Mh. Math. 94, 163—173 (1982)

Monatshefte für
Mathematik

Law of the Iterated Logarithm for Transitive C^2 Anosov Flows and Semiflows over Maps of the Interval

By

Sherman Wong, New York

(Received 23 December 1981)

Abstract. It is proved that a functional law of the iterated logarithm is valid for transitive C^2 Anosov flows on compact Riemannian manifolds when the observable belongs to a certain class of real-valued Hölder functions. The result is equally valid for semiflows over piecewise expanding interval maps that are similar to the Williams' Lorenz-attractor semiflows. Furthermore the observables need only be real-valued Hölder for these semiflows.

1. Introduction

In [7], M. Ratner proves a functional central limit theorem for transitive C^2 Anosov flows on compact Riemannian manifolds of any dimension. The proof requires the use of Markov partitions to represent the Anosov flow as a special flow, a flow built under a function. Ratner proceeds to prove that for a certain class of real-valued Hölder observables, the characteristic function (or Fourier transform) derived from the composition of the observable with the special flow can be approximated as the product of two other characteristic functions that converges to the two-dimensional normal characteristic function. The idea of representing the quantity of interest as a product of other quantities that are simpler in a sense is exploited in the present paper to prove a functional law of the iterated logarithm for transitive C^2 Anosov flows on compact Riemannian manifolds. Moreover this result extends to semiflows obtained from suspending certain piecewise expanding interval mappings, and by using the construction in [8], the result is equally valid for the flows obtained by extending the semiflows.

2. Preliminaries

Let $\{T^t\}$ be a transitive C^2 Anosov flow on a compact Riemannian manifold M for which M and the Riemannian metric on M are C^∞.

0026—9255/82/0094/0163/$02.20

For such flows there are smooth invariant measures, Gibbs measures that maximize a variational principle [2]. By means of a Markov partition [7], one gets a special representation of the flow $\{T^t\}$. This partition determines a matrix $A = [a_{ij}]$, $a_{ij} = 0, 1$, of order r, such that for some integer $s > 0$, all the entries of A^s are positive. Using this matrix, one constructs the space $X \subset \{1, 2, \ldots, r\}^Z$ of sequences $\mathfrak{x} = (x_i)_{i=-\infty}^{\infty}$ with $a_{x_i x_{i+1}} = 1$ for all $i \in Z$. Define $\varphi: X \mapsto X$ by $\varphi \mathfrak{x} = (x'_i)_{i=-\infty}^{\infty}$ where $x'_i = x_{i+1}$.

The Markov partition enables one to define:

(i) a continuous positive function l on X satisfying a Hölder condition,

(ii) a special flow $\{S^t\}$ acting in the space

$$W = (X, l) = \{(\mathfrak{x}, s) : \mathfrak{x} \in X,\ 0 \leqslant s < l(\mathfrak{x}),\ (x, l(\mathfrak{x})) = (\varphi \mathfrak{x}, 0)\}$$

so that for $t < \inf_{\mathfrak{x} \in X} l(\mathfrak{x})$,

$$S^t(\mathfrak{x}, s) = \begin{cases} (\mathfrak{x}, s + t) & \text{for } t < l(\mathfrak{x}) - s \\ (\varphi_{\mathfrak{x}}, t + s - l(\mathfrak{x})) & \text{for } t \geqslant l(\mathfrak{x}) - s \end{cases}$$

and $\{S^t\}$ is uniquely determined for other values of t by the condition that it be a one-parameter transformation group;

(iii) a continuous mapping $\psi: W \mapsto M$ such that $\psi S^t = T^t \psi$.

If ν is an $\{S^t\}$-invariant Borel measure on W such that the set on which ψ fails to be one-to-one has ν-measure 0, then the flows $\{S^t\}$ on (W, ν) and $\{T^t\}$ on $(M, \psi * \nu)$ are isomorphic (for a Borel set $A \subset M$, $\psi * \nu(A) = \nu(\psi^{-1} A)$). It is pointed out in [7], the method above is used by Ya. Sinai to construct invariant Gibbs measures for transitive C^2 Anosov flows. A Gibbs measure ν on W induces a φ-invariant measure μ on X such that $d\nu = (l^*)^{-1}(d\mu \times dt)$ where $l^* = \int_X l(\mathfrak{x})\, d\mu$, and the shift φ on (X, μ) is a Bernoulli automorphism with uniformly strong mixing, specifically, for any sets $B_i \in \mathcal{M}_{k+n}^{\infty}$, $B_i \cap B_j = \emptyset$ $(i \neq j)$ and $A \in \mathcal{M}_{-\infty}^{k}$,

$$\sum_i |\mu(B_i | A) - \mu(B_i)| < C\gamma^{n^\alpha} \tag{1}$$

where $0 < \gamma < 1$, $\alpha > 0$, $C > 0$ a constant, and $\mathcal{M}_a^b$ is the σ-algebra of the sets measurable with respect to $\{x_i : i = a, \ldots, b\}$.

For a continuous $h: X \mapsto \mathbb{R}$, let

$$\mathrm{var}_n h = \sup\{|h(\mathfrak{x}) - h(\mathfrak{x}')| : \mathfrak{x}, \mathfrak{x}' \in X,\ x_i = x_i' \text{ for } |i| \leqslant n\}.$$

One says that $h \in \mathscr{F}_X$ if for some $C > 0$ and $0 < \gamma < 1$, $\mathrm{var}_n h \leqslant C\gamma^n$ for all $n \geqslant 0$. (For a continuous $H: W \mapsto \mathbb{R}$, define $\mathrm{var}_n H$ and $\mathscr{F}_W$ analogously.) It will be assumed that l belongs to $\mathscr{F}_X$.

Let V be the infinitesimal operator corresponding to the group $\{V_t\}$ of unitary operators adjoint to the flow $\{S^t\}$, i. e., $V_t = \exp(i\,t\,V)$. Consider the equation:

$$V h(\mathfrak{w}) = f(\mathfrak{w}) - f^* \text{ where } f^* = \int_W f\,d\nu \text{ and } f \in L_\nu^2(W). \tag{2}$$

3. Statement and Proof of Theorem

Theorem (*Law of the Iterated Logarithm*). *Let f belong to $\mathscr{F}_W$ and suppose that equation* (2) *has no solution in $L_\nu^2(W)$. Then*

$$\nu(\{\mathfrak{w} \in W: \limsup_{t\to\infty} (|\int_0^t [f(S_{\mathfrak{w}}^{-u}) - f^*]\,d\mu\,|(2\sigma^2 t \log\log \sigma^2 t)^{-1/2}) = 1\}) = 1 \tag{3}$$

with $\sigma^2 = 2\pi(l^*)^{-1} r_{F-(F^*/l^*)l}(0) > 0$ *and* $F(\mathfrak{x}) = \int_0^{l(\mathfrak{x})} f(\mathfrak{x}, s)\,ds$

($r_G(\varrho)$ is the spectral density of G).

(For brevity, "$\log_2$" will be used in place of "log log".)

The theorem is proven by a sequence of propositions and results from [5], [7], [9].

Proposition 1. *Let $F \in \mathscr{F}_X$ and*

$$\sigma_N^2 F = \int_X [\sum_{i=0}^{N-1} (F(\varphi^{-i}\mathfrak{x}) - F^*)]^2\,d\mu.$$

If $\sigma_N^2 F \to \infty$ as $N \to \infty$, then F satisfies the law of the iterated logarithm for φ with respect to (X, μ) and $\sigma_N^2 F \sim \sigma_F N$ for some $\sigma_F > 0$. Moreover $\sigma^2 = \sigma_F$ in the law of the iterated logarithm (abbreviated "LIL").

Proof: For $\mathfrak{x} \in X$, let $\Delta_k(\mathfrak{x}) = \{\mathfrak{x}' \in X: x_i' = x_i,\ |i| \leqslant k\}$.

Denote

$$F_k(\mathfrak{x}) = [\mu(\Delta_k(\mathfrak{x}))]^{-1} \int_{\Delta_k(\mathfrak{x})} F\,d\mu.$$

Because $F \in \mathscr{F}_X$, it follows that in $L_\mu^2(X)$

$$\|F(\mathfrak{x}) - F_k(\mathfrak{x})\|_2 < C\gamma^k. \tag{4}$$

From [5], (4) implies that when (1) holds, $\sigma_N^2 F \sim \sigma_F N$ for $\sigma_F > 0$, and from [9], F satisfies the LIL with $\sigma^2 = \sigma_F$. □

As RATNER points out, since for Gibbs measures of transitive Anosov flows, $\{S^t\}$ is a Kolmogorov flow in (W, ν), the equation $UG - G = l - l^*$ has no solution in $L_\mu^2(X)$ where U is the unitary operator in $L_\mu^2(X)$ adjoint to φ, and consequently $\sigma_n^2 l \sim \sigma_l n$ for $\sigma_l > 0$. Since $l \in \mathscr{F}_X$, l satisfies the LIL for φ with $\sigma^2 = \sigma_l$. For $f \in \mathscr{F}_W$, define

$$F(\mathfrak{x}) = \int_0^{l(\mathfrak{x})} f(\mathfrak{x}, t)\, dt \text{ and } F^\sim(\mathfrak{x}) = F(\mathfrak{x}) - (F^*/l^*)\, l(\mathfrak{x}).$$

In [7], it is proved that the equation $Vh(\mathfrak{w}) = f(\mathfrak{w}) - f^*$ not having a solution in $L_\nu^2(W)$ is equivalent to the equation $UH - H = (F^*/l^*)\, l$ not having a solution in $L_\mu^2(X)$. Thus $\sigma_n^2 F^\sim \sim \sigma_{F^\sim} n$ as $n \to \infty$ where $\sigma_{F^\sim} = 2\pi r_{F^\sim}(0) > 0$ with $r_G(\varrho)$ the spectral density of G. Further because $F^\sim \in \mathscr{F}_X$, $F^\sim$ satisfies the LIL for φ.

In the central limit theorem for special flows, RATNER proves that if $\sigma_t^2 f = \int_W [\int_0^t (f(S^{-u}\mathfrak{w}) - f^*)\, du]^2\, d\nu$, then $\sigma_t^2 f \sim \sigma_f t$ for $\sigma_f = 2\pi (l^*)^{-1} r_{F^\sim}(0) > 0$. In the present situation of the LIL, the same σ_f is used for σ^2 in (3). One notices that because $\sigma_{F^\sim} = 2\pi r_{F^\sim}(0)$, $\sigma_{F^\sim} = l^* \sigma_f = l^* \sigma^2$.

Notation. 1. Let

$$\Lambda = \{\mathfrak{x} \in X:\ \limsup_{n\to\infty} |\sum_{i=0}^{n-1} F^\sim(\varphi^{-i}\mathfrak{x})|\, (2\sigma_{F^\sim} n \log_2 \sigma_{F^\sim} n)^{-1/2} = 1\}.$$

2. For $\mathfrak{x} \in X$, define $n(\mathfrak{x}, t)$ by

$$\sum_{i=0}^{n(\mathfrak{x},t)} l(\varphi^{-i}\mathfrak{x}) < t \leqslant \sum_{i=0}^{n(\mathfrak{x},t)+1} l(\varphi^{-i}\mathfrak{x}). \tag{5}$$

3. Let

$$\mathscr{H} = \{\mathfrak{x} \in X: \limsup_{t\to\infty} (l^*)^{3/2} |n(\mathfrak{x}, t) - (l^*)^{-1} t| \cdot (2\sigma_l t \log_2 \sigma_l t)^{-1/2} = 1\}.$$

4. Let $\sigma = \sigma_f = (l^*)^{-1} \sigma_{F^\sim}$ and

$$\mathscr{L} = \{\mathfrak{w} \in W: \limsup_{t\to\infty} (2\sigma t \log_2 \sigma t)^{-1/2} |\int_0^t [f(S^{-u}\mathfrak{w}) - f^*]\, du| = 1\}.$$

Proposition 2. $\mu(\mathscr{H}) = 1$.

Proof: For $\mathfrak{x} \in X$, (5) implies

$$(2\sigma_l n(\mathfrak{x}, t) \log_2 \sigma_l n(\mathfrak{x}, t))^{-1/2} \{\sum_{i=0}^{n(\mathfrak{x}, t)} [l(\varphi^{-i}\mathfrak{x}) - l^*] + l^*\} <$$
$$< (2\sigma_l n(\mathfrak{x}, t) \log_2 \sigma_l n(\mathfrak{x}, t))^{-1/2} [t - l^* n(\mathfrak{x}, t)]$$
$$\leqslant (2\sigma_l n(\mathfrak{x}, t) \log_2 \sigma_l n(\mathfrak{x}, t))^{-1/2} \{\sum_{i=0}^{n(\mathfrak{x}, t)+1} [l(\varphi^{-i}\mathfrak{x}) - l^*] + 2l^*\}.$$

Because φ is ergodic, $n(\mathfrak{x}, t) \to \infty$ as $t \to \infty$ for almost every $\mathfrak{x}$ with respect to μ, and because l satisfies the LIL for φ,

$$\limsup_{t \to \infty} (2\sigma_l n(\mathfrak{x}, t) \log_2 \sigma_l n(\mathfrak{x}, t))^{-1/2} |t - l^* n(\mathfrak{x}, t)| = 1$$

for almost every $\mathfrak{x}$ with respect to μ.

Thus for t sufficiently large,

$$|t - l^* n(\mathfrak{x}, t)| < 2(2\sigma_l n(\mathfrak{x}, t) \log_2 \sigma_l n(\mathfrak{x}, t))^{1/2}.$$

For some $C > 0$ that depends on l only,

$$|t(l^* n(\mathfrak{x}, t))^{-1} - 1| < 2(l^* n(\mathfrak{x}, t))^{-1} (2\sigma_l n(\mathfrak{x}, t) \log_2 \sigma_l n(\mathfrak{x}, t))^{1/2}$$
$$< C(n(\mathfrak{x}, t))^{-1/2}.$$

Consequently, $(l^*)^{-1} t \sim n(\mathfrak{x}, t)$ and

$$1 = \limsup_{t \to \infty} |t - l^* n(\mathfrak{x}, t)| (2\sigma_l n(\mathfrak{x}, t) \log_2 \sigma_l n(\mathfrak{x}, t))^{-1/2}$$
$$= \limsup_{t \to \infty} l^* |(l^*)^{-1} t - n(\mathfrak{x}, t)| (2\sigma_l t (l^*)^{-1} \log_2 \sigma_l t (l^*)^{-1})^{-1/2}$$
$$= \limsup_{t \to \infty} (l^*)^{3/2} |(l^*)^{-1} t - n(\mathfrak{x}, t)| (2\sigma_l t \log_2 \sigma_l t)^{-1/2}$$

for almost every $\mathfrak{x}$ with respect to μ, i. e., $\mu(\mathscr{H}) = 1$. □

Proposition 3. *If* $\mathfrak{w} = (\mathfrak{x}, t_0) \in \mathscr{L}$ *with* $t_0 \in [0, l(\mathfrak{x}))$, *then* $\{\mathfrak{x}\} \times \times [0, l(\mathfrak{x})) \subset \mathscr{L}$, *i. e., the "fiber" over* $\mathfrak{x}$ *belongs to* $\mathscr{L}$ *if one point of the "fiber" belongs to* $\mathscr{L}$.

Proof: For $0 \leqslant s < t_0$,

$$\int_0^t [f(S^{-u}\mathfrak{w}) - f^*]\,du = \int_0^t [f(S^{-u}(\mathfrak{x},s)) - f^*]\,du \tag{6}$$

$$- \int_0^{t_0-s} [f(S^{-u}(\mathfrak{x},s)) - f^*]\,du + \int_0^{t+(t_0-s)} [f(S^{-u}(\mathfrak{x},s)) - f^*]\,du$$

$$= \int_0^t [f(S^{-u}(\mathfrak{x},s)) - f^*]\,du - [f(S^{-t_1}(\mathfrak{x},s)) - f(S^{-t_2}(\mathfrak{x},s))]\,(t_0 - s)$$

for some t_1 and t_2 with $0 < t_1 < t$ and $t < t_2 < t + (t_0 + s)$. From (6), one can imply that

$$(2\sigma t \log_2 \sigma t)^{-1/2} \left| \int_0^t [f(S^{-u}\mathfrak{w}) - f^*]\,du \right| \geqslant$$

$$\geqslant (2\sigma t \log_2 \sigma t)^{-1/2} \left| \int_0^t [f(S^{-u}(\mathfrak{x},s)) - f^*]\,du \right| - (2\sigma t \log_2 \sigma t)^{-1/2} C\, l(\mathfrak{x})$$

for some $C \geqslant \max_W |f|$.

Because $\mathfrak{w} \in \mathscr{L}$, one concludes

$$1 \geqslant \limsup_{t\to\infty} (2\sigma t \log_2 \sigma t)^{-1/2} \left| \int_0^t [f(S^{-u}(\mathfrak{x},s)) - f^*]\,du \right| .$$

Furthermore since by (6)

$$(2\sigma t \log_2 \sigma t)^{-1/2} \left| \int_0^t [f(S^{-u}\mathfrak{w}) - f^*]\,du \right| \leqslant$$

$$\leqslant (2\sigma t \log_2 \sigma t)^{-1/2} \left| \int_0^t [f(S^{-u}(\mathfrak{x},s)) - f^*]\,du \right| + (2\sigma t \log_2 \sigma t)^{-1/2} C\, l(\mathfrak{x}),$$

$$\limsup_{t\to\infty} (2\sigma t \log_2 \sigma t)^{-1/2} \left| \int_0^t [f(S^{-u}(\mathfrak{x},s)) - f^*]\,du \right| \geqslant 1 .$$

Hence for $0 \leqslant s < t_0$, $(\mathfrak{x},s) \in \mathscr{L}$.

By an analogous argument, $(\mathfrak{x},s) \in \mathscr{L}$ for $t_0 \leqslant s < l(\mathfrak{x})$. □

Proposition 4. *If* $\mathfrak{x} \in \Lambda \cap \mathscr{H}$, *then* $(\mathfrak{x},0) \in \mathscr{L}$.

Proof: For $\mathfrak{x} \in X$, let $c(n) = \sum_{i=0}^{n} l(\varphi^{-i}\mathfrak{x})$. From some t_0 with $c(n(\mathfrak{x},t)) < t_0 < t$,

$$\int_0^t [f(S^{-u}(x,0)) - f^*]\, du =$$

$$= \int_0^{c(n(x,t))} [f(S^{-u}(x,0)) - f^*]\, du + \int_{c(n(x,t))}^{t} [f(S^{-u}(x,0)) - f^*]\, du \qquad (7)$$

$$= \sum_{i=0}^{n(x,t)} \int_{c(i-1)}^{c(i)} [f(S^{-u}(x,0)) - f^*]\, du + [f(S^{-t_0}(x,0)) - f^*]\ (t - c(n(x,t)))$$

$$= \sum_{i=0}^{n(x,t)} \int_0^{l(\varphi^{-i}x)} [f(S^{-u}(\varphi^{-i+1}x,0)) - f^*]\, du + \Delta_2$$

$$= \sum_{i=0}^{n(x,t)} F^{\sim}(\varphi^{-i}x) + \Delta_2 = \Delta_1 + \Delta_2$$

where $\Delta_2 = [f(S^{-t_0}(x,0)) - f^*](t - c(n(x,t)))$.

It is now claimed that

$$\mu(\{x \in X\colon \lim_{t\to\infty} (2\sigma t \log_2 \sigma t)^{-1/2}[t - c(n(x,t))] = 0\}) = 1\,. \qquad (8)$$

Let

$$E = \{x \in X\colon \limsup_{t\to\infty} (2\sigma t \log_2 \sigma t)^{-1/2}[t - c(n(x,t))] > 0\}\,.$$

It is enough to show that $\mu(E) = 0$.

$$E \subset \{x \in X\colon \limsup_{t\to\infty} (2\sigma t \log_2 \sigma t)^{-1/2} l(\varphi^{-n(x,t)-1}x) > 0\}$$

$$\subset \{x \in X\colon \limsup_{n\to\infty} (2\sigma n \log_2 \sigma n)^{-1/2} l(\varphi^{-(n+1)}x) > 0\}\,.$$

By using Markov's inequality and the invariance of φ with respect to μ, one has for a given $\varepsilon > 0$

$$\mu(\{x \in X\colon (2\sigma n \log_2 \sigma n)^{-1/2} l(\varphi^{-(n+1)}x) > \varepsilon\}) \leqslant$$

$$\leqslant \|l\|_{2+\delta}^{2+\delta}\, \varepsilon^{-2-\delta}\, (2\sigma n \log_2 \sigma n)^{-1/2(2+\delta)}$$

for a fixed $\delta > 0$ and

$$\sum_{n\geqslant 1} \mu(\{x \in X\colon (2\sigma n \log_2 \sigma n)^{-1/2} l(\varphi^{-(n+1)}x) > \varepsilon\}) \leqslant$$

$$\leqslant \|l\|_{2+\delta}^{2+\delta}\, \varepsilon^{-(2+\delta)} \sum_{n\geqslant 1} (2\sigma n \log_2 \sigma n)^{-1/2(2+\delta)} < \infty\,.$$

By the Borel-Cantelli 0-1 Law, one concludes that

$$\mu(\{\mathfrak{x}\in X: \limsup_{n\to\infty}(2\sigma n\log_2\sigma n)^{-1/2}\, l(\varphi^{-(n+1)}\mathfrak{x})>\mathfrak{x})>0\})=0.$$

Because of (8), one concludes that for almost every $\mathfrak{x}$,

$$\lim_{t\to\infty}(2\sigma t\log_2\sigma t)^{-1/2}|\Delta_2|=0. \tag{9}$$

$$(2\sigma t\log_2\sigma t)^{-1/2}|\Delta_1|=(2\sigma_{F^\sim}n(\mathfrak{x},t)\log_2\sigma_{F^\sim}n(\mathfrak{x},t))^{-1/2}\cdot$$
$$\cdot|\Delta_1|\,[(\sigma t\log_2\sigma t)^{-1}(\sigma_{F^\sim}n(\mathfrak{x},t)\log_2\sigma_{F^\sim}n(\mathfrak{x},t))]^{1/2}.$$

Because $\sigma_{F^\sim}=l^*\sigma$ and $n(\mathfrak{x},t)\sim(l^*)^{-1}t$ for $\mathfrak{x}\in\mathscr{H}$,

$$\begin{aligned}&(\sigma t\log_2\sigma t)^{-1}(\sigma_{F^\sim}n(\mathfrak{x},t)\log_2\sigma_{F^\sim}n(\mathfrak{x},t))\sim\\ &\sim(\sigma t\log_2\sigma t)^{-1}[l^*\sigma t(l^*)^{-1}\log_2 l^*\sigma t(l^*)^{-1}]=1.\end{aligned}\tag{10}$$

Using (9), (10) and the fact that $F^\sim$ satisfies the LIL for φ, one has for $\mathfrak{x}\in\Lambda\cap\mathscr{H}$,

$$\limsup_{t\to\infty}(2\sigma t\log_2\sigma t)^{-1/2}\left|\int_0^t[f(S^{-u}(\mathfrak{x},0))-f^*]\,du\right|\leqslant$$
$$\leqslant\limsup_{t\to\infty}(2\sigma t\log_2\sigma t)^{-1/2}|\Delta_1|+\limsup_{t\to\infty}(2\sigma t\log_2\sigma t)^{-1/2}|\Delta_2|=1.$$

Similarly,

$$\limsup_{t\to\infty}(2\sigma t\log_2\sigma t)^{-1/2}\left|\int_0^t[f(S^{-u}(\mathfrak{x},0))-f^*]\,du\right|\geqslant$$
$$\geqslant\limsup_{t\to\infty}(2\sigma t\log_2\sigma t)^{-1/2}|\Delta_1|-\limsup_{t\to\infty}(2\sigma t\log_2\sigma t)^{-1/2}|\Delta_2|=1.$$

Thus $(\mathfrak{x},0)\in\mathscr{L}$. □

Proposition 5. $\nu(\mathscr{L})=1$.

Proof: Recall that $d\nu=(l^*)^{-1}(d\mu\times dt)$.

$$\nu(\mathscr{L})=\int_L d\nu=(l^*)^{-1}\int_{\mathscr{L}}d\mu\times dt=(l^*)^{-1}\int_{\Lambda\cap\mathscr{H}}d\mu\int_0^{l(\mathfrak{x})}dt$$
$$=(l^*)^{-1}\int_{\Lambda\cap\mathscr{H}}l(\mathfrak{x})\,d\mu=1\quad\text{since }\mu(\Lambda\cap\mathscr{H})=1.\ \square$$

Remarks. 1. If $\psi: W\mapsto M$ is the isomorphism relating $\{S^t\}$ on (W,ν) to (M,ν^*) (ν^* the Gibbs measure on M) and if $h: M\mapsto\mathbb{R}$ is such that $|h(z)-h(z')|<C\gamma^{\varkappa|\log d(z,z')|}$ for some $C>0$, $\varkappa>0$, $0<\gamma<1$, and d the metric on M, then $f(\mathfrak{w})=h(\psi\mathfrak{w})\in\mathscr{F}_W$ [7]. Thus the theorem is valid for the class of all such h on M.

2. If $\{T^t\}$ is a transitive C^2 Axiom-A flow on a compact Riemannian manifold M and L is a basic hyperbolic set with $m(L) > 0$ (m is the measure on M derived from the Riemannian metric), then L is a connected component of M and $\{T^t\}$ restricted to L is an Anosov flow [3]. Consequently, the law of the iterated logarithm is valid when ν is an equilibrium state for $\{T^t\}$ restricted to L.

4. Semiflows from Interval Mappings

Let $\varphi: [0, 1] \mapsto [0, 1]$ possess a partition of $[0, 1]$, $\mathscr{P} = \{0 = a_0 < a_1 < a_2 < \ldots < a_{m-1} < a_m = 1\}$, for which φ is C^1 on (a_i, a_{i+1}) and $|d\varphi/dx|^{-1}$ is of bounded variation on $[a_i, a_{i+1}]$ for $i = 0, 1, \ldots, m-1$. Suppose that $\inf_x |d\varphi/dx| = \varrho^{-1} > 1$ and that there exists a φ-invariant weak-mixing measure μ absolutely continuous with respect to Lebesgue measure on [0, 1]. In [8], RATNER proves that $\mathscr{P}$ is weakly Bernoulli and hence the natural extension of φ is Bernoulli. Moreover if $l \in L^1_\mu$ and l^{-1}, the reciprocal of l, is Hölder with exponent α $(0 < \alpha \leqslant 1)$, then either $\{S^t\}$, the special flow build under l using the extension of φ and $d\nu = (l^*)^{-1} d\mu \times dt$, is Bernoulli, or for some $t_0 > 0$, S^{t_0} is not ergodic. In this section, it is shown that the law of iterated logarithm is valid for f Hölder and for $l \in L^{2+\delta}_\mu$ with l^{-1} being Hölder. (To simplify the discussion, this section deals with the semiflow obtained from suspending φ, not its extension, by l. The proof of the LIL can be modified to handle the case of the semiflow.)

To begin, from HOFBAUER's and KELLER's work [4], the φ-invariant weak-mixing measure μ, described above, actually satisfies (1).

If W is the space on which the semiflow is defined, then the requirement that $f \in \mathscr{F}_W$ can be relaxed to f being Hölder. The condition "$f \in \mathscr{F}_W$" is used so that (4) is satisfied for $F^\sim$. However as is shown in [1], it is sufficient for $\sum_{k \geqslant 1} \| F^\sim - F_k^\sim \|_2 < \infty$. From [11], by knowing that $\varrho < 1$ and G is Hölder on $[0, 1]$, $\sum_{k \geqslant 1} \| G - G_k \|_2 < \infty$. Consequently, if $\sum_{k \geqslant 1} \| l - l_k \|_2 < \infty$ and f is Hölder on W, one has the desired series converging.

Proposition 6. *Let φ and μ be as above. If $l \in L^2_\mu$ and l^{-1} is Hölder of exponent α $(0 < \alpha \leqslant 1)$ on $[a_i, a_{i+1}]$ for $i = 0, 1, \ldots, m-1$, then* $\sum_{k \geqslant 1} \| l - l_k \|_2 < \infty$.

Proof: Observe that for each Δ_k,

$$0 \leqslant \int_{\Delta_k} (l - l_k)^2 \, d\mu = \int_{\Delta_k} l^2 \, d\mu - \mu(\Delta_k)^{-1} \left(\int_{\Delta^k} l \, d\mu\right)^2 = \int_{\Delta_k} l^2 \, d\mu - \int_{\Delta_k} l_k^2 \, d\mu,$$

$$\text{i.e.,} \int_{\Delta_k} l_k^2 \, d\mu \leqslant \int_{\Delta_k} l^2 \, d\mu$$

and hence $\|l_k\|_2 \leqslant \|l\|_2$. For a fixed $k \geqslant 1$, for each Δ_k, there is an $x_k \in \Delta_k$ such that $l(x_k) = l_k(x_k)$. Consequently, for some constant $C > 0$,

$$\int_{\Delta_k} (l - l_k)^2 \, d\mu = \int_{\Delta_k} (l - l(x_k))^2 \mu = \int_{\Delta_k} l^2 l(x_k)^2 |l(x_k)^{-1} - l^{-1}|^2 \, d\mu \leqslant$$

$$\leqslant C^2 \varrho^{2\alpha k} \int_{\Delta_k} l^2 l_k^2 \, d\mu .$$

By the Cauchy-Schwarz inequality,

$$\|l - l_k\|_2 \leqslant C \varrho^{\alpha k} \|l l_k\|_2 \leqslant C \varrho^{\alpha k} \|l\|_2 \|l_k\|_2 \leqslant C \varrho^{\alpha k} \|l\|_2^2 .$$

The conclusion follows directly from this last inequality. □

As to the supposition that equation (2) has no solution in $L_\nu^2(W)$, it is noted in §3 that this condition is equivalent to the equation $UH - H = F - (F^*/l^*)\,l$ not having a solution in L_μ^2. In the present situation, U is not a unitary operator but an isometric operator adjoint to φ [6]. Thus if $UH - H = F - (F^*/l^*)\,l$ has no solution in L_μ^2, the analogue to equation (2) has no solution in $L_\nu^2(W)$.

Because all of the conditions for the theorem are fulfilled for semiflows obtained by suspending φ with an l described in Proposition 6 and a Hölder f on W, the law of the iterated logarithm is valid for such semiflows.

Remarks. 3. The reason why it is of interest to permit l to escape to infinity is that the semiflow obtained from the Williams' Lorenz attractor has such a return-time function (see [8], [10]). The escape to infinity means that the point never returns to the cross-section and is caught on the stable manifold of the origin, a stationary solution to the Lorenz equations. (Because of this desire to have infinite return times, $l \in L_\mu^{2+\delta}$ for some $\delta > 0$ in order that (8) is valid. (8) is used to imply (9) rather than deriving (9) from the assumption that l is Hölder.)

4. If one considers all the properties that have been proven for the Williams' Lorenz-attractor flow: the Bernoulli isomorphism, the functional central limit theorem (by modifying the proof in [7]), and the functional law of the iterated logarithm, one is led to believe that the flow exhibits "chaotic" behavior from the probabilistic aspect (with "random" or "stochastic" replacing "chaotic").

References

[1] BILLINGSLEY, P.: Convergence of Probability Measures. New York: Wiley. 1968.

[2] BOWEN, R.: Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms. Lecture Notes Math. 470. Berlin–Heidelberg–New York: Springer. 1975.

[3] BOWEN, R., RUELLE, D.: The ergodic theory of Axiom-A flows. Invent. Math. **18**, 181—202 (1975).

[4] HOFBAUER, F., KELLER, G.: Ergodic properties of invariant measures for piecewise monotonic transformations. Preprint. Universität Göttingen. 1981.

[5] IBRAGIMOV, I. A.: Some limit theorems for stationary processes. Th. Prob. Appl. **7**, 349—382 (1962).

[6] LEONOV, V. P.: On the dispersion of time-dependent means of a stationary stochastic process. Th. Prob. Appl. **6**, 87—93 (1961).

[7] RATNER, M.: The central limit theorem for geodesic flows on n-dimensional manifolds of negative curvature. Israel J. Math. **16**, 181—197 (1973).

[8] RATNER, M.: Bernoulli flows over maps of the interval. Israel J. Math. **31**, 298—314 (1978).

[9] REZNIK, M. KH.: The law of the iterated logarithm for some classes of stationary processes. Th. Prob. Appl. **13**, 606—621 (1968).

[10] WILLIAMS, R. F.: The structure of Lorenz attractors. IHES Publ. Math. **50**, 321—347 (1979).

[11] WONG, S.: Two probabilistic properties of weak-Bernoulli interval maps. (To appear.)

SH. WONG
The City University of New York
Baruch College
17 Lexington Avenue
New York, NY 10010, U.S.A.