^0(x), \qquad \eta(x, 0) = \eta^0(x), \qquad x \in \bar{B}, \end{aligned} \tag{2.27}$$

where u_i^0, v_i^0, φ^0, ζ^0, θ^0, η^0 are prescribed functions.

From (2.4), (2.11), (2.24) and (2.25) we obtain the field equations in terms of the displacement, volume fraction and temperature, for homogeneous and isotropic bodies

$$\begin{aligned} &\mu \Delta u_i + (\lambda + \mu)\, u_{r,ri} + b\varphi_{,i} - \beta\theta_{,i} + \varrho f_i = \varrho \ddot{u}_i, \\ &\alpha \Delta \varphi - b u_{r,r} - \xi\varphi + m\theta + \varrho l = \varrho \varkappa \ddot{\varphi}, \\ &k \Delta \theta - \beta T_0 \dot{u}_{r,r} - a T_0 \dot{\theta} - m T_0 \dot{\varphi} = -\varrho s, \end{aligned} \tag{2.28}$$

where Δ is the Laplacian. Let us introduce the notations

$$\begin{aligned} &c_1^2 = (\lambda + 2\mu)/\varrho, \qquad c_2^2 = \mu/\varrho, \\ &c_3^2 = \alpha/\varrho\varkappa, \qquad c = a T_0. \end{aligned} \tag{2.29}$$

The Eqs. (2.28) may be written in the alternative form

$$\begin{aligned} &c_2^2 \Delta u_i + (c_1^2 - c_2^2)\, u_{j,ji} + \frac{b}{\varrho}\, \varphi_{,i} - \frac{\beta}{\varrho}\, \theta_{,i} + f_i = \ddot{u}_i, \\ &c_3^2 \Delta \varphi - \frac{b}{\varrho\varkappa}\, u_{r,s} - \frac{\xi}{\varrho\varkappa}\, \varphi + \frac{m}{\varrho\varkappa} \theta + \frac{1}{\varkappa}\, l = \ddot{\varphi}, \\ &k \Delta \theta - \beta T_0 \dot{u}_{r,r} - m T_0 \dot{\varphi} - c\dot{\theta} = -\varrho s. \end{aligned} \tag{2.30}$$

3. Uniqueness and Variational Theorems

In this section we establish some general theorems in the dynamic theory of thermoelastic materials with voids.

Theorem 3.1. Suppose that

$$2W = C_{ijrs}e_{ij}e_{rs} + 2B_{ij}e_{ij}\varphi + 2D_{ijr}e_{ij}\varphi_{,r} + 2d_i\varphi\varphi_{,i} + \xi\varphi^2 + A_{ij}\varphi_{,i}\varphi_{,j} \geqq 0, \tag{3.1}$$

$$\varrho > 0, \qquad \varkappa > 0, \qquad a > 0, \qquad T_0 > 0.$$

Then the boundary-initial-value problem of thermoelasticity has at most one solution.

Proof. With the help of (2.22) we obtain

$$t_{ij}\dot{e}_{ij} + h_i\dot{\varphi}_{,i} - g\dot{\varphi} + \varrho\theta\dot{\eta} = \dot{W} + a\theta\dot{\theta}. \tag{3.2}$$

On the other hand, from (2.4), (2.10), (2.11) and (2.25), we find that

$$\begin{aligned} &t_{ij}\dot{e}_{ij} + h_i\dot{\varphi}_{,i} - g\dot{\varphi} + \varrho\theta\dot{\eta} \\ &= \left(t_{ij}\dot{u}_i + h_j\dot{\varphi} + \frac{1}{T_0}q_j\theta\right)_{,j} + \varrho\left(f_i\dot{u}_i + l\dot{\varphi} + \frac{1}{T_0}s\theta\right) \\ &\quad - \frac{1}{T_0}q_i\theta_{,i} - \varrho(\dot{u}_i\ddot{u}_i + \varkappa\dot{\varphi}\ddot{\varphi}). \end{aligned} \tag{3.3}$$

By the divergence theorem and (2.3), (2.15) it follows that

$$\begin{aligned} &\int_B (t_{ij}\dot{e}_{ij} + h_i\dot{\varphi}_{,i} - g\dot{\varphi} + \varrho\theta\dot{\eta})\,dV \\ &= \int_{\partial B}\left(t_i\dot{u}_i + h\dot{\varphi} + \frac{1}{T_0}q\theta\right)dA + \int_B \varrho\left(f_i\dot{u}_i + l\dot{\varphi} + \frac{1}{T_0}s\theta\right)dV \\ &\quad - \int_B \varrho(\dot{u}_i\ddot{u}_i + \varkappa\dot{\varphi}\ddot{\varphi})\,dV - \frac{1}{T_0}\int_B q_i\theta_{,i}\,dV. \end{aligned} \tag{3.4}$$

If we introduce the *total energy* U on $[0, t_0)$ by

$$U = \frac{1}{2}\int_B (\varrho\dot{u}_i\dot{u}_i + \varkappa\dot{\varphi}^2 + 2W + a\theta^2)\,dV, \tag{3.5}$$

then, from (3.2), (3.4) and (2.19) we conclude that

$$\dot{U} - \int_B \varrho \left(f_i \dot{u}_i + l\dot{\varphi} + \frac{1}{T_0} s\theta\right) dV - \int_{\partial B} \left(t_i \dot{u}_i + h\dot{\varphi} + \frac{1}{T_0} q\theta\right) dA$$
$$= -\frac{1}{T_0} \int_B q_i \theta_{,i} \, dV \leqq 0. \tag{3.6}$$

Suppose that there are two solutions $\{u_i^{(\alpha)}, \varphi^{(\alpha)}, \theta^{(\alpha)}, e_{ij}^{(\alpha)}, \eta^{(\alpha)}, t_{ij}^{(\alpha)}, h_i^{(\alpha)}, g^{(\alpha)}, q_i^{(\alpha)}\}$, $(\alpha = 1, 2)$. Then their difference $\overline{\Pi} = \{\bar{u}_i = u_i^{(1)} - u_i^{(2)}, \ \bar{\varphi} = \varphi^{(1)} - \varphi^{(2)}, \bar{\theta} = \theta^{(1)} - \theta^{(2)}, \ldots, \bar{q}_i = q_i^{(1)} - q_i^{(2)}\}$ corresponds to null data. If $\overline{U}$ is the total energy corresponding to $\overline{\Pi}$ then from (3.6) we obtain

$$\overline{U}(t) \leqq \overline{U}(0), \qquad 0 \leqq t < t_0 .$$

The initial conditions imply $\overline{U}(0) = 0$. By hypothesis we find $\overline{U}(t) = 0, 0 \leqq t < t_0$. Hence $\dot{\bar{u}}_i = 0$, $\dot{\bar{\varphi}} = 0$, $\bar{\theta} = 0$ on $\overline{B} \times [0, t_0)$. But $\bar{u}_i$ and $\bar{\varphi}$ vanish initially; thus $\bar{u}_i = 0$, $\bar{\varphi} = 0$ on $B \times [0, t_0)$.

The uniqueness of the solution in the theory of elastic materials with voids has been proved in [4].

With a view toward establishing the variational characterization of the solution we first give an alternative formulation of the boundary-initial-value problem and prove a reciprocity relation.

Let u and v be scalar fields on $B \times [0, t_0)$ that are continuous in time. We denote by $u * v$ the convolution of u and v

$$[u * v]\,(x, t) = \int_0^t u(x, t - \tau)\, v(x, \tau)\, d\tau .$$

Let us introduce the notations

$$\begin{gathered} e(t) = t, \qquad j(t) = 1, \\ F_i = \varrho(e * f_i + t v_i^0 + u_i^0), \\ G = \varrho[e * l + \varkappa(t\zeta^0 + \varphi^0)], \\ S = \varrho j * s + \varrho T_0 \eta^0 . \end{gathered} \tag{3.7}$$

Following [9, p. 337], [10, p. 370] one can prove

Theorem 3.2. The functions u_i, φ, η, t_{ij}, h_i, g, q_i satisfy the Eqs. (2.4), (2.11), (2.25) and the initial conditions (2.27) if, and only if

$$\begin{gathered} e * t_{ji,j} + F_i = \varrho u_i, \\ e * (h_{i,i} + g) + G = \varrho \varkappa \varphi, \\ j * q_{i,i} + S = \varrho T_0 \eta . \end{gathered} \tag{3.8}$$

This theorem enables us to give an alternative formulation of the boundary-initial-value problem in which the initial conditions are incorporated into the field equations. Thus, the admissible process $\Pi = \{u_i, \varphi, \theta, e_{ij}, \eta, t_{ij}, h_i, g, q_i\}$ is a solution of the boundary-initial-value problem if and only if Π satisfies the Eqs. (2.10), (2.22), (3.8) and the boundary conditions (2.26).

Let us consider the body subjected to two-different systems of loadings $L^{(\alpha)} = \{f_i^{(\alpha)}, l^{(\alpha)}, s^{(\alpha)}, \tilde{u}_i^{(\alpha)}, \tilde{t}_i^{(\alpha)}, \tilde{\varphi}^{(\alpha)}, \tilde{h}^{(\alpha)}, \tilde{\theta}^{(\alpha)}, \tilde{q}^{(\alpha)}, u_i^{0(\alpha)}, v_i^{0(\alpha)}, \zeta^{0(\alpha)}, \eta^{0(\alpha)}\}$ $(\alpha = 1, 2)$, and the two corresponding solutions $\Pi^{(\alpha)} = \{u_i^{(\alpha)}, \varphi^{(\alpha)}, \theta^{(\alpha)}, \ldots, q_i^{(\alpha)}\}$. The functions F_i, G, S corresponding to the system $L^{(\alpha)}$ will be denoted by $F_i^{(\alpha)}$, $G^{(\alpha)}$, $S^{(\alpha)}$.

Theorem 3.3. (Reciprocal theorem). If a thermoelastic solid with symmetric conductivity tensor is subjected to two systems of loadings $L^{(\alpha)}$ $(\alpha = 1, 2)$, then between the corresponding solutions $\Pi^{(\alpha)}$ there is the following reciprocity relation

$$\begin{aligned}
&\int_B \left(F_i^{(1)} * u_i^{(2)} + G^{(1)} * \varphi^{(2)} - \frac{1}{T_0}\, e * S^{(1)} * \theta^{(2)}\right) dV \\
&\quad + \int_{\partial B} e * \left(t_i^{(1)} * u_i^{(2)} + h^{(1)} * \varphi^{(2)} - \frac{1}{T_0}\, j * q^{(1)} * \theta^{(2)}\right) dA \\
&= \int_B \left(F_i^{(2)} * u_i^{(1)} + G^{(2)} * \varphi^{(1)} - \frac{1}{T_0}\, e * S^{(2)} * \theta^{(1)}\right) dV \\
&\quad + \int_{\partial B} e * \left(t_i^{(2)} * u_i^{(1)} + h^{(2)} * \varphi^{(1)} - \frac{1}{T_0}\, j * q^{(2)} * \theta^{(1)}\right) dA .
\end{aligned} \tag{3.9}$$

Proof. On the basis of the relations (2.23) we conclude from the constitutive equations that

$$\begin{aligned}
&t_{ij}^{(1)} * e_{ij}^{(2)} + h_i^{(1)} * \varphi_{,i}^{(2)} - g^{(1)} * \varphi^{(2)} + \varrho\theta^{(1)} * \eta^{(2)} \\
&= t_{ij}^{(2)} * e_{ij}^{(1)} + h_i^{(2)} * \varphi_{,i}^{(1)} - g^{(2)} * \varphi^{(1)} + \varrho\theta^{(2)} * \eta^{(1)} .
\end{aligned} \tag{3.10}$$

If we introduce the notation

$$\begin{aligned}
I_{\alpha\beta} = \int_B e * (t_{ij}^{(\alpha)} * e_{ij}^{(\beta)} + h_i^{(\alpha)} * \varphi_{,i}^{(\beta)} - g^{(\alpha)} * \varphi^{(\beta)} \\
- \varrho\eta^{(\alpha)} * \theta^{(\beta)})\, dV ,
\end{aligned} \tag{3.11}$$

then from (3.10) we get

$$I_{12} = I_{21} . \tag{3.12}$$

By using (2.10) and (3.8) we can write

$$\begin{aligned}
& e * (t_{ij}^{(\alpha)} * e_{ij}^{(\beta)} + h_i^{(\alpha)} * \varphi_{,i}^{(\beta)} - g^{(\alpha)} * \varphi^{(\beta)} - \varrho\eta^{(\alpha)} * \theta^{(\beta)}) \\
& = e * \left(t_{ki}^{(\alpha)} * u_i^{(\beta)} + h_k^{(\alpha)} * \varphi^{(\beta)} - \frac{1}{T_0} j * q_k^{(\alpha)} * \theta^{(\beta)}\right)_{,k} \\
& + F_i^{(\alpha)} * u_i^{(\beta)} + G^{(\alpha)} * \varphi^{(\beta)} - \frac{1}{T_0} e * S^{(\alpha)} * \theta^{(\beta)} \\
& - \varrho(u_i^{(\alpha)} * u_i^{(\beta)} + \varkappa\varphi^{(\alpha)} * \varphi^{(\beta)}) + \frac{1}{T_0} e * j * k_{rs}\theta_{,s}^{(\alpha)} * \theta_{,r}^{(\beta)}.
\end{aligned} \tag{3.13}$$

In view of the divergence theorem we find that

$$\begin{aligned}
I_{\alpha\beta} = & \int_B \left(F_i^{(\alpha)} * u_i^{(\beta)} + G^{(\alpha)} * \varphi^{(\beta)} - \frac{1}{T_0} e * S^{(\alpha)} * \theta^{(\beta)}\right) dV \\
& + \int_{\partial B} e * \left(t_i^{(\alpha)} * u_i^{(\beta)} + h^{(\alpha)} * \varphi^{(\beta)} - \frac{1}{T_0} j * q^{(\alpha)} * \theta^{(\beta)}\right) dA \\
& - \int_B \left(\varrho u_i^{(\alpha)} * u_i^{(\beta)} + \varrho\varkappa\varphi^{(\alpha)} * \varphi^{(\beta)} - \frac{1}{T_0} e * j * k_{rs}\theta_{,r}^{(\alpha)} * \theta_{,s}^{(\beta)}\right) dV.
\end{aligned} \tag{3.14}$$

Finally, (3.12) and (3.14) imply the desired result.

This type of reciprocal theorem in the thermoelastodynamics has been established in [11].

From (2.10), (2.22) and (3.8) we obtain the field equations in terms of the displacement, volume fraction and temperature. These equations can be written in the form

$$\mathscr{L}u = \mathscr{F}, \tag{3.15}$$

where the five-dimensional vectors u, $\mathscr{L}u$, $\mathscr{F}$ are defined by

$$\begin{aligned}
& u = (u_1, u_2, u_3, \varphi, \theta), \qquad \mathscr{F} = \left(F_1, F_2, F_3, G, -\frac{1}{T_0} e * S\right), \\
& \mathscr{L}_i u = \varrho u_i - e * (C_{ijrs}u_{r,s} + B_{ij}\varphi + D_{ijk}\varphi_{,k} - \beta_{ij}\theta)_{,j}, \\
& \mathscr{L}_4 u = \varrho\varkappa\varphi - e * [(A_{ij}\varphi_{,j} + D_{rsi}u_{r,s} + d_i\varphi - a_i\theta)_{,i} \\
& \qquad - B_{ij}u_{i,j} - \xi\varphi - d_i\varphi_{,i} + m\theta], \\
& \mathscr{L}_5 u = e * \left[\frac{1}{T_0} j * (k_{rs}\theta_{,s})_{,r} - \beta_{rs}u_{r,s} - a\theta - m\varphi - a_i\varphi_{,i}\right].
\end{aligned} \tag{3.16}$$

In what follows we consider the case of homogeneous boundary conditions, i.e. $\tilde{t}_i = \tilde{u}_i = \tilde{h} = \tilde{\varphi} = \tilde{q} = \tilde{\theta} = 0$. If $\hat{a}$ and $\hat{b}$ are five-dimensional vectors, then

$$\hat{a} * \hat{b} = \sum_{i=1}^{5} a_i * b_i.$$

If we introduce the notations $u = (u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, \varphi^{(1)}, \theta^{(1)})$, $v = (u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, \varphi^{(2)}, \theta^{(2)})$ then by (3.15), (3.16) the relation (3.9) can be written in the form

$$\int_B u * \mathcal{L} v \, dV = \int_B v * \mathcal{L} u \, dV .$$

This relation shows that the operator $\mathcal{L}$ is symmetric in convolution. Let us denote by $\mathcal{D}$ the domain of the definition of the operator $\mathcal{L}$. Following [12], we are led to the variational theorem

Theorem 3.4. Let $\mathcal{K} \subset \mathcal{D}$ be the set of all admissible vectors $u = (u_i, \varphi, \theta)$ which satisfy the homogeneous boundary conditions, and for each $t \in [0, t_0)$ define the functional $\Gamma_t\{\cdot\}$ on $\mathcal{K}$ by

$$\begin{aligned} \Gamma_t\{u\} = \int_B e * (C_{ijrs} u_{r,s} * u_{i,j} + 2B_{ij}\varphi * u_{i,j} \\ + 2D_{ijk} u_{i,j} * \varphi_{,k} + 2d_i \varphi * \varphi_{,i} + \xi \varphi * \varphi + A_{ij} \varphi_{,i} * \varphi_{,j} \\ - 2\beta_{ij} u_{i,j} * \theta - 2m\theta * \varphi - 2a_i \varphi_{,i} * \theta - a\theta * \theta) \, dV \\ + \int_B \varrho (u_i * u_i + \varkappa \varphi * \varphi) \, dV \\ - 2 \int_B \left(F_i * u_i + G * \varphi - \frac{1}{T_0} e * S * \theta \right) dV \\ - \frac{1}{T_0} \int_B e * j * k_{rs} \theta_{,r} * \theta_{,s} \, dV, \end{aligned} \tag{3.17}$$

for every $u \in \mathcal{K}$. Then

$$\delta \Gamma_t\{u\} = 0, \qquad t \in [0, t_0),$$

at $u \in \mathcal{K}$ if and only if u is a solution of the boundary-initial-value problem with homogeneous boundary conditions.

The Theorem 3.4 is a variational theorem of Gurtin type. In the classical theory of thermoelastodynamics, the variational theorems of this type are presented in [9, p. 338], [10, p. 370]. Variational theorems for nonhomogeneous boundary conditions can be derived by the method given in [12].

4. Acceleration Waves

Let Σ be a moving surface defined by

$$x_i = x_i(\theta^1, \theta^2, t), \tag{4.1}$$

where θ^1, θ^2 are curvilinear coordinate on the surface. We assume that the functions (4.1) are continuously differentiable with respect to their arguments and that Σ is smooth in the sense that the matrix $(\partial x_i / \partial \theta^\alpha)$ has rank two. The metric tensor of the surface is given by $a_{\alpha\beta} = x_{i,\alpha} x_{i,\beta}$.

In this section we will denote by n_i the unit normal to Σ. We note that [14]

$$n_i n_i = 1, \qquad n_i x_{i;\alpha} = 0, \qquad x_{i;\alpha\beta} = b_{\alpha\beta} n_i, \qquad n_{i;\alpha} = -a^{\lambda\varrho} b_{\varrho\alpha} x_{i;\lambda}, \tag{4.2}$$

where indices followed by a semicolon represents covariant partial differentiation based on the metric of Σ, $b_{\alpha\beta}$ is the second fundamental form of the surface and $a^{\alpha\beta}$ are the elements of the inverse of matrix $(a_{\alpha\beta})$. We also note that

$$a^{\alpha\beta} x_{i;\alpha} x_{j;\beta} = \delta_{ij} - n_i n_j, \qquad H = \frac{1}{2} a^{\alpha\beta} b_{\alpha\beta}, \tag{4.3}$$

where H is the mean curvature of the surface.

The propagating surface Σ is said to be an acceleration wave if u_i, φ, θ have the following properties:

i) u_i, φ, θ, $\dot{u}_i$, $\dot{\varphi}$, $\dot{\theta}$, $u_{i,j}$, $\varphi_{,i}$, $\theta_{,i}$ are continuous functions everywhere;

ii) $\ddot{u}_i$, $\ddot{\varphi}$, $\ddot{\theta}$, $u_{i,jk}$, $\varphi_{,ij}$, $\theta_{,ij}$, $\dot{u}_{i,j}$, $\dot{\varphi}_{,i}$, $\dot{\theta}_{,i}$ and all higher-order derivatives of these quantities have, at most, jump discontinuities across Σ but are continuous functions everywhere else.

In this section we study the acceleration waves in homogeneous and isotropic bodies. Let f be one of the functions u_i, φ, θ. We list for future reference the compatibility conditions satisfied by the jumps on Σ of the second and third derivatives of this function

$$[f_{,ij}] = C n_i n_j, \qquad [\dot{f}_{,i}] = -CV n_i, \qquad [\ddot{f}] = V^2 C,$$

$$[f_{,kij}] = a^{\alpha\beta}(C n_k)_{;\alpha}(n_i x_{j;\beta} + n_j x_{i;\beta}) - a^{\alpha\beta} a^{\nu\varrho} b_{\alpha\nu} C n_k x_{i;\beta} x_{j;\varrho} + [f_{,kpq}] n_p n_q n_i n_j,$$

$$[\dot{f}_{,si}] = -a^{\alpha\beta}(VC n_s)_{;\alpha} x_{i;\beta} + n_i \frac{\delta}{\delta t}(C n_s) - V[f_{,spq}] n_p n_q n_i, \tag{4.4}$$

$$[\ddot{f}_{,s}] = -C n_s \frac{\delta V}{\delta t} - 2V \frac{\delta}{\delta t}(C n_s) + V^2 [f_{,sij}] n_i n_j,$$

$$[f_{,jjs}] = -2HC n_s + [f_{,sij}] n_i n_j,$$

where

$$\frac{\delta}{\delta t} = \frac{\partial}{\partial t} + V n_i \frac{\partial}{\partial x_i},$$

is the displacement derivative, V is the speed of propagation of Σ in the direction of the normal $\boldsymbol{n}$, and $C = [f_{,ij} n_i n_j]$. A detailed account of compatibility conditions holding on singular surfaces has been given in [15], [16, pp. 491—529]. Derivation of the Eqs. (4.4) can be constructed in a straightforward manner from these sources. We assume that $f_i = l = s = 0$. On forming the jump of each term in the Eq. (2.30) on the singular surface Σ and using the compatibility condition

(4.4), we obtain

$$(V^2 - c_2^2)\,\lambda_i - (c_1^2 - c_2^2)\,\lambda_j n_j n_i = 0, \tag{4.5}$$

$$(V^2 - c_3^2)\,\eta = 0, \tag{4.6}$$

$$k\zeta + \beta T_0 V \lambda_i n_i = 0, \tag{4.7}$$

where $\lambda_i = [u_{i,rs} n_r n_s]$, $\eta = [\varphi_{,rs} n_r n_s]$, $\zeta = [\theta_{,ij} n_i n_j]$.

Equations (4.5) admit a non trivial solution for λ_i if, and only if

$$(c_2^2 - V^2)^2\,(c_1^2 - V^2) = 0.$$

It is an immediate consequence of Eq. (4.5) that longitudinal waves of first kind (for which $\lambda_i = A n_i$) propagate with the speed c_1. The speed of propagation of transverse waves (for which $\lambda_i n_i = 0$) is c_2. From Eq. (4.7) we note that $\zeta = -\beta T_0 \cdot c_1 A/k$ for a longitudinal wave of first kind, and $\zeta = 0$ for a transverse wave.

If $\eta \neq 0$, the wave is an acceleration wave of compaction or distension. This wave is called the longitudinal wave of second kind. The possible speed of propagation of this wave is c_3.

Let us study now the growth of the acceleration waves. We first apply to each term of Eqs. (2.30) the operator $\partial/\partial x_s$ and form jumps across the wave Σ. Next we make use of the compatibility condition (4.4), then multiply by n_s and sum on the repeated index s. We obtain the equations

$$\begin{aligned}
&(V^2 - c_2^2)\,\mu_i - (c_1^2 - c_2^2)\,\{\mu_j n_j n_i + a^{\alpha\beta} x_{i;\beta}(\lambda_{j;\alpha} n_j - a^{\nu\varrho} b_{\alpha\nu} \lambda_j x_{j;\varrho}) \\
&\quad + a^{\alpha\beta} \lambda_{j;\alpha} n_i x_{j;\beta}\} - 2V \frac{\delta \lambda_i}{\delta t} + 2H c_2^2 \lambda_i - \frac{b}{\varrho}\,\eta n_i + \frac{\beta}{\varrho}\,\zeta n_i = 0, \\
&(V^2 - c_3^2)\,\tau + 2H c_3^2 \eta - 2V \frac{\delta \eta}{\delta t} + \frac{b}{\varrho\varkappa}\,\lambda_j n_j = 0,
\end{aligned} \tag{4.8}$$

$$\beta T_0 \left(V \mu_j n_j - n_i\,\frac{\delta \lambda_i}{\delta t} + V a^{\alpha\beta} \lambda_{j;\alpha} x_{j;\beta} \right) + k\gamma + (cV - 2kH)\,\zeta + m T_0 V \eta = 0,$$

where

$$\mu_i = [u_{i,pqr}]\,n_p n_q n_r, \qquad \tau = [\varphi_{,ijk}]\,n_i n_j n_k, \qquad \gamma = [\theta_{,ijk}]\,n_i n_j n_k,$$

and simplifications have been effected with the aid of Eqs. (4.2) and the fact that V is constant for acceleration waves. It is known [17, pp. 43—45] that $\delta n_i/\delta t = -a^{\alpha\beta} x_{i;\alpha} V_{,\beta}$. Using this result and the fact that V is constant for all waves we find

$$\frac{\delta A}{\delta t} = n_j \frac{\delta \lambda_j}{\delta t}. \tag{4.9}$$

With the help of (4.2) and (4.3) we obtain

$$a^{\alpha\beta} \lambda_{j;\alpha} x_{j;\beta} = a^{\alpha\beta} (\lambda_j x_{j;\beta})_{;\alpha} - 2HA, \qquad \lambda_{j;\alpha} n_j - \lambda_j a^{\nu\varrho} b_{\alpha\nu} x_{j;\varrho} = A_{,\alpha}. \tag{4.10}$$

Using (4.9) and (4.10), we can write (4.8) in the form

$$(V^2 - c_2^2)\,\mu_i - (c_1^2 - c_2^2)\,\{\mu_j n_j n_i + a^{\alpha\beta}(\lambda_j x_{j;\beta})_{;\alpha}\, n_i + a^{\alpha\beta} \Lambda_{,\alpha} x_{i;\beta}\}$$

$$- 2V \frac{\delta \lambda_i}{\delta t} + 2H\{c_2^2 \lambda_i + (c_1^2 - c_2^2)\,\Lambda n_i\} - \frac{b}{\varrho}\,\eta n_i + \frac{\beta}{\varrho}\,\zeta n_i = 0,$$

$$(V^2 - c_3^2)\,\tau - 2V \frac{\delta \eta}{\delta t} + 2c_3^2 H \eta + \frac{b}{\varrho \varkappa}\,\Lambda = 0, \qquad (4.11)$$

$$\beta T_0 V \mu_j n_j + k\gamma - \beta T_0 \left\{\frac{\delta \Lambda}{\delta t} - V a^{\alpha\beta}(\lambda_j x_{j;\beta})_{;\alpha} + 2HV\Lambda\right\}$$

$$+ (cV - 2Hk)\,\zeta + mT_0 V \eta = 0.$$

On multiplying throughout Eq. $(4.11)_1$ by n_i, summing on the repeated index i and using Eqs. (4.2) and (4.9), we obtain

$$(V^2 - c_1^2)\,\mu_j n_j - (c_1^2 - c_2^2)\, a^{\alpha\beta}(\lambda_j x_{j;\beta})_{;\alpha} - 2V \frac{\delta \Lambda}{\delta t} + 2Hc_1^2 \Lambda - \frac{b}{\varrho}\,\eta + \frac{\beta}{\varrho}\,\zeta = 0. \qquad (4.12)$$

In what follows we assume that $c_1 \neq c_2$, $c_1 \neq c_3$, $c_2 \neq c_3$. In the case of longitudinal waves of first kind, $V = c_1$, $\lambda_i = \Lambda n_i$, $\zeta = -\beta T_0 c_1 \Lambda / k$, and $\eta = 0$.

The Eq. (4.12) yields the growth equation

$$\frac{1}{c_1} \frac{\delta \Lambda}{\delta t} = \frac{d\Lambda}{dn} = \Lambda \left(H - \frac{\varepsilon \omega^*}{2c_1}\right), \qquad (4.13)$$

where n is the distance measured along the normal to the wave, measured from the wavefront at $t = t_0$, and

$$\varepsilon = \beta^2 T_0 / c\varrho c_1^2, \qquad \omega^* = cc_1^2/k.$$

In view of (4.13), Eqs. (4.11) reduce to

$$\mu_i - \mu_j n_j n_i = a^{\alpha\beta} \Lambda_{,\alpha} x_{i;\beta}, \qquad (4.14.1)$$

$$(c_1^2 - c_3^2)\,\tau = \frac{b}{\varrho \varkappa}\,\Lambda, \qquad (4.14.2)$$

$$\beta T_0 c_1 \mu_j n_j + k\gamma = \beta T_0 c_1 \Lambda \left\{H + \frac{\omega^*}{c_1}\left(1 - \frac{1}{2}\,\varepsilon\right)\right\}. \qquad (4.14.3)$$

Suppose that, at $t = t_0$, the mean and Gaussian curvature of the wavefront are H_0 and K_0, respectively. Then, the mean curvature at a subsequent time t is given by [14, pp. 108—113]

$$H = (H_0 - nK_0)/(1 - 2nH_0 + n^2 K_0),$$

and the transport Eq. (4.13) can be integrated to give the decay law for Λ,

$$\Lambda = \Lambda_0(1 - 2nH_0 + n^2K_0)^{-1/2} \exp(-\varepsilon\omega^* n/2c_1), \tag{4.15}$$

where Λ_0 is the strength of the wave at time $t = t_0$. From (4.14.2) and (4.15) we obtain

$$\tau = \frac{b\Lambda_0}{\varrho\varkappa(c_1^2 - c_3^2)} (1 - 2nH_0 + n^2K_0)^{-1/2} \exp(-\varepsilon\omega^* n/2c_1), \tag{4.16}$$

so that a disturbance in the porosity, which is of third order is induced by a longitudinal wave of first kind. This discontinuity propagates with the same speed as the inducing longitudinal wave. The decay law (4.15) contains the exponential factor $\exp(-\varepsilon\omega^* n/2c_1)$ which ensures that the strength Λ, and hence the jumps λ_i, ζ, τ tend to zero as the interval $t - t_0$ increases indefinitely.

It follows that λ_i, ζ, τ are completely determined if Λ_0 is known. To determine the jumps μ_i and γ, we see from Eqs. (4.14.1, 3) that it is necessary to have prior knowledge of one of these jumps.

In the case of transverse wave we have $V = c_2$, $\Lambda = 0$, $\eta = 0$, $\zeta = 0$ and the Eq. (4.12) reduces to

$$\mu_j n_j = -a_{\alpha\beta}(\lambda_r x_{r;\beta})_{;\alpha}.$$

Using this result in (4.11), we get

$$\frac{1}{c_2} \frac{\delta\lambda_i}{\delta t} = H\lambda_i, \tag{4.17}$$

together with the result

$$\tau = \gamma = 0.$$

As before, we have

$$\lambda_i = \lambda_i^0(1 - 2nH_0 + n^2K_0)^{-1/2},$$

where $\lambda_i = \lambda_i^0$ when $t = t_0$. To determine the jumps μ_j it is necessary to have prior knowledge of two of the jumps μ_i.

In the case of an acceleration wave of compaction or distension we have $V = c_3$, $\lambda_i = 0$, $\zeta = 0$, and the Eq. (4.12) reduces to

$$(c_3^2 - c_1^2)\,\mu_j n_j = \frac{b}{\varrho}\,\eta. \tag{4.18}$$

The Eq. (4.11) become

$$\{(c_3^2 - c_2^2)\,\delta_{ij} - (c_1^2 - c_2^2)\,n_j n_i\}\,\mu_j = \frac{b}{\varrho}\,\eta n_i,$$

$$\frac{1}{c_3} \frac{\delta\eta}{\delta t} = H\eta, \tag{4.19}$$

$$\beta T_0 c_3 \mu_j n_j + k\gamma + mT_0 c_3 \eta = 0.$$

The solution of the Eqs. (4.18), (4.19) is

$$\eta = \eta_0(1 - 2nH_0 + n^2K_0)^{-1/2},$$

$$\mu_i = \frac{bn_i\eta_0}{\varrho(c_3^2 - c_1^2)}(1 - 2nH_0 + n^2K_0)^{-1/2}, \tag{4.20}$$

$$\gamma = -\frac{mc_3T_0\eta_0}{k}\left(1 + \frac{\beta b}{m(c_3^2 - c_1^2)}\right)(1 - 2nH_0 + n^2K_0)^{-1/2},$$

where $\eta = \eta_0$ when $t = t_0$. We note that an acceleration wave of compaction or distension is accompanied by third order discontinuities in mechanical and thermal fields.

5. Equilibrium Theory

Let us consider the linear theory of thermoelastostatics for homogeneous and isotropic materials with voids. As in the classical theory we assume that the mechanical loadings are absent, the principal attention being devoted to the deformation due to the temperature field. The Eq. (2.28) reduce to

$$\begin{aligned} &\mu \Delta u_i + (\lambda + \mu)\, u_{r,ri} + b\varphi_{,i} - \beta\theta_{,i} = 0, \\ &\alpha\Delta\varphi - bu_{r,r} - \xi\varphi + m\theta = 0, \end{aligned} \tag{5.1}$$

and

$$k\Delta\theta = -\varrho s, \tag{5.2}$$

in B. Let us consider the boundary conditions

$$t_{ji}n_j = 0, \qquad \frac{\partial\varphi}{\partial n} = 0, \qquad \theta = \tilde{\theta} \text{ on } \partial B. \tag{5.3}$$

The Eq. (5.2) with the corresponding boundary condition determine the temperature variation θ.

In what follows we study the effect of a concentrated source of heat in a body which occupies the entire space and the thermal stresses in a thick walled spherical shell and a hollow cylinder.

i) *Concentrated source of heat.* Let us consider the case of a concentrated source of heat applied at the point $y(y_i)$ of a body occupying the entire space. Let $s = Q\delta(x - y)$, where Q is a constant and $\delta(\cdot)$ is the Dirac delta. From (5.2) we get

$$\theta = \frac{\varrho Q}{4\pi kr}, \tag{5.4}$$

where $r^2 = (x_i - y_i)(x_i - y_i)$. We assume now that $u_i = \Phi_{,i}$ where Φ is an unknown function. The Eq. (5.1) are satisfied if the functions Φ and φ satisfy the equations

$$\Delta\Phi = \frac{1}{\lambda + 2\mu}(\beta\theta - b\varphi),$$
$$\Delta\varphi - \tau^2\varphi = -f\theta, \tag{5.5}$$

where

$$\tau^2 = \frac{1}{\alpha(\lambda + 2\mu)}[(\lambda + 2\mu)\xi - b^2],$$
$$f = \frac{m}{\alpha} - \frac{b\beta}{\alpha(\lambda + 2\mu)}. \tag{5.6}$$

It is known [6] that $\tau^2 > 0$. This is a consequence of the fact that the internal energy density is a positive definite quadratic form. From (5.4) and (5.5) we find

$$\Phi = \frac{\varrho Q}{4\pi k\tau^2(\lambda + 2\mu)}\left[\frac{\beta\xi - mb}{2\alpha}r - \frac{fb}{\tau^2 r}(1 - e^{-\tau r})\right],$$
$$\varphi = \frac{\varrho f Q}{4\pi k\tau^2 r}(1 - e^{-\tau r}). \tag{5.7}$$

Let us note that in the classical theory of thermoelasticity [18, p. 162] the function Φ has the form Cr where C is a constant. In the case of thermoelastic materials with voids the displacement field contains new terms characterizing the influence of the material porosity.

ii) *Thermal stresses in a thick walled spherical shell.* We assume that B is a thick walled shell. Let the internal and external radii of the shell be a_1 and a_2, respectively. We take the center of the shell to be at the origin. We assume that the heat source is absent, the flow of heat in the shell being produced by maintaining the outer surface of the shell at the constant temperature T_2 and the inner surface at the constant temperature T_1. In this case

$$\theta = \frac{1}{r}A_1 + A_2, \tag{5.8}$$

where $r^2 = x_i x_i$ and

$$A_1 = \frac{a_1 a_2(T_2 - T_1)}{a_2 - a_1}, \qquad A_2 = \frac{T_2 a_2 - T_1 a_1}{a_2 - a_1}.$$

We seek the solution of the system (5.1) in the form

$$u_i = x_i\psi(r), \qquad \varphi = \varphi(r). \tag{5.9}$$

The substitution from (5.9) in (5.1) yields the equations

$$(r^3\psi)' = \frac{r^2}{\lambda + 2\mu}(\beta\theta - b\varphi) + A_1 r^2,$$
$$\Delta\varphi - \tau^2\varphi = \frac{b}{\alpha} A_1 - f\theta, \tag{5.10}$$

where prime denotes the derivative with respect to r, and A_1 is an unknown constant. From (5.10) we obtain

$$\psi = \frac{\xi}{3\alpha\tau^2} A_1 + \frac{1}{r^3} A_2 + H(r),$$
$$\varphi = \varphi_0(r) + \frac{1}{\alpha\tau^2}(\alpha f\theta - bA_1), \tag{5.11}$$

where

$$\varphi_0 = \frac{1}{r}(B_1 e^{-\tau r} + B_2 e^{\tau r}),$$

$$H = \frac{\gamma}{r^3(\lambda + 2\mu)} \int_0^r r^2\theta(r)\, dr - \frac{b\varphi_0'(r)}{\tau^2(\lambda + 2\mu) r}, \tag{5.12}$$

$$\gamma = \frac{(\lambda + 2\mu)(\beta\xi - bm)}{(\lambda + 2\mu)\xi - b^2},$$

and A_2, B_1, B_2 are unknown constants.

A simple calculation shows that the stress $T_{rr} = t_{ij}n_i n_j$ in the radial direction $n_i = x_i/r$ is

$$T_{rr} = pA_1 - \frac{4\mu}{r^3} A_2 - 4\mu H(r), \tag{5.13}$$

where

$$p = \frac{1}{3\alpha\tau^2}[(3\lambda + 2\mu)\xi - 3b^2].$$

For the determination of the constants B_α we have the boundary conditions

$$\varphi'(r) = 0 \quad \text{for} \quad r = a_1, \qquad r = a_2, \tag{5.14}$$

and, on solving these equations, we get

$$B_1 = -\frac{fA_1}{\tau^2 D}[e^{\tau a_2}(1 + \tau a_2) - e^{\tau a_1}(1 + \tau a_1)],$$
$$B_2 = -\frac{fA_1}{\tau^2 D}[e^{-\tau a_1}(1 - \tau a_1) - e^{-\tau a_2}(1 - \tau a_2)], \tag{5.15}$$

where

$$D = e^{\tau(a_2-a_1)}(1 - \tau a_1)(1 + \tau a_2) - e^{-\tau(a_2-a_1)}(1 - \tau a_2)(1 + \tau a_1).$$

By using (5.8), we obtain

$$H = \frac{\gamma}{6(\lambda + 2\mu)}\left(\frac{3}{r}A_1 + 2A_2\right) + \frac{b}{\tau^2(\lambda + 2\mu)\,r^3}\left[B_1 e^{-\tau r}(\tau r + 1) - B_2 e^{\tau r}(\tau r - 1)\right]. \tag{5.16}$$

The surface of the shell is free of external loads if

$$T_{rr} = 0 \quad \text{for} \quad r = a_1, \qquad r = a_2. \tag{5.17}$$

From (5.13) and (5.17) we find that

$$A_1 = \frac{4\mu[a_2^3 H(a_2) - a_1^3 H(a_1)]}{p(a_2^3 - a_1^3)},$$
$$A_2 = \frac{a_1^3 a_2^3[H(a_2) - H(a_1)]}{a_2^3 - a_1^3}. \tag{5.18}$$

Let us note that $p > 0$. This is a consequence of the fact that the internal energy density is a positive definite quadratic form [4]. With the help of (5.14), we obtain

$$H(a_\alpha) = \frac{\gamma}{6(\lambda + 2\mu)}\left[3\left(\frac{1}{a_\alpha} - \frac{2bf}{a_\alpha^3\tau^4\gamma}\right)A_1 + 2A_2\right], \qquad (\alpha = 1, 2). \tag{5.19}$$

By using (5.19) we can express the constants A_α in terms of the temperatures T_1 and T_2. Thus, the functions ψ and φ are given by (5.11) where φ_0, H are defined by (5.12), (5.15), (5.16), and the constants A_α are uniquely determined from (5.18), (5.19).

The radial stress T_{rr} can be calculated from (5.13), (5.16), (5.18) and (5.19). The "hoop stress" $T_{\theta\theta}$, in the tangential direction, is

$$T_{\theta\theta} = (3\lambda + 2\mu)\,\psi + \lambda r\psi' - \beta\theta + b\varphi.$$

If $T_1 = T_2 = T^*$, then $A_1 = 0$, $A_2 = T^*$ and (5.15) implies $\varphi_0 = 0$. In this case we obtain

$$\theta = T^*, \qquad \psi = \frac{\beta\xi - mb}{(3\lambda + 2\mu)\,\xi - 3b^2}\,T^*, \qquad \varphi = \frac{[(3\lambda + 2\mu)\,m - 3b\beta]}{(3\lambda + 2\mu)\,\xi - 3b^2}\,T^*.$$

The radial displacement $u = u_i x_i / r$ is given by

$$u = \frac{\beta\xi - mb}{(3\lambda + 2\mu)\,\xi - 3b^2}\,T^* r.$$

Let us note that in the classical theory of thermoelasticity the radial displacement is $\beta T^* r/(3\lambda + 2\mu)$.

The result established here can be used in order to obtain the solution in the case of the elastic space with a spherical cavity.

iii) *Thermal stresses in a hollow cylinder.* Let B be a right hollow cylinder with the generic cross-section Σ. We assume that the domain Σ is bounded by two concentric circles of radius a_1 and a_2, where $a_1 < a_2$. The rectangular Cartesian coordinate frame is chosen such that the x_3 axis coincides with the center line of the cylinder. We assume that the heat source is absent. We suppose that the temperature on the inner surface of the cylinder is T_1 and that the temperature on the outer surface is T_2, where T_α are constants. In this case the body is in a state of thermoelastic plane strain, parallel to the x_1x_2-plane.

We have

$$\theta = G_1 \ln r + G_2, \tag{5.20}$$

where $r^2 = x_\alpha x_\alpha$ and

$$G_1 = \frac{T_2 - T_1}{\ln(a_2/a_1)}, \qquad G_2 = \frac{T_1 \ln a_2 - T_2 \ln a_1}{\ln(a_2/a_1)}.$$

We seek the solution in the form

$$u_\alpha = x_\alpha \chi(r), \qquad u_3 = 0, \qquad \varphi = \varphi(r). \tag{5.21}$$

The Eq. (5.1) are satisfied if the functions χ and φ satisfy the equations

$$\begin{aligned} (r^2\chi)' &= \frac{r}{\lambda + 2\mu}(\beta\theta - b\varphi) + C_1 r, \\ \Delta\varphi - \tau^2\varphi &= \frac{b}{\alpha} C_1 - f\theta, \end{aligned} \tag{5.22}$$

where C_1 is an unknown constant. From (5.22) we obtain

$$\begin{aligned} \chi &= \frac{\xi}{2\alpha\tau^2} C_1 + \frac{1}{r} C_2 + F(r), \\ \varphi &= \varphi_0(r) + \frac{1}{\alpha\tau^2}(\alpha f\theta - bC_1), \end{aligned} \tag{5.23}$$

where

$$\varphi_0 = D_1 I_0(\tau r) + D_2 K_0(\tau r),$$

$$F = \frac{\gamma}{(\lambda + 2\mu) r^2} \int_0^r r\theta \, dr - \frac{b\varphi_0'(r)}{(\lambda + 2\mu)\tau^2 r},$$

C_2, D_1, D_2 are unknown constants, and I_n, K_n are modified Bessel functions of order n.

The radial stress τ_{rr} is given by

$$\tau_{rr} = 2\left(\zeta C_1 - \frac{\mu}{r^2} C_2 - \mu F\right), \tag{5.24}$$

where

$$\zeta = \frac{(\lambda + \mu)\,\xi - b^2}{2\alpha\tau^2}.$$

The constants D_1 and D_2 are determined by thec onditions $\varphi'(a_\alpha) = 0$, $(\alpha = 1, 2)$. We obtain

$$D_1 = \frac{fG_1}{a_1 a_2 d}\,[a_2 K_1(\tau a_2) - a_1 K_1(\tau a_1)],$$

$$D_2 = \frac{fG_1}{a_1 a_2 d}\,[a_2 I_1(\tau a_2) - a_1 I_1(\tau a_1)],$$

where

$$d = I_1(\tau a_2)\,K_1(\tau a_1) - I_1(\tau a_1)\,K_1(\tau a_2).$$

The constants C_1 and C_2 are found from the condition that the surfaces $r = a_1$ and $r = a_2$ are free from forces. We find that

$$C_1 = \frac{\mu[a_2^2 F(a_2) - a_1^2 F(a_1)]}{\zeta(a_2^2 - a_1^2)},$$

$$C_2 = \frac{a_1^2 a_2^2 [F(a_2) - F(a_1)]}{a_2^2 - a_1^2}.$$

Thus, the functions χ and φ are determined. By a simple calculation we can find the other components of the stress tensor.

If $T_1 = T_2 = T^*$, then

$$\theta = T^*, \qquad \chi = \frac{(\beta\xi - mb)\,T^*}{2[(\lambda + \mu)\,\xi - b^2]}, \qquad \varphi = \frac{(\lambda + \mu)\,m - b\beta}{(\lambda + \mu)\,\xi - b^2}\,T^*,$$

and the radial displacement is given by

$$u = \frac{\beta\xi - mb}{2[(\lambda + \mu)\,\xi - b^2]}\,T^* r.$$

In the classical theory of thermoelasticity the radial displacement is $\beta T^* r/2(\lambda + \mu)$.

In the isothermal case, the problems of the thick walled spherical and circular cylinder shells under internal and external pressure are solved by Cowin and Puri [6].

6. Conclusions

The results established in this paper can be summarized as follows:

a) *General dynamic theory.* We have derived a linear theory of thermoelastic materials with voids. To obtain the field equations we have used the balance of energy, the entropy production inequality and the invariance requirements under superposed rigid body motions. Some basic theorems concerning the uniqueness appropriate to the fundamental boundary-initial-value problem, the reciprocity relation and the variational characterization of the solution are proved.

b) *Acceleration waves.* The propagation conditions and growth equations, which govern the propagation of acceleration waves in homogeneous and isotropic materials with voids, are derived and discussed. The couplings between the discontinuities are studied. In general, three speeds of propagation are possible:

$$c_1 = [(\lambda + 2\mu)/\varrho]^{1/2}, \qquad c_2 = (\mu/\varrho)^{1/2}, \qquad c_3 = (\alpha/\varrho\varkappa)^{1/2}.$$

It is shown that the longitudinal wave which propagates with the speed c_1 induces a disturbance in the porosity which is of third order. This discontinuity propagates with the same speed as the inducing longitudinal wave. The foregoing longitudinal wave is accompanied by a second order discontinuity in thermal field. The acceleration wave of compaction or distension, which propagates with the speed c_3, is accompanied by third order discontinuities in mechanical and thermal fields. The transverse wave propagates without affecting the temperature and the porosity of the material.

c) *Thermoelastostatics.* The solutions for the traditional problems of concentrated source of heat, the deformation of a thick walled spherical shell and a hollow cylinder have been developed for a thermoelastic material with voids. The salient feature of these solutions is that the displacement field, the temperature and the stresses contain new terms characterizing the influence of the material porosity and their values are therefore modified from the values predicted by the classical theory of thermoelasticity.

References

[1] Nunziato, J. W., Cowin, S. C.: A nonlinear theory of elastic materials with voids. Arch. Rational Mech. Anal. 72, 175—201 (1979).

[2] Jarić, J., Golubović, Z.: Theory of thermoelasticity of granular materials. Rev. Roum. Sci. Techn., Méc. Appl. 24, 793—805 (1979).

[3] Jarić, J., Ranković, S.: Acceleration waves in granular materials. Mehanika 6, 66—76 (1980).

[4] Cowin, S. C., Nunziato, J. W.: Linear elastic materials with voids. J. Elasticity 13, 125—147 (1983).

[5] Markov, K. Z.: On the dilatation theory of elasticity. ZAMM **61**, 349—358 (1981).
[6] Cowin, S. C., Puri, P.: The classical pressure vessel problems for linear elastic materials with voids. J. Elasticity **13**, 157—163 (1983).
[7] Ieşan, D.: Some results in the theory of elastic materials with voids. J. Elasticity **15**, 215—224 (1985).
[8] Green, A. E., Rivlin, R. S.: On Cauchy's equation of motion. ZAMP **15**, 290—294 (1964).
[9] Carlson, D. E.: Linear thermoelasticity, in: Handbuch der Physik VIa/2. Berlin—Heidelberg—New York: Springer 1972.
[10] Lebon, G.: Variational principles in thermomechanics, in: Recent developments in thermomechanics of solids (Eds.: Lebon, G., Perzyna, P.). Wien—New York: Springer 1980.
[11] Ieşan, D.: On the linear coupled thermoelasticity with two temperatures. ZAMP **21**, 583—591 (1970).
[12] Ieşan, D.: On reciprocity theorems and variational theorems in linear elastodynamics. Bull. Acad. Polon. Sci., Sér. Sci. Techn. **22**, 273—281 (1974).
[13] Gurtin, M. E.: Variational principles for linear elastodynamics. Arch. Rational Mech. Anal. **16**, 34—50 (1964).
[14] Thomas, T. Y.: Concepts from tensor analysis and differential geometry. New York, London: Academic Press 1961.
[15] Chadwick, P., Powdrill, B.: Singular surfaces in linear thermoelasticity. Int. J. Engng. Sci. **3**, 561—595 (1965).
[16] Truesdell, C., Toupin, R. A.: The classical field theories. Handbuch der Physik, III/1, Berlin, Göttingen, Heidelberg: Springer 1960.
[17] Thomas, T. Y.: Plastic flow and fracture in solids. New York, London: Academic Press 1961.
[18] Nowacki, W.: Thermoelasticity. Oxford: Pergamon Press 1962.

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