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The Sub-Semi-Groups Excluding Zero of a Near-Ring

By

George Szeto, Peoria, Ill., USA

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Introduction. Let R be a finite near-ring with no zero divisors. Then the additive group of R and the possible multiplications defined on $(R, +)$ are studied in [3] and [4]. The purpose of the present paper is to study the semi-group $(R—0)$ where R is not necessarily finite. In section one, it is proved that the semi-group $(R—0)$ is a union of the following two types of sub-semi-groups: (1) a sub-semi-group whose elements do not have identities and (2) a sub-semi-group each of whose elements has an identity. A structural theorem of type 2 is also derived. Then in section two, some applications of the above results are given. By using the structural theorem, we can compute the number of all sub-semi-groups excluding zero of R when R is finite. A property of a quotient near-ring with no zero divisors is obtained when each element of R is a power of itself as studied by Bell in [1].

Throughout the present paper we assume that R is a near-ring with $0r = 0$ for all r in R. All definitions can be found in [1], [2] and [3].

Section 1. First we recall the Pierce decomposition theorem for near-rings which will help us to develope a series of lemmas for our theorem in this section. R has no zero divisors except for the proposition below.

Proposition 1.1. *Let R be a near-ring with idempotent e. Then $R \cong eR + R'$ such that $eR' = 0$.*

Lemma 1.2. *Let* $\mathrm{Id}(R) = \{a \mid a \neq 0$, *there exists a non-zero element* b *such that* $ba = b\}$. *Then* $\mathrm{Id}(R) = \{x \mid x$ *is a non-zero idempotent in* R; *that is,* $x^2 = x\} = \{x \mid x$ *is a left identity of* $R\}$.

Proof. For each non-zero idempotent x in R, since $x^2 = xx = x$, x is in $\mathrm{Id}(R)$. Conversely, for each element a in $\mathrm{Id}(R)$, there exists a non-zero element b such that $ba=b$. Hence $ba^2=ba$; and so $b(a^2-a)=0$. Noting that R has no zero divisors we have $a^2=a$. Thus the element a is a non-zero idempotent. Moreover, for each left identity x of R, $xx=x$, so x is an idempotent. Conversely, for each non-zero idempotent x, $R\cong xR+R'$ with $xR'=0$ by the Pierce decomposition theorem for near-rings. But x is not a zero divisor then $xR'=0$. Hence $xR\cong R$. Thus, for any r in R, $xr=r_1$ implies $x^2r=xr_1$, therefore $xr=xr_1$. Since R is free of divisors of zero, $r=r_1=xr$. This shows that x is a left identity of R.

Lemma 1.3. *Let S be a subset of $(R-0)$ such that each element in S has no right identity. Then S is a sub-semi-group of $(R-0)$.*

Proof. For each x,y in S, suppose to the contrary that xy has a right identity a. Then $xya=xy$; and so $x(ya-y)=0$. Since x is not a zero divisor, $ya=y$. This contradicts to that y has no right identity. Thus xy is in S.

Lemma 1.4. *Let $S'=\{x\,|\,x$ has a right identity$\}$. Then*

(1) S' is a sub-semi-group of $(R-0)$,

(2) S' is a union of non-overlapping sub-semi-groups each having an identity; that is, $S'=\bigcup S'_\alpha$ where α is in some index set U and S'_α is a semi-group with identity,

(3) $S'_\alpha\cong S'_\beta$ as semi-group isomorphism for each α,β in U.

Proof. For a, b in S' such that x is a right identity of b, $(ab)x=a(bx)=ab$, so (1) is proved. Here we note that the right identity of each element x in S' is unique because $xa=x=xb$ implies $x(a-b)=0$. Hence $a=b$. Then, we can define a relation "$\sim$" on S' by $a\sim b$ if and only if the right identity of a is equal to the right identity of b. It is easy to see that this relation is an equivalence relation. Denote the equivalence classes by S'_α for α in some index set U. It is clear that each class is a semi-group with identity. This proves part (2). For part (3), let S'_α be an equivalence class with the identity a'. We claim that $S'_\alpha=(R-0)a'$. In fact, since for each b in $S'_\alpha, b=ba'$ which is in $(R-0)a'$ and since for each s in $(R-0)$, $(sa')a'=sa'$ which is in S'_α; $S'_\alpha=(R-0)a'$. Let S'_β be $(R-0)b'$ for an β in U, where b' is the right identity of the class S'_β. Define a map λ from S'_α to S'_β by $sa'\to sb'$. It is easy to see that λ is well defined. Also, for each sa', ta' in S'_α, we have $[(sa')(ta')]\lambda=[s(a't)a']\lambda=(sta')\lambda=(st)b'=(sb')(tb')$ (for a' and b' are left identities of R by lemma 1.2). But this is equal to

$(sa')\lambda(ta')\lambda$ then λ is a homomorphism. Moreover, λ is onto because for each sb' in S'_β, $(sa')\lambda = sb'$ with (sa') in S'_α. Furthermore, let (sa') be an element in S'_α such that $(sa')\lambda = b'$. Then $(sa')b' = b'$. On the other hand, since b' is a left identity of R by lemma 1.2, so is $(sa')b'$; and so (sa') is a left identity of R in S'_α. But there is only one left identity of R in S'_α which is the identity of S'_α, then $sa' = a'$. Thus λ is a one to one map. Therefore is a semi-group isomorphism.

From lemma 1.3 and lemma 1.4 the following theorem is immediate:

Theorem 1.5. *If R is a near-ring with no zero divisors, then $(R-0)$ is a union of a semi-group whose elements have no right identities and a semi-group which is a union of the isomorphic sub-semi-groups each with an identity.*

Bell studied such a kind of near-rings that each element is a power of itself in [1]. For this R with no zero divisors we have:

Corollary 1.6. *If for each element x in R there is an integer $n(x)$ depending on x such that $x^{n(x)+1} = x$, then*

(1) each non-zero element has a right identity,

(2) let S denote $(R-0)$; then S is a union of non-overlapping groups. That is, $S = \bigcup S_\alpha$, where α is in some index set V and S_α is a group,

(3) $S_\alpha \cong S_\beta$ as group isomorphism for each α, β in V.

Proof. Since for each non-zero element x in R there is an integer $n(x)$ such that $x^{n(x)+1} = x, x^{n(x)}$ is the identity of x; and so $x^{n(x)-1}$ is the inverse of x. But then this corollary is immediate from the above theorem.

Corollary 1.7. *If R is finite then the three consequences in corollary* 1.6 *hold.*

Remark 1. If R has zero divisors then $(R-0)$ is not a semi-group and lemma 1.2 does not hold.

Remark 2. If there is a non-zero right distributive element in R and if $\mathrm{Id}(R)$ is not empty in lemma 1.2; then $\mathrm{Id}(R)$ has only one element. Thus there is only one equivalence class in theorem 1.5 and a distributively-generated near-ring in corollaries 1.6 and 1.7 is a field ([1], Th. 2).

Remark 3. In lemma 1.4, S'_α is a maximal sub-semi-group with identity and in corollary 1.6, S_α is a maximal subgroup for each α.

Section 2. In this section we shall give two applications of the results in section one. The first one is a property of a quotient near-ring; the second one is the number of all sub-semi-groups excluding 0 in a finite near-ring with no zero divisors. In [1], BELL studied a near-ring R with the property that for each $x \neq 0$ in R there exists an integer $n(x) > 0$ depending on x such that $x^{n(x)+1} = x$. Then he constructed a collection of maximal sub-semi-groups M excluding 0 of R such that (1) $A(M)$ is an ideal for each M where $A(M) = \{r \mid \text{there is an element } b \text{ in } M \text{ such that } br = 0\}$, (2) $\bigcap_M A(M) = 0$ for all M and (3) M and $A(M)$ are set complements of each other. By the same proof of BELL we have a general case for the same kind of near-rings ([1], Lemma 3.1).

Lemma 2.1. *If S is a sub-semi-group of R excluding 0, then*

(1) $A(S)$ is an ideal of R where $A(S) = \{r / \text{there is an element } a \text{ in } S \text{ such that } ar = 0\}$;

(2) $S + A(S)$ is a sub-semi-group excluding 0 of R:

(3) $S \cap A(S) = \emptyset$, a void set.

The part (1) of the above lemma gives a quotient near-ring $R/A(S)$. In particular, let M be a maximal sub-semi-group excluding 0. If there is an element a in M with $a^{n(a)+1} = a$, then we have a quotient near-ring $R/A(Ma^{n(a)})$.

Theorem 2.2. *If a and b are two elements with $ab \neq 0$, then there is a maximal sub-semi-group M excluding 0 such that*

(1) a and b are in M:

(2) $(Ma^{n(a)} + A(Ma^{n(a)}) / A(Ma^{n(a)}) \cong (\mathrm{M\,b}^{m(b)}) + \mathrm{A}(Mb^{m(b)}) / A(Mb^{m(b)})$ as group isomorphism, where $a^{n(a)+1} = a$ and $b^{m(b)+1} = b$.

Proof. The existence of such an M is from lemma 1.3 in [1]. For part (2), since $\bar{a} \neq \bar{0}$ in $R/A(M)$ and since $R/A(M)$ has no zero divisors; $(Ma^n + A(M))/A(M) \cong (Mb^m + A(M))/A(M)$ as group isomorphism by corollary 1.6. Hence it suffices to show that $A(M) = A(Ma^n) = A(Mb^m)$. Obviously, $A(Ma^n)$ is contained in $A(M)$ because Ma^n is contained in M. Conversely, for each x in $A(M)$ there is an element s in M such that $sx = 0$. Hence by the intersection-of-factors property (IFP) in [1] ([1], Lemma 1), $srx = 0$ for all r in R; and so $sa^n x = 0$. Noting that sa^n is in Ma^n we have that x is in $A(Ma^n)$. Therefore $A(Ma^n) = A(M)$. Similarly $A(Mb^m) = A(M)$. This completes the proof.

When R is a finite near-ring with no zero divisors, the number of all non-isomorphic sub-semi-groups excluding 0 can be computed by using corollary 1.7.

Theorem 2.3. *Let x be a non-zero element in R with $x^{n+1}=x$ for some integer n. Then*

(1) $(R-0)x^n$ is a group with identity x^n;

(2) there are $m(2^k-1)$ non-isomorphic sub-semi-groups excluding 0 for R, where m = the number of non-isomorphic subgroups of $(R-0)x^n$ and k = the number of all left identities of R.

Proof. By corollary 1.7, $(R-0)x^n$ is a group with identity x^n and $(R-0)$ is a union of non-overlapping groups each isomorphic to $(R-0)x^n$, so part (1) is proved. This also implies that the number of such groups is equal to the number of $(R-0)$ divided by the number of $(R-0)x^n$. On the other hand, this number is the number k of all left identities of R by lemma 1.2. Moreover, by the argument as given in corollary 1.7, let S be a sub-semi-group excluding 0; then for each t, s in S with $s^{p+1}=s$ and $t^{u+1}=t$, $Ss^p \cong St^u$. Hence S is a union of non-overlapping isomorphic groups $\{Ss^p$ with some integer p and s in $S\}$. Thus a subset B of left identities of R and a subgroup of $(R-0)x$ for an x in B are determined by a given sub-semi-group excluding 0. Conversely, given a subset B of left identities of R and a subgroup H with identity s^p in B for some integer p, we have a sub-semi-group $S=\bigcup Hs^p$ for all s^p in B. It is easy to see that Hs^p is a subgroup of $(R-0)s^p$ for each s^p in B and that $Ss^p \cong Hs^p \cong Ht^u \cong St^u$ for all s^p and t^u in B. Thus a sub-semi-group S is determined by a subset B of left identities of R and a subgroup H of $(R-0)s^p$ for an element s^p in B. Now let k be the number of all left identities of R. Then there are (2^k-1) subsets of left identities. Therefore, let m be the number of non-isomorphic subgroups of $(R-0)x^n$ for some left identity x^n of R. We conclude that the number of non-isomorphic sub-semi-groups excluding 0 of R is equal to $m(2^k-1)$.

Remark. If R is infinite, then the fact is still true that a sub-semi-group excluding 0 is determined up to isomorphism by a subset B of left identities and a subgroup of $(R-0)x^n$ for an element x^n in B, and vice versa.

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Author's address:

Prof. G. SZETO
Mathematics Department
Bradley University
Peoria, IL 61606, USA