ON THE NUMBER OF ISOLATING INTEGRALS IN RESONANT SYSTEMS WITH 3 DEGREES OF FREEDOM

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Abstract. The 2: 2: 1-resonance case for a potential problem with three degrees of freedom is characterized by the existence of two isolating approximate integrals apart from the energy. This result completes a statement by Gustavson concerning the number of formal integrals in resonant Hamiltonian systems.

1. Introduction

A contribution towards understanding the problem of the disappearance of formal integrals in the case of systems with three or more degrees of freedom has been given recently by Contopoulos (1978). Some misunderstanding however may have arisen from a statement by Gustavson (1966) concerning the number of formal integrals in Hamiltonian systems with N degrees of freedom, when r independent resonance relations exist between the unperturbed oscillation frequencies of the system. Gustavson states that, apart from exceptional cases, only $N-r$ formal independent integrals exist apart from the energy integral, starting with (independent) quadratic terms. In fact, this statement does not imply a disappearance of integrals: it does not exclude the existence of other algebraic formal integrals or transcendental ones, obtained by other methods. For three degrees of freedom systems with two resonance relations ($N = 3$, $r = 2$) this is still an important open problem; cf. van der Aa and Sanders (1979) who studied the 1: 2: 1-resonance. We shall show in the sequel that for the 2: 2: 1-resonance the general potential problem has apart from the energy two formal integrals, a quadratic and a cubic one. We shall use the concepts of integral, formal integral and approximate or asymptotic integral; for a discussion of this terminology see Verhulst (1979) and Sanders and Verhulst (1979), Section 7.

2. An Example of a 2:2:1-Resonance

Consider the system with three degrees of freedom characterized by the potential

$$\phi = \tfrac{1}{2}(Ax^2 + By^2 + Cz^2) - \varepsilon xz^2 - \eta yz^2 \tag{1}$$

with $\sqrt{A} : \sqrt{B} : \sqrt{C} = \omega_1 : \omega_2 : \omega_3 = 2 : 2 : 1$.

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By introducing a canonical transform to action-angle variables, we obtain for the Hamiltonian

$$H = \omega_1 I_1 + \omega_2 I_2 + \omega_3 I_3 + \varepsilon\alpha(I_1, I_3)[\cos(\theta_1 - 2\theta_3) + \\ + 2\cos\theta_1 + \cos(\theta_1 + 2\theta_3)] + \eta\beta(I_2, I_3) \times \\ \times [\cos(\theta_2 - 2\theta_3) + 2\cos\theta_2 + \cos(\theta_2 + 2\theta_3)].$$

Among the first order terms, all will be rapidly varying, except the terms involving $\cos(\theta_1 - 2\theta_3)$ and $\cos(\theta_2 - 2\theta_3)$. After elimination of the fast variables, we remain with two combinations of angles only. A new transformation of variables $(I, \theta) \to (J, \chi)$ realized by means of the generating function

$$S = (\theta_1 - 2\theta_3)J_1 + (\theta_2 - 2\theta_3)J_2 + \theta_3 J_3$$

leads to

$$H = \omega_1 J_1 + \omega_2 J_2 + \omega_3(-2J_1 - 2J_2 + J_3) + \varepsilon\alpha(J_1, J_3)\cos\chi_1 + \\ + \eta\beta(J_2, J_3)\cos\chi_2$$

The new angle χ_3 is ignorable so that the corresponding action J_3 is an integral.

$$J_3 = I_3 + 2I_1 + 2I_2.$$

If we write $\omega_1' = \omega_1 - 2\omega_3$ and $\omega_2' = \omega_2 - 2\omega_3$ the Hamiltonian takes the form

$$H = \omega_1' J_1 + \omega_2' J_2 + \omega_3 J_3 + \varepsilon\alpha(J_1, J_3)\cos\chi_1 + \eta\beta(J_2, J_3)\cos\chi_2.$$

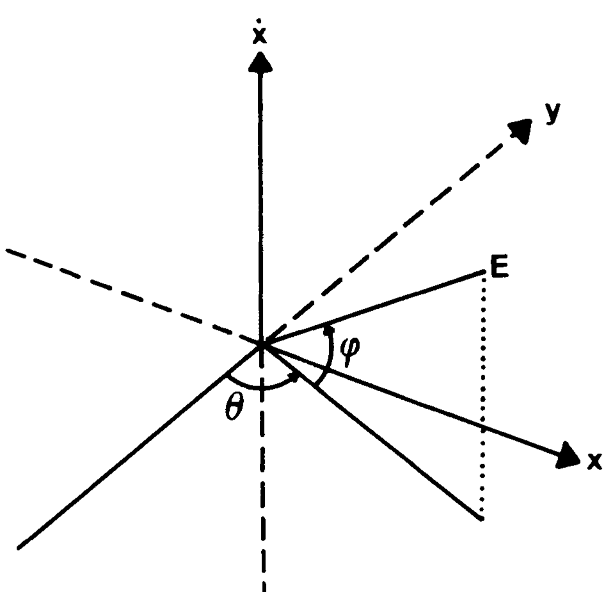

Fig. 1. Definition of the direction (θ, ϕ) of the observer E with respect to the system of axes used in Figure 2.

We have assumed $\omega'_1 = \omega'_2 = 0$. In this case, following the arguments given by Gustavson (1966), only two formal integrals may be constructed. In fact, the reduced Hamiltonian is

$$H' = \varepsilon\alpha(J_1, J_3)\cos\chi_1 + \eta\beta(J_2, J_3)\cos\chi_2 .$$

In cases where α and β would have a more general dependence on J_i, there were effectively no isolating integrals besides H' itself. So the system would be, to a large extent, stochastic. But the present example belongs to a case previoulsy discussed by

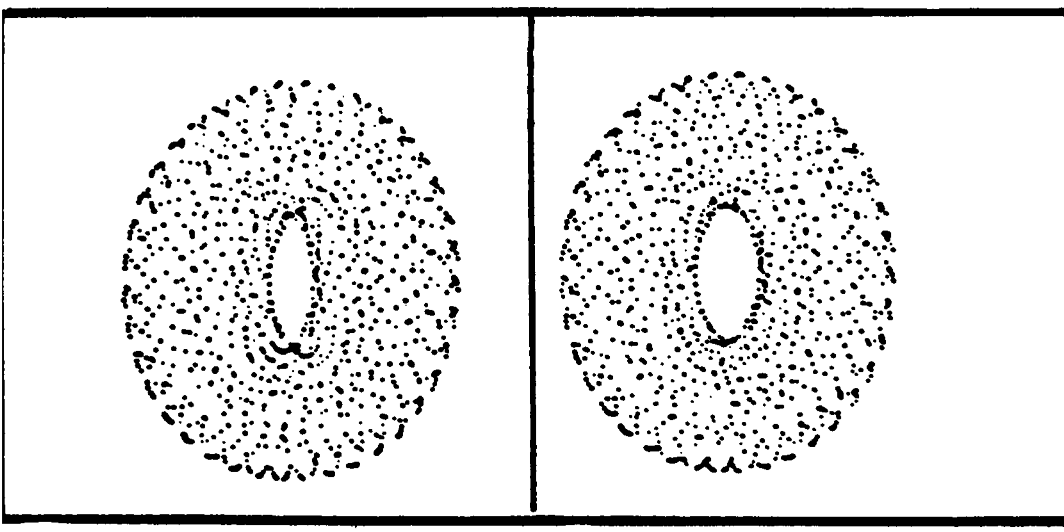

Fig. 2a.

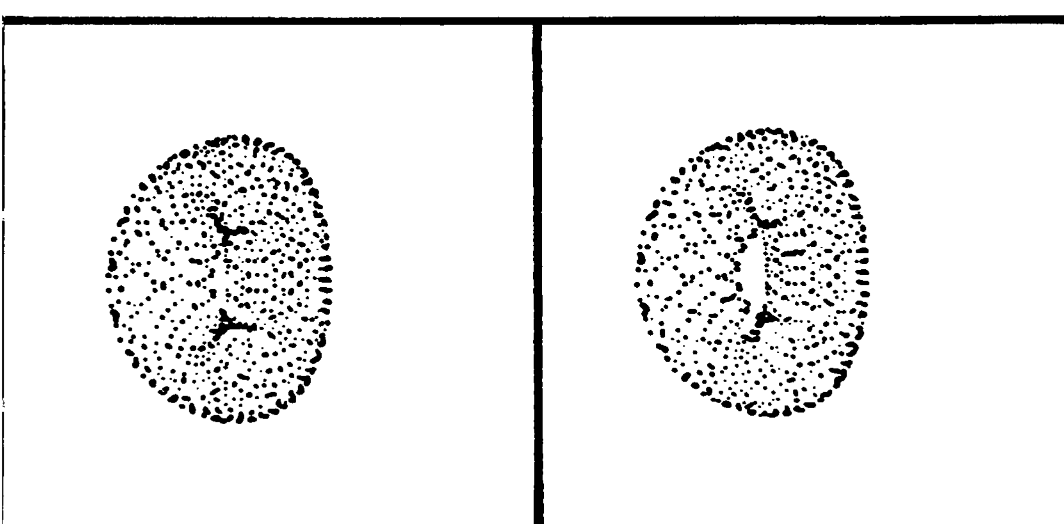

Fig. 2b.

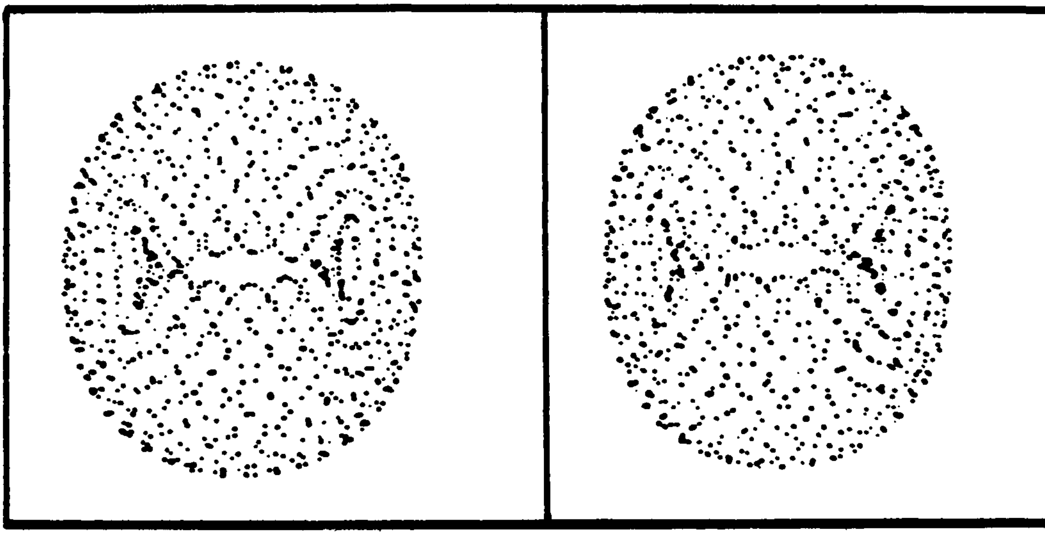

Fig. 2c.

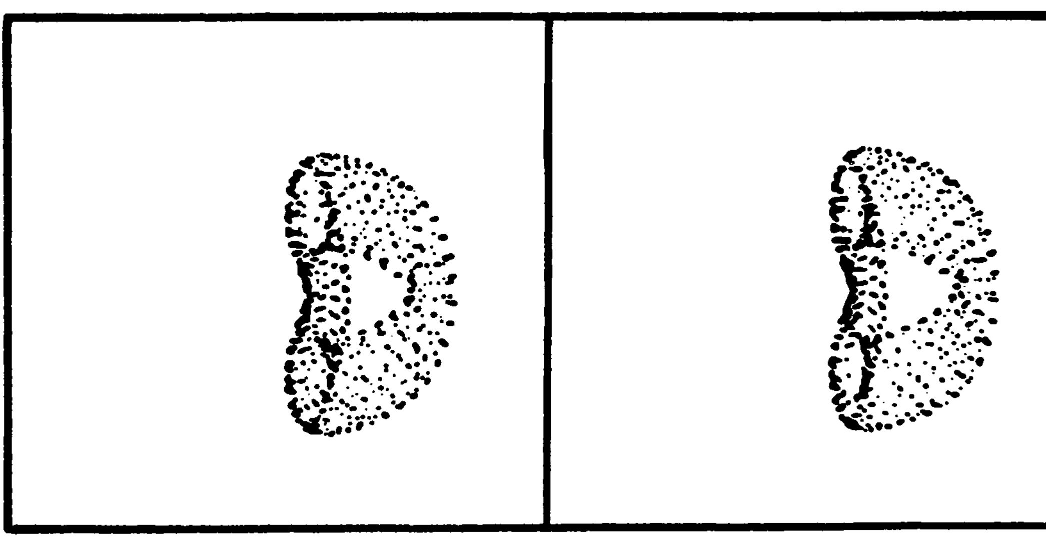

Fig. 2d.

Fig. 2a–d. Stereoscopic views of invariant surfaces represented in the system $(x, y, \dot{x})$ and observed from the direction $\theta = 30°$, $\phi = 0°$. Different initial conditions were chosen: With $A = 0.4$, $B = 0.4$, $C = 0.1$ and the energy $h = 0.00765$.

(a) $x_0 = -0.01$, $\dot{x}_0 = 0$, $y_0 = 0.030$, $\dot{y}_0 = 0$, $\varepsilon = 0.05$, $\eta = 0.03$.
(b) $x_0 = -0.055$, $\dot{x}_0 = 0$, $y_0 = -0.03$, $\dot{y}_0 = 0$, $\varepsilon = 0.05$, $\eta = 0.03$.
(c) $x_0 = -0.0367$, $\dot{x}_0 = 0$, $y_0 = -0.01$, $\dot{y}_0 = 0$, $\varepsilon = 0.1$ $\eta = 0.1$.
(d) $x_0 = -0.01$, $\dot{x}_0 = 0$, $y_0 = 0.059$, $\dot{y}_0 = 0$, $\varepsilon = 0.1$ $\eta = 0.1$.

Contopoulos (1979): α is a function of J_1 only and β a function of J_2 only, since J_3 is constant; then both quantities $\alpha \cos \chi_1$ and $\beta \cos \chi_2$ are integrals. Therefore the system under consideration has 3 integrals. A numerical investigation based on the method of 'surface of section' and of 'stereoscopic views' described by Martinet and Magnenat (1981) has been undertaken. For a large set of initial conditions chosen in different domains of the phase space and for not too large values of ε and η we effectively found good invariant surfaces of the type shown in Figures 2a – d, which agrees with the existence of 2 isolating integrals besides the energy in this problem. The order in which the coordinate axes are given in the caption of Figure 2 corresponds to the order indicated in Figure 1. Let the points of interest be near the origin of the axes of the representation. The direction of the observer E with respect to this origin is defined by the angles θ and ϕ as shown in Figure 1 and his distance to the origin is constant. A constant value of the angle of convergence of the two eyes towards the origin, $\alpha = 15°$, is adopted.

3. The 2: 2: 1-Resonance for the General Potential Problem

In this section we shall show that the particular example which we discussed in Section 2 presents a generic case for potential problems in 2: 2: 1-resonance.

I.e. perturbation of the Hamiltonian of Section 2 produces the same qualitative picture as has been sketched above. Consider the Hamiltonian

$$H = \tfrac{1}{2}(4x^2 + 4y^2 + z^2) + \tfrac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - \varepsilon\phi_3(x, y, z) + 0(\varepsilon^2). \tag{2}$$

ϕ_3 is a homogeneous cubic in x, y and z; the term of $0(\varepsilon^2)$ contains terms of degree 4 and higher in x, y and z. The Hamiltonian has been obtained by the usual scaling process in the vicinity of a stable, critical point of a Hamiltonian vector field, cf. for instance Verhulst (1979).

In Section 2 the Hamiltonian was treated by transforming to action-angle variables and averaging out the rapidly varying terms. Here we shall do essentially the same thing but by slightly different transformations. This produces the same result, as does Birkhoff transformation, but it has the additional advantage that it becomes transparent that the approximations are not formal but asymptotic in the mathematical sense.

We write explicitly

$$\phi_3 = a_1 x^3 + a_2 y^3 + a_3 z^3 + a_4 xy^2 + a_5 xz^2 + a_6 yz^2 + \\ + a_7 x^2 y + a_8 x^2 z + a_9 xyz + a_{10} y^2 z$$

and transform

$$x = r_1 \cos(2t + \Psi_1) \qquad \dot{x} = -2r_1 \sin(2t + \Psi_1)$$

$$y = r_2 \cos(2t + \Psi_2) \qquad \dot{y} = -2r_2 \sin(2t + \Psi_2)$$

$$z = r_3 \cos(t + \Psi_3) \qquad \dot{z} = -r_3 \sin(t + \Psi_3)$$

in which $r_i, \Psi_i, i = 1, 2, 3$ are functions of t. This is a generalized van der Pol-transformation which produces the standard form for averaging

$$\frac{dr_i}{dt} = \varepsilon f_i(r, \psi, t; a), \qquad \frac{d\psi_i}{dx} = \varepsilon g_i(r, \psi, t; a), \quad i = 1, 2, 3$$

where $r = (r_1, r_2, r_3)$, $\Psi = (\Psi_1, \Psi_2, \Psi_3)$, $a = (a_1, \ldots, a_{10})$. The equations in the standard form are 2π-periodic in t and we can perform first-order averaging over t; for details and other applications, see Verhulst (1979). The averaged equations produce solutions which approximate the phase-flow induced by the original Hamiltonian (2) to order ε on the time-scale $1/\varepsilon$.

We now obtain the remarkable result that of the coefficients $a_1 \ldots a_{10}$ only a_5 and a_6 are retained in the averaged equations. The same result is obtained by applying canonical Birkhoff transformation.

We conclude that the example treated in Section 2 represents the approximate behaviour of general potential problems with three degrees of freedom in 2:2:1-resonance.

We shall now derive the approximate integrals in the original coordinate system as this adds some elements to the discussion. The phase-flow induced by Hamiltonian (2) is approximated by the solutions of system

$$\ddot{x} + 4x = \varepsilon a_5 z^2 \tag{3}$$

$$\ddot{y} + 4y = \varepsilon a_6 z^2, \tag{4}$$

$$\ddot{z} + z = 2\varepsilon(a_5 x + a_6 y)z. \tag{5}$$

One easily deduces from Equations (3, 4)

$$\frac{d^2}{dt^2}(a_6 x - a_5 y) + 4(a_6 x - a_5 y) = 0.$$

This equation has the integral

$$(a_6 \dot{x} - a_5 \dot{y})^2 + 4(a_6 x - a_5 y)^2 \tag{6}$$

which is a second independent integral of system (3–5), the first integral being the Hamiltonian.

On introducing $u = a_5 x + a_6 y$ we derive from system (3–5)

$$\ddot{u} + 4u = \varepsilon(a_5^2 + a_6^2)z^2, \tag{7}$$

$$\ddot{z} + z = 2\varepsilon u z.$$

The system (7) becomes Hamiltonian after transforming $(a_5^2 + a_6^2)z^2 = \bar{z}^2$. Apart from the energy another approximate integral of system (7) is (see Verhulst, 1979)

$$(a_5 x + a_6 y)z^2 - (a_5 x + a_6 y)\dot{z}^2 + (a_5 \dot{x} + a_6 \dot{y})z\dot{z}, \tag{8}$$

where we replaced u, $\bar{z}$ again by x, y, z.

We note that the asymptotic validity of the integrals derived here is restricted to the time-scale $1/\varepsilon$. They may be valid for all time. The integral (6) is exact, the integral (8) is an approximate integral of system (3–5). Both (6) and (8) are approximate integrals of the general potential problem (Hamiltonian (2)).

We finally remark that if one performs Birkhoff transformation of the Hamiltonian (2) the cubic integral (8) is again produced by normalization up to order three; this result has been obtained by van der Aa and will be published in a more general setting later on.

References

Contopoulos, G.: 1978, *Celes. Mech.* **17**, 167.
Gustavson, F. G.: 1966, *Astron. J.* **71**, 670.
Martinet, L. and Magnenat, P.: 1981, *Astron. Astrophys.* **96**, 68.
Sanders, J. A. and Verhulst, F.: 1979, *Lecture Notes in Math.* **711**, 209, Springer-Verlag.
Van der Aa, E. and Sanders, J. A.: 1979, *Lecture Notes in Math.* **711**, 187, Springer-Verlag.
Verhulst, F.: 1979, *Phil. Trans. Roy. Soc. London* A **290**, 435.