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Radial Functions and Regularity of Solutions to the Schrödinger Equation

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Abstract. Let f be a radial function and set $T^*f(x) = \sup_{0<t<1} |T_t f(x)|$, $x \in \mathbb{R}^n$, $n \geqslant 2$, where $(T_t f)\hat{}\,(\xi) = e^{it|\xi|^a}\hat{f}(\xi)$, $a > 1$. We show that, if B is the ball centered at the origin, of radius 100, then $\int_B |T^*f(x)|\,dx \leqslant c(\int |\hat{f}(\xi)|^2 (l + |\xi|^2)^s\,ds)^{1/2}$ if and only if $s \geqslant \frac{1}{4}$.

Let f belong to the Schwartz class $\mathscr{S}(\mathbb{R}^n)$ and consider

$$Tf(x) = \sup_{0<t<1} \Big| \int_{\mathbb{R}^n} e^{2\pi i x \xi} e^{it|\xi|^a} \hat{f}(\xi)\,d\xi \Big|, \quad x \in \mathbb{R}^n,\ n \geqslant 2,$$

for every $a > 1$, where

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i \xi x} f(x)\,dx.$$

Let $H_s(\mathbb{R}^n)$ denote the closure of $\{f \in \mathscr{S}: [(1 + |\xi|^2)^s \hat{f}(\xi)]\check{} \in L^2\}$ under the norm $\|f\|_{H_s}^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2\,d\xi$. We shall prove the following

Theorem. *Let us assume f radial and let $B = \{x: |x| \leqslant 100\}$. Then the inequality*

$$\|Tf\|_{L^1(B)} \leqslant c_n \|f\|_{H^s} \tag{1}$$

holds if and only if $s \geqslant \frac{1}{4}$.

In the particular case $a = 2$, the theorem has as a consequence that if radial f belongs to H^s, $s \geqslant \frac{1}{4}$, then

$$u(x, t) = \int_{\mathbb{R}^n} e^{2\pi i x \xi} e^{it|\xi|^2} \hat{f}(\xi)\,d\xi,$$

a solution to the Schrödinger equation $i\frac{\partial u}{\partial t} = \Delta u$, converges a.e. to $f(x)$ as $t \to 0$.

Inequality (1) and the a.e. convergence of $u(x, t)$ have been studied by several authors, namely L. CARLESON, B. DAHLBERG and C. KENIG, C. KENIG and A. RUIZ, A. CARBERY, M. COWLING, P. SJÖLIN. In one dimension, inequality (1), with $a = 2$, has been established for $s \geqslant \frac{1}{4}$ in [2] and such a result has been proved to be sharp in [4]. In dimension $n > 1$ the best result up to now can be found in [7]. It states that

$$\| T^* f\|_{L^2(B)} \leqslant c \|f\|_{H^s} \quad \text{for} \quad s > \tfrac{1}{2}.$$

Our proof is in the spirit of [2]. We shall need the following Lemma 1, for the proof of which we refer to [6] and Lemma 2, the proof of which we postpone.

Lemma 1. *Let $k \geqslant 0$ be half of an integer and let $\mathscr{J}_k$ denote the Bessel function of order k. Then*

(a) $|\mathscr{J}_k(t)| \leqslant c_k,\ t \geqslant 0$;

(b) $\mathscr{J}_k(t) = \sqrt{2/\pi t}\cos\left(t - \frac{\pi}{2}k - \frac{\pi}{4}\right) + E_k(t)$

where $|E_k(t)| \leqslant \frac{\bar{c}_k}{t^{3/2}},\ t > A(k) > 0.$

Lemma 2. *Let $a > 1$ and let s, s' belong to* $(0, 100]$. *Set*

$$F(s, s') = \int \frac{\exp(i(2\pi r(s - s') + (t(s) - t(s'))r^a))}{(1 + r^2)^{\frac{1}{4}}} dr$$

where the interval of integration is $\left(\max\left(\frac{A(n)}{s}, \frac{A(n)}{s'}\right), +\infty\right)$. *Then*

$$|F(s, s')| \leqslant \frac{c}{|s - s'|^{\frac{1}{2}}}.$$

By c_n, c_a, c we denote constants not necessarily the same in all instances.

Proof of the theorem. Let $s = |x|$ and $r = |\xi|$. We linearize the operator T, by making t into a function of s, $t(s)$ and we obtain (see [8], p. 155)

$$Tf(s) = \frac{2\pi}{s^{(n-2)/2}} \int_0^\infty \mathscr{J}_{(n-2)/2}(2\pi r s)\, e^{i r^a t(s)} \hat{f}(r)\, r^{n/2}\, dr. \tag{2}$$

(We still write T for the above operator.) All we have to prove is

$$\int_0^{100} |\, Tf(s) \,|\, s^{n-1}\, ds \leqslant c_n \|f\|_{H^{\frac{1}{4}}}. \tag{3}$$

We break up the domain of integration in (2) as follows $(0, A(n)/s)$ and $(A(n)/s, +\infty)$. So we can write $Tf = T_1 f + T_2 f$. We can prove easily estimate (3) for T_1. For by (a) and Schwarz inequality we have

$$|\, T_1 f(s) \,| \leqslant \frac{c_n}{s^{(n-2)/2}} \int_0^{A(n)/s} |\hat{f}(r)|\, r^{n/2}\, dr \leqslant \frac{c_n}{s^{(n-2)/2}} \|f\|_{H^{\frac{1}{4}}} B(s)^{\frac{1}{2}},$$

where $B(s) = \int_0^{A(n)/s} \frac{r}{(1+r^2)^{\frac{1}{4}}}\, dr \leqslant \frac{c_n}{s^{3/2}}$. Therefore T_1 satisfies (3).

Now we turn our attention to T_2. We shall use the equality

$$\mathscr{J}_{(n-2)/2}(r) = \left[\mathscr{J}_{(n-2)/2}(r) - \frac{i\, e^{i(r-(\pi/4)n-(\pi/4))}}{\sqrt{2\pi r}} + \frac{i\, e^{-i(r-(\pi/4)n-(\pi/4))}}{\sqrt{2\pi r}}\right] + \\ + \frac{i\, e^{i(r-(\pi/4)n-(\pi/4))}}{\sqrt{2\pi r}} - \frac{i\, e^{-i(r-(\pi/4)n-(\pi/4))}}{\sqrt{2\pi r}}$$

and write $T_2 f = T_3 f + T_4 f + T_5 f$. By (b) and Schwarz inequality we have

$$|\, T_3 f(s) \,| \leqslant \frac{c_n}{s^{(n-2)/2}} \cdot \int_{A(n)/s}^{\infty} \frac{1}{(r s)^{3/2}} |\hat{f}(r)|\, r^{n/2}\, dr \leqslant \frac{c_n s^{3/4}}{s^{(n+1)/2}} \|f\|_{H^{\frac{1}{4}}}.$$

So T_3 satisfies (3).

Now we are going to deal with $T_4 f$ and similarly one can estimate $T_5 f$. We have

$$|\, T_4 f(s) \,| = \frac{c}{s^{(n-1)/2}} \left| \int_{A(n)/s}^{\infty} e^{2\pi i r s} e^{i r^a t(s)} \hat{f}(r)\, r^{(n-1)/2}\, dr \right|$$

and

$$\| T_4 f \|_{L^1(B)} = \int_0^{100} \theta(s)\, T_4 f(s)\, s^{n-1}\, ds$$

where $\theta(s)\, T_4 f(s) = |\, T_4 f(s) \,|$ and so $|\, \theta(s) \,| = 1$. By exchanging the order of integration and by Schwartz inequality we obtain

$$\| T_4 f\|_{L^1(B)} \leqslant \|f\|_{H^{\frac{1}{4}}} \Big(\int_0^\infty \Theta(r)\, dr\Big)^{\frac{1}{2}}$$

where

$$\int_0^\infty \Theta(r)\, dr = \int_0^\infty \frac{1}{(1+r^2)^{\frac{1}{4}}} \left| \int_{A(n)/r}^{100} \theta(s)\, e^{i(2\pi r s + t(s) r^a)}\, s^{(n-1)/2}\, ds \right|^2 dr =$$

$$= \int_0^\infty \frac{1}{(1+r^2)^{\frac{1}{4}}} \int_{A(n)/r}^{100} \int_{A(n)/r}^{100} \theta(s)\, \overline{\theta(s')}\, e^{i[2\pi r(s-s') + (t(s)-t(s')) r^a]} (s s')^{(n-1)/2}\, ds\, ds'\, dr$$

$$= \int_0^{100} \theta(s)\, s^{(n-1)/2} \int_0^{100} \overline{\theta(s')}\, (s')^{(n-1)/2} F(s,s')\, ds\, ds'.$$

By Lemma 2 we have $\big(\int_0^\infty \Theta(r)\, dr\big)^{\frac{1}{2}} \leqslant c$. So T_4 satisfies (3). This completes the proof of the positive part of the theorem.

The counterexample goes as follows. We shall define even functions $f_v(x)\colon \mathbb{R} \to \mathbb{R}$ such that

(i) $\lim_{v\to 0} \|f_v\|_{H_s} = 0$ if $s < \frac{1}{4}$;

(ii) if we choose $t(x) = x v^{2a-2}/a$ then there exists a small number $b(a)$ such that $|Tf_v(x)| \geqslant \bar{c}$ for all x, $b(a)/2 \leqslant |x| \leqslant b(a)$ and v sufficiently small, depending upon a and $\bar{c}$.

This evidently shows that (3) and therefore (1) cannot hold unless $s \geqslant \frac{1}{4}$.

Let g be a positive even function in $C_0^\infty(\mathbb{R})$ supported in $[-1,1]$, such that $\int_{\mathbb{R}} g(\xi)\, d\xi = D \neq 0$. We set

$$\hat{f}_v(\xi) = \hat{f}_{1,v}(\xi) + \hat{f}_{2,v}(\xi), \quad 0 < v < 1,$$

where $\hat{f}_{1,v}(\xi) = v g(v\xi + 1/v)$ and $\hat{f}_{2,v}(\xi) = v g(v\xi - 1/v)$. In [7] the following has been proved

(i′) $\lim_{v\to 0} \|f_{1,v}\|_{H_s} = 0$ if $s < \frac{1}{4}$;

(ii′) if $t(x) = x v^{2a-2}/2$ then $|Tf_{1,v}(x)| \geqslant c$ for every x in a neighbourhood $I(a)$ of the origin and v small.

From now on we denote by c exactly the constant that appears in (ii′). Now (i) is easy to prove, while (ii) will be proved by (ii′) and the following

(ii″) $|Tf_{2,v}(x)| \leqslant \dfrac{c}{10^2}$ for all x, $\dfrac{b(a)}{2} \leqslant |x| \leqslant b(a)$ and v small, that we proceed to show. By Taylor's formula we have

$$|Tf_{2,v}(x)| \leqslant \left| \int_{-\infty}^{+\infty} e^{i(2(x/v)\xi + \frac{1}{2}(a-1)x\xi^2)} g(\xi)\, d\xi \right| +$$

$$+ \left| \int_{-\infty}^{+\infty} e^{i(2(x/v)\xi + \frac{1}{2}(a-1)x\xi^2)} (e^{i\mathcal{O}(|x|v)} - 1) g(\xi)\, d\xi \right| =$$

$$= M(x) + E(x).$$

For all $x, |x| \leqslant 1$ and v small $|E(x)| \leqslant c|x|v \leqslant c/10^{10}$. Now we write

$$M(x) \leqslant \left| \int e^{i2(x/v)\xi} g(\xi)\, d\xi \right| + \left| \int e^{i2(x/v)\xi} (e^{i\frac{1}{2}(a-1)x\xi^2} - 1) g(\xi)\, d\xi \right| =$$

$$= M_1(x) + M_2(x).$$

We have $M_1(x) = |\hat{g}(x/\pi v)| \leqslant 10^{-10} c$ if $|x| \geqslant |I(a)|/2$ and v sufficiently small. Moreover

$$M_2(x) \leqslant \sum_{h=1}^{M} \left| \int e^{i2(x/v)\xi} \frac{1}{h!} \left(\frac{a-1}{2} x \xi^2 \right)^h g(\xi)\, d\xi \right| + |\mathrm{Tail}(x)|$$

where

$$|\mathrm{Tail}(x)| \leqslant \frac{2}{(M+1)!} \int g(\xi)\, d\xi \leqslant \frac{2D}{(M+1)!} \leqslant \frac{c}{10^{10}},$$

if $\left| \frac{a-1}{2} x \xi^2 \right| < 1$, that is $|x| < \frac{2}{a-1}$ and M large enough. On the other hand if $|x| < \frac{2}{a-1}$ then

$$\sum_{1}^{M} \frac{1}{h!} \left(\frac{a-1}{2} |x| \right)^h \left| \int e^{2i(x/v)\xi} g(\xi)\, \xi^{2h}\, d\xi \right| \leqslant \sum_{1}^{M} \frac{1}{h!} \left| (g(\xi)\, \xi^{2h})\hat{} \left(\frac{x}{\pi v} \right) \right|.$$

Now being g smooth we have $|(g(\xi)\,\xi^{2h})\hat{}(x)| < c_h/|x|$ if $|x| > 1$ and so

$$\sum_{1}^{M} \frac{1}{h!} \left| (g(\xi)\, \xi^{2h})\hat{} \left(\frac{x}{\pi v} \right) \right| \leqslant \frac{c}{10^{10}} \sum \frac{1}{h!} \leqslant \frac{c}{10^{10}}$$

if $|x| > |I(a)|/2$ and v is sufficiently small. Since we can assume $|I(a)| \leqslant \min(2/(a-1), 1)$, we proved that (ii'') holds for $|I(a)|/2 \leqslant$ $\leqslant |x| \leqslant |I(a)|$ and v small. This ends the proof of the theorem.

Now we turn to the

Proof of Lemma 2. First we assume $t(s) \neq t(s')$ and change variables as follows $\eta = |t(s) - t(s')|^{1/a} r$. Then

$$F(s,s') = |t(s) - t(s')|^{-(1/a)} \int_{m(s,s')}^{\infty} \frac{e^{i(2\pi(s-s')|t(s)-t(s')|^{-(1/a)}\eta \pm \eta^a)}}{(1 + |t(s) - t(s')|^{-(2/a)}\eta^2)^{\frac{1}{4}}} d\eta$$

where $m(s,s') = |t(s) - t(s')|^{+(1/a)} \max(A(n)/s, A(n)/s')$. Now we set $2\pi |t(s) - t(s')|^{-(1/a)}(s-s') = v$ and so

$$F(s,s') = |t(s) - t(s')|^{-(1/a)} \int_{m(s,s')}^{\infty} \frac{e^{iv\eta} e^{\pm i\eta^a}}{(1 + |t(s) - t(s')|^{-(2/a)}\eta^2)^{\frac{1}{4}}} d\eta.$$

To prove Lemma 2 all we have to show is

$$|Q(v)| = \left| \int_{m(s,s')}^{\infty} \frac{e^{iv\eta} e^{\pm i\eta^a}}{(1 + P(s,s')^{-(2/a)}\eta^2)^{\frac{1}{4}}} d\eta \right| \leqslant c \frac{P(s,s')^{1/2a}}{|v|^{\frac{1}{2}}}, \qquad (4)$$

where $P(s,s') = |t(s) - t(s')|$. We write

$$Q(v) = \int_{m(s,s')}^{100} d\eta + \int_{100}^{\infty} d\eta = Q_1(v) + Q_2(v).$$

First assume $|v| \leqslant 1$. Then $|Q_1(v)| \leqslant c P^{1/2a}$. Since it is always $|v \pm a\eta^{a-1}| \geqslant c_a > 0$, integrating by parts we obtain

$$|Q_2(v)| \leqslant c_a P^{1/2a} + \int_{100}^{\infty} \left(\frac{a(a-1)\eta^{a-2}}{|v \pm a\eta^{a-1}|^2 (1 + P^{-(2/a)}\eta^2)^{\frac{1}{4}}} + \right.$$
$$\left. + \frac{2P^{-(2/a)}\eta}{|v \pm a\eta^{a-1}|(1 + P^{-(2/a)}\eta^2)^{5/4}} \right) d\eta \leqslant c_a P^{1/2a}.$$

So estimate (4) has been proved for $|v| \leqslant 1$. Secondly assume $|v| > 1$. We start by $Q_1(v)$ assuming $m(s,s') \leqslant |v|^{-1}$. (At those points (s,s') such that $m(s,s') > |v|^{-1}$ the estimates are slightly simpler and they are left to the interested reader).

An integration by parts shows that

$$|Q_1(v)| \leqslant \int_{m(s,s')}^{1/|v|} \frac{1}{P^{-1/2a}\eta^{\frac{1}{2}}} d\eta + \left| \frac{e^{iv\eta}}{v} \left[\frac{e^{\pm i\eta^a}}{(1 + P^{-(2/a)}\eta^2)^{\frac{1}{4}}}\right]_{1/|v|}^{100} \right| +$$
$$+ \frac{c_a}{|v|} \left| \int_{1/|v|}^{100} \frac{e^{iv\eta} e^{\pm i\eta^a} \eta^{a-1}}{(1 + P^{-(2/a)}\eta^2)^{\frac{1}{4}}} d\eta \right| + \frac{c_a}{|v|} \left| \int_{1/|v|}^{100} \frac{e^{iv\eta} e^{\pm i\eta^a} P^{-(2/a)}\eta}{(1 + P^{-(2/a)}\eta^2)^{5/4}} d\eta \right| \leqslant$$
$$\leqslant P^{1/2a} \left(\frac{1}{|v|^{\frac{1}{2}}} + \frac{c_a}{|v|} + \frac{1}{|v|^{\frac{1}{2}+a}} \right) \leqslant c_a \frac{P^{1/2a}}{|v|^{\frac{1}{2}}}.$$

Now we are going to consider $Q_2(v)$. Again we shall integrate by parts, but this time it might be that $v \pm a\eta^{a-1} = 0$, namely for

$\eta = (\pm v/a)^{1/(a-1)} = \eta_v$. Assume $\eta_v \geqslant 100$ (otherwise we are in a simpler case). Then we subdivide the interval of integration into three pieces

$$I_3 = (100, \eta_v - (|v|/a)^{(2-a)/2(a-1)}),$$
$$I_4 = (\eta_v - (|v|/a)^{(2-a)/2(a-1)},\ \eta_v + (|v|/a)^{(2-a)/2(a-1)}),$$
$$I_5 = (\eta_v + (|v|/a)^{(2-a)/2(a-1)}, +\infty).$$

Accordingly $Q_2 = Q_3 + Q_4 + Q_5$.

We start by the estimate of Q_4.

$$|Q_4(v)| \leqslant \int_{I_4} \frac{P^{1/2a}}{\eta^{\frac{1}{2}}} d\eta \leqslant c\, P^{1/2a} \left(\frac{|v|}{a}\right)^{(2-a)/2(a-1)} \eta_v^{-\frac{1}{2}} \leqslant c_a \frac{P^{1/2a}}{|v|^{\frac{1}{2}}}.$$

Now let us consider $Q_5(v)$ and integrate by parts.

$$|Q_5(v)| \leqslant \frac{1}{|v \pm a\Gamma^{a-1}|} \cdot \frac{P^{1/2a}}{\Gamma^{\frac{1}{2}}} + c_a \int_{I_5} \frac{\eta^{a-2}}{|v \pm a\eta^{a-1}|^2 (1 + P^{-(2/a)}\eta^2)^{\frac{1}{4}}} d\eta +$$
$$+ c_a \int_{I_5} \frac{P^{-(2/a)}\eta}{|v \pm a\eta^{a-1}| (1 + P^{-(2/a)}\eta^2)^{5/4}} d\eta.$$

where $\Gamma = \eta_v + 2(|v|/a)^{(2-a)/2(a-1)}$. We are going to show that each one of the three terms in the above formula is dominated by $c_a P^{1/2a}/|v|^{\frac{1}{2}}$. First we claim that

$$|v \pm a\Gamma^{a-1}| \geqslant c_a |v|^{(a-2)/2(a-1)}. \tag{5}$$

This easily implies the estimate we want for the first term. To prove our claim we write $|v \pm a\Gamma^{a-1}| > ||v| - a\Gamma^{a-1}|$. Now

$$|v| - a\Gamma^{a-1} = |v| - |v| \left\{1 + \left(\frac{|v|}{a}\right)^{-a/2(a-1)}\right\}^{a-1} =$$
$$= |v| - |v| \left\{1 + (a-1)\left(\frac{|v|}{a}\right)^{-a/2(a-1)} + \right.$$
$$\left. + \frac{(a-1)(a-2)}{2} \left(\frac{|v|}{a}\right)^{-2a/2(a-1)} + \dots \right\}$$

by the binomial series expansion. This expansion from a certain point on, depending upon a, will show alternating terms. So if $1 < a < 2$

$$||v| - a\Gamma^{a-1}| \geqslant (a-1)\left(\frac{|v|}{a}\right)^{(a-2)/2(a-1)} \left(1 + \frac{a-2}{a}\left(\frac{|v|}{a}\right)^{-a/2(a-1)}\right) \geqslant$$
$$\geqslant \frac{a-1}{2}\left(\frac{|v|}{a}\right)^{(a-2)/2(a-1)},$$

while if $a \geqslant 2$, quite simply $|\,|v| - a\Gamma^{a-1}| \geqslant (a-1)\left(\frac{|v|}{a}\right)^{(a-2)/2(a-1)}$.

This proves the claim.

Now let us go back to the original estimate for Q_5 and consider the third term. By (5) we know that if $\eta \in I_5 = (\Gamma, +\infty)$ then $|v \pm a\eta^{a-1}| \geqslant c_a |v|^{(a-2)/2(a-1)}$. Moreover $\Gamma \geqslant (|v|/a)^{1/(a-1)}$. From these two inequalities the estimate we want for the third term follows easily. We are left with the second term. As we already pointed out if $\eta \in I_5$ then $|v \pm a\eta^{a-1}| \geqslant c_a|v|^{(a-2)/2(a-1)}$, but at infinity it is more convenient to use the estimate $|v \pm a\eta^{a-1}| \sim a\eta^{a-1}$. Therefore we decompose I_5 as follows

$$I_5 = (\Gamma, 2\eta_v) \cup (2\eta_v, +\infty)$$

and

$$(\Gamma, 2\eta_v) = \bigcup_{i=1}^{d}\left(\left(\frac{|v|}{a}\right)^{1/a-1} + 2^{i-1}\left(\frac{|v|}{a}\right)^{(2-a)/2(a-1)},\right.$$

$$\left.\left(\frac{|v|}{a}\right)^{1/a-1} + 2^{i}\left(\frac{|v|}{a}\right)^{2-a/2(a-1)}\right) = \bigcup_{i=1}^{d} J_i.$$

Now as in the proof of (5) one can show that for every $i \leqslant d$ we have $|v - a\eta^{a-1}| \geqslant c_a 2^i |v|^{(a-2)/2(a-1)}$ if $\eta \in J_i$ and that if $\eta \in (2(|v|/a)^{1/a-1}, +\infty)$ we have $|v - a\eta^{a-1}| \geqslant c_a\eta^{a-1}$. From these inequalities the correct estimate for the second term follows. Let us show it for instance for

$$\int_{J_i} \frac{\eta^{a-2} P^{1/2a}}{|v \pm a\eta^{a-1}|^2 \eta^{\frac{1}{2}}}\, d\eta \leqslant c_a P^{1/2a} \frac{2^{-2i}\eta_v^{a-5/2}|J_i|}{|v|^{(a-2)/(a-1)}} \leqslant c_a \frac{P^{1/2a}}{2^i |v|^{1/2}}.$$

So we proved that $|Q_5(v)| \leqslant c_a \dfrac{P^{1/2a}}{|v|^{\frac{1}{2}}}$. In a similar way one can estimate $Q_3(v)$. Namely we use the same integration by parts as in the estimate of $Q_5(v)$, eventually writing

$$I_3 = \left(100, \frac{\eta_v}{2}\right) \cup \left(\frac{\eta_v}{2}, \eta_v - \left(\frac{|v|}{a}\right)^{(2-a)/2(a-1)}\right).$$

On $(100, \eta_v/2)$ we use the inequality $|v \pm a\eta^{a-1}| \geqslant c_a|v|$ and eventually we furthermore subdivide $(\eta_v/2, \eta_v - (|v|/a)^{(2-a)/2(a-1)})$ with a dyadic grid. So we showed that $|Q(v)| \leqslant c_a \dfrac{P^{1/2a}}{|v|^{\frac{1}{2}}}$ for all v's.

Now suppose $t(s) = t(s')$. Then $F(s, s') = \int\limits_{\max\left(\frac{A(n)}{s}, \frac{A(n)}{s'}\right)}^{\infty} \frac{e^{2\pi i r(s-s')}}{(1+r^2)^{\frac{1}{4}}} dr.$

By an integration by parts we have that

$$c\left|\int \frac{e^{2\pi i r s}}{(1+r^2)^{\frac{1}{4}}} dr\right| \leqslant \int\limits_0^{1/s} \frac{1}{(1+r^2)^{\frac{1}{4}}} dr + \frac{1}{s^{\frac{1}{2}}} + \frac{1}{s}\left|\int\limits_{1/s}^{\infty} e^{2\pi i r s} \frac{2r}{(1+r^2)^{5/4}} dr\right| \leqslant \frac{10}{s^{\frac{1}{2}}}.$$

So we proved that $|F(s, s')| \leqslant \frac{c}{|s-s'|^{\frac{1}{2}}}$ at those points (s, s') such that $t(s) = t(s')$. This ends the proof of Lemma 2.

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