Mh. Math. 93, 117—126 (1982)

Monatshefte für
Mathematik

Isometries of Spaces of Compact or Compact Convex Subsets of Metric Manifolds

By

P. M. Gruber and **R. Tichy**, Wien

(Received 15 May 1981)

Abstract. The isometries with respect to the Hausdorff metric of spaces of compact or compact convex subsets of certain compact metric spaces are precisely the mappings generated by isometries of the underlying spaces. In particular this holds when the underlying space is a finite dimensional torus or a sphere in a finite dimensional strictly convex smooth normed space.

1. Introduction

Let E^d, S^d, B^d denote the Euclidean space, its unit sphere and the solid unit ball respectively of dimension d. In recent papers (e. g. [5]—[9], [11]) the isometries of some of the spaces of compact or compact convex subsets of E^d, S^d, B^d endowed with the Hausdorff metric or the symmetric difference metric were completely described. We say that a mapping I of some space of subsets of a given space into itself is *generated* by a mapping i of the underlying space if $I(C) = i(C) = \{i(x) \mid x \in C\}$ for each subset C. It turned out that in the cases mentioned before the isometries of the spaces of subsets into themselfes were either generated by or were strongly related to isometries or measure preserving mappings of the underlying space.

Our aim is to prove some general theorems of this type (section 2) and to deduce from them the corresponding results for the torus T^d and the unit sphere U^d of any strictly convex smooth normed space of dimension d (section 3). Tools for the proofs are Brouwer's theorem on invariance of domain and a result of Danzer and Grünbaum on the maximal number of points with equal mutual distance in a finite dimensional normed space.

For a metric space $\langle M, \varrho \rangle$ define the *Hausdorff metric* δ on the space $\mathscr{K} = \mathscr{K}(M)$ of all nonempty compact subsets of M or on any subspace of $\mathscr{K}$ by

0026—9255/82/0093/0117/\$ 02.00

$$\delta(C,D) := \max\{\sup_{x\in C}\inf_{y\in D}\varrho(x,y),\ \sup_{y\in D}\inf_{x\in C}\varrho(x,y)\} \text{ for } C,D\in\mathscr{K}.$$

Call the metric space $\langle M,\varrho\rangle$ *antipodal* if it has finite diameter diam M and if for each $x\in M$ there exists a unique point $x^a\in M$ such that $\varrho(x,x^a) = \operatorname{diam} M$. The point x^a is called the antipode of x and the pair x, x^a an antipodal pair. We will not distinguish between $x\in M$ and the singleton $\{x\}\in\mathscr{K}(M)$. The symbol for the interior is int.

2. General Results

Theorem 1. *Let $\langle M,\varrho\rangle$ be an antipodal compact metric manifold. Denote by $\langle\mathscr{L},\delta\rangle$ a subspace of $\langle\mathscr{K}(M),\delta\rangle$ such that each set of $\mathscr{L}$ is connected and $\mathscr{L}$ contains all singletons. Then each isometry of $\langle\mathscr{L},\delta\rangle$ is generated by an isometry of $\langle M,\varrho\rangle$.*

Theorem 2. *Let $\langle M,\varrho\rangle$ be an antipodal compact metric space and suppose that there exists a number $m = m(M)$ such that some m points of M have pairwise equal distance (>0) but no $m+1$ points of M have this property. Let $\langle\mathscr{L},\delta\rangle$ be a subspace of $\langle\mathscr{K}(M),\delta\rangle$ with the following properties: $\mathscr{L}$ contains all singletons and for each singleton $L_0\in\mathscr{L}$ there exist sets $L_1,\dots,L_m\in\mathscr{L}$ all of them containing L_0 and such that $L_0,\dots,L_m$ have pairwise equal distance (>0). Then each isometry of $\langle\mathscr{L},\delta\rangle$ is generated by an isometry of $\langle M,\varrho\rangle$.*

Preliminaries. Here we collect some results which will be used in the proofs of the theorems.

It is well known that

each isometry of a compact metric space into itself actually is onto (1)

(see e.g. [10], p. 45). An immediate consequence of Brouwer's theorem on invariance of domain (see e.g. [1], p. 396) is:

A homeomorphism of an open subset of a manifold M into M has an open image. (2)

We prove,

let $\langle M,\varrho\rangle$ be an antipodal metric space and let $C,D\in\mathscr{K}(M)$ be such that $\delta(C,D) = \operatorname{diam} M$. Then $C=\{c\}$ and $c^a\in D$ or $D=\{d\}$ and $d^a\in C$. (3)

Suppose that D is not a singleton. This together with the compactness of C and D, the continuity of ϱ and the uniqueness of the antipode yields

$$\sup_{x\in C}\inf_{y\in D}\varrho(x,y)=\inf_{y\in D}\varrho(p,y)<\operatorname{diam}M,$$
$$\sup_{y\in D}\inf_{x\in C}\varrho(x,y)=\inf_{x\in C}\varrho(x,q)\leqslant\operatorname{diam}M \tag{4}$$

for suitable $p\in C$, $q\in D$. Since $\delta(C,D)=\operatorname{diam}M$, equality holds in (4). This implies $C=\{q^a\}$. Hence $C=\{c:=q^a\}$, $c^a=q\in D$, thus proving (3).

Let $\langle M,\varrho\rangle$ be an antipodal compact metric space. Then the antipodal mapping $a:M\to M$, defined by $a(x):=x^a$ is a homeomorphism of M onto M. (5)

Obviously a is onto and bijective. Since M is compact it suffices for the proof that a is a homeomorphism to show that a is continuous. Choose $x,x_1,x_2,\ldots\in M$ such that $x_1,x_2,\ldots\to x$. We are finished if we can show that each limit point of the sequence $x_1^a,x_2^a,\ldots$ coincides with x^a. (Note that M is compact.) Let y be any limit point of this sequence and assume that $x_{k_1}^a,x_{k_2}^a,\ldots\to y$. Then $\varrho(x,y)=$ $=\lim\varrho(x_{k_j},x_{k_j}^a)=\operatorname{diam}M$. Hence the uniqueness of the antipode implies $y=x^a$. This completes the proof of (5).

Let $\langle M,\varrho\rangle$ be an antipodal compact metric space. Suppose that $\langle\mathscr{L},\delta\rangle$ is a subspace of $\langle\mathscr{K}(M),\delta\rangle$ containing all singletons. Let I be an isometry of $\langle\mathscr{L},\delta\rangle$ such that $I(x)$ is a singleton for each $x\in M$. Then I is generated by an isometry of $\langle M,\varrho\rangle$. (6)

Since $\varrho(x,y)=\delta(x,y)$ for $x,y\in M$, the map

$i:M\to M$ defined by $i(x):=I(x)$ is an isometry of $\langle M,\varrho\rangle$. (7)

To prove (6) we show that

$$I(C)=i(C)\text{ for each } C\in\mathscr{L}. \tag{8}$$

Choose C. In order to see that the inclusion

$$i(C)\subset I(C) \tag{9'}$$

holds, let $x\in C$. By (7)

$$\operatorname{diam}M=\delta(x^a,C)=\delta(I(x^a),I(C))=\delta(i(x^a),I(C)).$$

Hence (3) yields $(i(x^a))^a \in I(C)$. Since i is an isometry it maps pairs of antipodal points onto pairs of antipodal points. Therefore $(i(x^a))^a = i(x^{aa}) = i(x)$. This shows that $i(x) \in I(C)$, thus proving (9′). To prove the converse inclusion

$$I(C) \subset i(C), \tag{9''}$$

let $y \in I(C)$. Then $y = i(x)$ for some $x \in M$ by (7) and (1). Hence $y^a = (i(x))^a = i(x^a)$ by the same argument as before. This yields

$$\operatorname{diam} M = \delta(y^a, I(C)) = \delta(i(x^a), I(C)) = \delta(I(x^a), I(C)) = \delta(x^a, C).$$

From this together with (3) we conclude that $x = x^{aa} \in C$ and thus $y = i(x) \in i(C)$. This proves (9″). It follows from (9′) and (9″) that $I(C) = i(C)$ thus finishing the proof (8). This concludes the proof of (6).

Proof of Theorem 1. Suppose that I is an isometry of $\langle \mathscr{L}, \delta \rangle$. Define

$$S := \{x \in M \mid I(x) \text{ is a singleton}\}.$$

Since the mapping $x \to \operatorname{diam} I(x)$ for $x \in M$ is continuous,

$$S \text{ is closed.} \tag{10}$$

We show

$$S \cup S^a = M. \tag{11}$$

For $x \in M$ we have $\{x\}, \{x^a\} \in \mathscr{L}$ and

$$\operatorname{diam} M = \delta(x, x^a) = \delta(I(x), I(x^a)).$$

Thus (3) implies that $I(x)$ or $I(x^a)$ is a singleton. Hence $x \in S$ or $x^a \in S$ and therefore $x \in S \cup S^a$. This proves (11).

$$T := \{y = I(x) \mid x \in \operatorname{int} S\} \text{ is open in } M. \tag{12}$$

The mapping $x \to y = I(x)$ for $x \in \operatorname{int} S$ is an isometry and hence a homeomorphism of the open subset $\operatorname{int} S$ of the manifold M onto $T \subset M$. By (2) this yields (12). To prove

$$\operatorname{int} S \subset S^a \tag{13}$$

choose $x \in \operatorname{int} S$. Let $y = I(x)$. Suppose $x \notin S^a$. Since

$$\operatorname{diam} M = \delta(x, x^a) = \delta(I(x), I(x^a)) = \delta(y, I(x^a)),$$

(2) yields that $y^a \in I(x^a)$. By definition of T we have $y \in T$ and thus $y^a \in T^a$. Since by (12) T is open, (4) implies that T^a is open too. By

assumption $x \notin S^a$ and thus $x^a \notin S$. This together with the definition of S and $y^a \in I(x^a)$ shows that $\{y^a\} \subsetneq I(x^a)$. By our assumption on $\mathscr{L}$, the set $I(x^a)$ is connected. Therefore we can choose a point $z^a \in I(x^a) \cap T^a \setminus \{y^a\}$. Since $z \in T$, we have $z = I(u)$ for some $u \in \operatorname{int} S$. From $I(u) = z \neq y = I(x)$ it follows that $u \neq x$. From

$$\operatorname{diam} M = \delta(I(x^a), z) = \delta(I(x^a), I(u)) = \delta(x^a, u)$$

we infer that x^a, u is an antipodal pair and thus $x = x^{aa} = u$, a contradiction. This finishes the proof of (13). Propositions (13) and (4) yield

$$\operatorname{int} S = \operatorname{int} S^a. \tag{14}$$

In order to prove that

$$S \text{ is dense in } M \tag{15}$$

it is sufficient to prove that any nonempty open subset E of M meets S. Let E be a nonempty open subset of M. If E does not meet S we have $E \subset S^a$ by (11) and thus $E \subset \operatorname{int} S^a$. Hence (14) implies $E \subset \operatorname{int} S \subset S$, a contradiction. This proves (15). It follows from (10) and (15) that $S = M$, that is,

$$I(x) \text{ is a singleton for each } x \in M.$$

Now (5) applies. It follows that I is generated by an isometry of $\langle M, \varrho \rangle$.

Proof of Theorem 2. Let I be an isometry of $\langle \mathscr{L}, \delta \rangle$. We show that

$$I(x) \text{ is a singleton for each } x \in M. \tag{16}$$

Assume the contrary. Then for some $x \in M$ the set $I(x)$ is not a singleton.

$$\operatorname{diam} M = \delta(x, x^a) = \delta(I(x), I(x^a)).$$

Hence $I(x^a) = y_0$ for suitable $y_0 \in M$ by (2). Let $L_0 := \{x^a\}$ and choose $L_1, \ldots, L_m \in \mathscr{L}$ such that

$$x^a \in L_1, \ldots, L_m, \tag{17}$$

$$\delta(x^a, L_k) = \delta(L_j, L_k) = \alpha > 0 \text{ for } j, k \in \{1, \ldots, m\}, j \neq k. \tag{18}$$

(17) implies that

$$\operatorname{diam} M = \delta(x, L_k) \text{ for } k \in \{1, \ldots, m\}. \tag{19}$$

Applying I to (18), (19) we obtain

$$\delta(y_0, I(L_k)) = \delta(I(L_j), I(L_k)) = \alpha \text{ for } j, k \in \{1, \ldots, m\}, j \neq k, \tag{20}$$

$$\operatorname{diam} M = \delta(I(x), I(L_k)) \text{ for } k \in \{1, \ldots, m\}. \tag{21}$$

Since $I(x)$ is not a singleton, it follows from (2) and (21) that $I(L_1), \ldots, I(L_m)$ are singletons. This together with (20) shows that there exist $m+1$ points of M having pairwise distance α which contradicts the definition of m and thus proves (16). (16) together with (4) yields that I is generated by an isometry of $\langle M, \varrho \rangle$.

3. Torus and Sphere

We denote the d-dimensional torus $E^d/\mathbb{Z}^d$ by T^d. One can represent T^d in the form $[0, 1[^d$ with addition mod 1 in each component. The quotient metric ϱ on T^d is defined by

$$\varrho(x, y) = \inf\{\|x - y + l\| \mid l \in \mathbb{Z}^d\} \text{ for } x, y \in T^d$$

where $\| \ \|$ denotes the Euclidean norm on E^d. The isometries of $\langle T^d, \varrho \rangle$ are the mappings of the form

$$x = (\xi_1, \ldots, \xi_d) \to (\varepsilon_1 \xi_{i_1} + \tau_1, \ldots, \varepsilon_d \xi_{i_d} + \tau_d) \bmod 1 \text{ for } x \in T^d$$

where $\varepsilon_1, \ldots, \varepsilon_d = \pm 1$, $(i_1, \ldots, i_d)$ is a permutation of $(1, \ldots, d)$ and $\tau_1, \ldots, \tau_d \in \mathbb{R}$. For any $(d+1)$-dimensional strictly convex smooth normed space denote its unit sphere by U^d and denote the metric on U^d induced by the norm by ϱ. It seems to be an open problem to characterize the isometries of $\langle U^d, \varrho \rangle$, unless U^d is of a very simple type, e. g. $U^d = S^d$. This problem belongs to the class of rigidity problems for convex surfaces.

A continuous curve in a metric space $\langle M, \varrho \rangle$ connecting two points is called a geodesic segment if it has minimal length among all such curves. Call a subset of M *geodesically convex* if any two of its points can be connected by a geodesic segment contained in the set (see e. g. [12]). Denote by $\mathscr{C}_g = \mathscr{C}_g(M)$ the space of all compact geodesically convex subsets of M. For special spaces there exist various concepts of convexity (see e. g. [4], p. 157—163, [2] and the references there). In particular we call a subset of U^d *strongly convex* if it can be represented as the intersection of U^d with a closed convex cone with unique apex at the origin. A subset of U^d is called *Robinson convex* if it is connected and can be presented as intersection of U^d with a family of closed half spaces, each containing the origin on its boundary plane. Denote the spaces of strongly convex and of Robinson convex subsets of U^d by $\mathscr{C}_s = \mathscr{C}_s(U^d)$ and $\mathscr{C}_r = \mathscr{C}_r(U^d)$.

Theorem 3. *The isometries of the spaces* $\langle \mathscr{C}_g(T^d), \delta \rangle$ *and* $\langle \mathscr{K}(T^d), \delta \rangle$ *are precisely the mappings generated by the isometries of* $\langle T^d, \varrho \rangle$.

Theorem 4. *The isometries of the spaces* $\langle \mathscr{C}_g(U^d), \delta \rangle$, $\langle \mathscr{C}_s(U^d), \delta \rangle$, $\langle \mathscr{C}_r(U^d), \delta \rangle$ *and* $\langle \mathscr{K}(U^d), \delta \rangle$ *are precisely the mappings generated by the isometries of* $\langle U^d, \varrho \rangle$.

Proof of Theorem 3. It is obvious that each isometry of $\langle T^d, \varrho \rangle$ generates isometries of $\langle \mathscr{C}_g(T^d), \delta \rangle$ and $\langle \mathscr{K}(T^d), \delta \rangle$.

To prove the converse we note that $\langle T^d, \varrho \rangle$ is a compact metric manifold (of dimension d) and for each $x = (\xi_1, \ldots, \xi_d) \in T^d$ the point $x^a := (\xi_1 + \frac{1}{2}, \ldots, \xi_d + \frac{1}{2}) \bmod 1$ is the unique antipode of x. It has distance $\sqrt{d}/2 = \operatorname{diam} T^d$ from x. Hence $\langle T^d, \varrho \rangle$ is antipodal.

Theorem 1 implies that each isometry of $\langle \mathscr{C}_g(T^d), \delta \rangle$ is generated by an isometry of $\langle T^d, \varrho \rangle$.

In the case of $\langle \mathscr{K}(T^d), \delta \rangle$ we will apply Theorem 2. Our first aim is to prove the following proposition:

Let $x_1, \ldots, x_m \in T^d$ be m points with equal mutual distance $\alpha > 0$. Then $m \leqslant 3^d$. (22)

Consider $y_1 := x_1$ as a point of E^d. Then by definition of ϱ there are points $y_2, \ldots, y_m \in E^d$ equivalent mod 1 to $x_2, \ldots, x_m$ respectively, such that

$$\|y_1 - y_j\| = \alpha, \ \|y_j - y_k\| \geqslant \alpha \text{ for } j, k \in \{2, \ldots, m\}, j \neq k.$$

Hence the m balls of radius $\alpha/3$ and centers at $y_1, \frac{1}{3}y_1 + \frac{2}{3}y_2, \ldots, \frac{1}{3}y_1 + \frac{2}{3}y_m$ have disjoint interior and are contained in the ball of radius α and center y_1. Thus $m \cdot (\frac{1}{3})^d \leqslant 1$. This proves (22). (22) implies that

$$m(T^d) \text{ exists and is } \leqslant 3^d. \tag{23}$$

We next show,

for each singleton $L_0 = \{x_0\} \in \mathscr{K}(T^d)$ there are $4^d - 1$ $(\geqslant m(T^d))$ sets in $\mathscr{K}(T^d)$ containing L_0 which have equal distance (>0) from L_0 and from each other. (24)

The $2d$ points

$$x_j = x_0 + (0, \ldots, 0, \underset{\substack{\uparrow \\ |j|\text{-th place}}}{\tfrac{1}{4} \operatorname{sign} j}, 0, \ldots, 0) \bmod 1, j \in \{\pm 1, \ldots, \pm d\}$$

have distance $\frac{1}{4}$ from x_0 and distance $>\frac{1}{4}$ from each other. Hence the $4^d - 1$ sets

$$L_J := \{x_0\} \cup \{x_j \mid j \in J\} \in \mathscr{K}(T^d), \ \emptyset \neq J \subset \{\pm 1, \ldots, \pm d\}$$

are as required in (24). Propositions (23) and (24) make sure that one can apply Theorem 2 to $\langle \mathscr{K}(T^d), \delta \rangle$. Thus each isometry of $\langle \mathscr{K}(T^d), \delta \rangle$ is generated by an isometry of $\langle T^d, \varrho \rangle$.

Proof of Theorem 4. As before each isometry of $\langle U^d, \varrho \rangle$ generates isometries of $\langle \mathscr{C}_g(U^d), \delta \rangle$, $\langle \mathscr{C}_s(U^d), \delta \rangle$, $\langle \mathscr{C}_r(U^d), \delta \rangle$ and $\langle \mathscr{K}(U^d), \delta \rangle$.

$\langle U^d, \varrho \rangle$ is a compact metric manifold of dimension d. Since U^d is the unit spere of a strictly convex normed space for each $x \in U^d$ the point $x^a := -x$ is the unique antipode and $\varrho(x, x^a) = 2 = \operatorname{diam} U^d$. Hence $\langle U^d, \varrho \rangle$ is antipodal.

By Theorem 1 each isometry of $\langle \mathscr{C}_g(U^d), \delta \rangle$, $\langle \mathscr{C}_s(U^d), \delta \rangle$, $\langle \mathscr{C}_r(U^d), \delta \rangle$ is generated by an isometry of $\langle U^d, \varrho \rangle$.

We now consider $\langle \mathscr{K}(U^d), \delta \rangle$. Since $\langle U^d, \varrho \rangle$ is isometrically embedded in a $(d+1)$-dimensional strictly convex normed space (with unit sphere U^d), there can be at most $2^{d+1} - 1$ points with equal mutual distance (> 0). This is an immediate consequence of a result of DANZER and GRÜNBAUM [3]. Hence

$$m(U^d) \text{ exists and is } \leqslant 2^{d+1} - 1. \tag{25}$$

We prove,

for each singleton $L_0 = \{x_0\} \in \mathscr{K}(U^d)$ there exist $4^d - 1$ $(\geqslant m(U^d))$ sets in $\mathscr{K}(U^d)$ all of which contain L_0 and have equal distance (> 0) from L_0 and from each other. (26)

Choose $L_0 = \{x_0\} \in U^d$. Since U^d is smooth there exists a unique supporting hyperplane of U^d containing x_0, say $x_0 + H$. Inscribe to $H \cap U^d$ a d-dimensional crosspolytope (i. e. d-octahedron) which is symmetric in the origin and has the following property: The supporting hyperplane of U^d at any pair of its vertices $\pm x_j$ is parallel to the subspace generated by the vertices $\pm x_1, \ldots, \pm x_{j-1}, \pm x_{j+1}, \ldots, \pm x_d$. Hence the strict convexity of the norm $|\ |$ with U^d as unit sphere yields

$$|x_j - (-x_j)| = 2 \geqslant 1 + 2\varepsilon, \ |\pm x_j - (\pm x_k)| \geqslant 1 + 2\varepsilon \quad \text{for } j, k \in \{1, \ldots, d\}, j \neq k \tag{27}$$

for suitable $\varepsilon > 0$. For $\mu > 0$ we consider the $2d$ points

$$x_0 \pm \mu x_j \in (x_0 + \mu U^d) \cap (x_0 + H), j \in \{1, \ldots, d\}.$$

By (27) these points have mutual distance $\geqslant \mu(1 + 2\varepsilon)$ and distance μ from x_0. Since U^d is smooth and thus differentiable at x_0, we can choose $\mu > 0$ so small that for suitable points

$$y_{\pm j} \in (x_0 + \mu U^d) \cap U^d, j \in \{1, \ldots, d\}$$

(chosen close to the corresponding points $x_0 + \mu x_j$) the following holds: These points have mutual distance $\geqslant \mu(1 + \varepsilon)$ and distance μ from x_0. Hence the $4^d - 1$ sets

$$L_J := \{x_0\} \cup \{y_k \mid k \in J\} \in K(U^d), \emptyset \neq J \subset \{\pm 1, \ldots, \pm d\}$$

satisfy (26). (25) and (26) together with Theorem 2 imply that each isometry of $\langle \mathscr{K}(U^d), \delta \rangle$ is generated by an isometry of $\langle U^d, \varrho \rangle$.

4. Final Remarks

As can be seen from the proofs, our theorems can be refined somewhat. In particular it is possible to replace U^d of Theorem 4 by any smooth closed convex surface of constant width of a finite-dimensional strictly convex normed space.

It remains an open question to prove the corresponding results for the Euclidean unit ball B^d and the hyperbolic space H^d. Furthermore many problems of this type for the case of the symmetric difference metric are not yet settled.

In general one cannot expect that each isometry with respect to the Hausdorff metric of a space of closed bounded subsets of a metric space is generated by an isometry of the underlying space. One example for this is provided by E^d (see [7]). Another example is as follows: Let S be the unit sphere of l_2 endowed with the metric ϱ induced by the norm of l_2. Let $\mathscr{C}$ be the space of closed circular caps $C(m, \varepsilon)$ on S with center $m \in S$ and radius $\varepsilon \geqslant 0$ endowed with the Hausdorff metric δ. Define an isometry i of $\langle S, \varrho \rangle$ by $i(x) := (0, \xi_1, \xi_2, \ldots)$ for $x = (\xi_1, \xi_2, \ldots) \in l_2$. Then the isometry I of $\langle \mathscr{C}, \delta \rangle$ defined by $I(C(m, \varepsilon)) := C(i(m), \varepsilon)$ is not generated by an isometry of $\langle S, \varrho \rangle$.

Acknowledgement. For many discussions we thank Dr. LETTL. Many thanks are due to Prof. WEISS for his help in the preparation of the manuscript.

References

[1] ALEXANDROFF, P., HOPF, H.: Topologie I. Berlin: Springer. 1935.

[2] BOLTYANSKI, V. G., SOLTAN, P. S.: Combinatorial geometry and convexity classes. Uspekhi Mat. Nauk **33**, 3—42 (1978) and Russian Math. Surveys **33**, 1—45 (1978).

[3] DANZER, L., GRÜNBAUM, B.: Über zwei Probleme bezüglich konvexer Körper von P. Erdös und V. Klee. Math. Z. **79**, 95—99 (1962).

[4] DANZER, L., GRÜNBAUM, B., KLEE, V.: Helly's theorem and its relatives. In: Convexity. Proc. Sympos. Pure Math., Seattle 1961, pp. 101—180. Providence: Amer. Math. Soc. 1963.

[5] GRUBER, P. M.: Isometries of the space of convex bodies of E^d. Mathematika **25**, 270—278 (1978).

[6] GRUBER, P. M.: Isometries of the space of convex bodies contained in a Euclidean ball. Manuscript.

[7] GRUBER, P. M., LETTL, G.: Isometries of the space of convex bodies in Euclidean space. Bull. London Math. Soc. **12**, 455—460 (1980).

[8] GRUBER, P. M., LETTL, G.: Isometries of the space of compact subsets of E^d. Studia Math. Hungar. (In print.)

[9] LETTL, G.: Isometrien des Raumes der konvexen Teilmengen der Sphäre. Archiv Math. **35**, 471—475 (1980).

[10] NABER, G. L.: Topological Methods in Euclidean Spaces. Cambridge: University Press. 1980.

[11] SCHNEIDER, R.: Isometrien des Raumes der konvexen Körper. Coll. Math. **33**, 219—224 (1975).

[12] WALTER, R.: Konvexität in Riemannschen Mannigfaltigkeiten. Jber. Deutsch. Math. Ver. **83**, 1—31 (1981).

Prof. Dr. P. GRUBER
and
Dr. R. TICHY
Institut für Analysis,
Technische Mathematik und
Versicherungsmathematik,
Technische Universität Wien
Gusshausstrasse 27
A-1040 Wien, Austria