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HANKEL OPERATORS ON THE BERGMAN SPACES OF STRONGLY PSEUDOCONVEX DOMAINS[1]

Huiping Li

We characterize functions $f \in L^2(D)$ such that the Hankel operators H_f are, respectively, bounded and compact on the Bergman spaces of bounded strongly pseudoconvex domains.

1. **INTRODUCTION** Let D be a bounded strongly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$, $n \geq 2$. The Bergman space $H^2(D)$, consisting of holomorphic L^2 functions, is a closed subspace of the Hilbert space $L^2(D)$. The Bergman projection P is the orthogonal projection from $L^2(D)$ onto $H^2(D)$ defined by $Pf(z)= \int K(z,w)f(w)dv(w)$. Here $K(z,w)$ is the Bergman kernel of D , and dv the usual Lebesgue measure. For $f \in L^2(D)$, the Hankel operator H_f from $H^2(D)$ into $L^2(D)$ is defined by $H_f(g)=(I-P)(f\cdot g)$. H_f is densely defined on $H^2(D)$. In [3], Bekolle, Berger, Coburn and Zhu give necessary and sufficient conditions for the boundedness and compactness of both H_f and $H_{\bar{f}}$ with $f \in L^2(\Omega)$ on the bounded symmetric domains Ω . In [9], we proved that the conditions in [3] are sufficient for the boundedness and compactness of both H_f and $H_{\bar{f}}$ on bounded strongly pseudoconvex domains in $\mathbb{C}^n$. Recently, D. Luecking [12] characterized functions $f \in L^2(\Delta)$ such that H_f are, respectively, bounded and compact on the unit disc Δ of the complex plane $\mathbb{C}$. At the end of the paper, Luecking pointed out the difficulties in extending his results to the unit ball and to the so called Fock space in $\mathbb{C}^n$. In this paper, we overcome those difficulties by using the integral representations of solutions to the $\bar{\partial}$-equations. In fact, we characterize the functions $f \in L^2(D)$ such that H_f are, respectively,

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bounded and compact on strongly pseudoconvex domains in $\mathbb{C}^n$. Since there is no non trivial holomorphic automorphism for general strongly pseudoconvex domains in $\mathbb{C}^n$, our theory is more subtle. To state our results more precisely, we need some definitions and notations.

For $z \in D$ and $\xi \in \mathbb{C}^n$, let $F_B(z,\xi)$ be the infinitesimal form of the Bergman metric of D. Let $\beta(z,w)$ be the Bergman distance of two points $z,w \in D$. Denote by $B(z,r)$ the Bergman metric ball $B(z,r) = \{ w \in D: \beta(z,w) < r \}$. For any set $S \subset D$, let $|S|$ denote the usual Lebesgue measure of S. For $f \in L^2(D)$ and $r > 0$, write

$$F_r(z)^2 = \inf\{ 1/|B(z,r)| \cdot \int_{B(z,r)} |f-h|^2 \, dv : h \in H^2(D) \}.$$

For a (p,q)– form $H(z) = \Sigma H_{I,J}(z)\, dz_I \wedge d\bar{z}_J$ with locally integrable coefficients $H_{I,J}$ on D, where $dz_I = dz_{i_1} \wedge dz_{i_2} \wedge \cdots \wedge dz_{i_p}$ and $d\bar{z}_J = d\bar{z}_{j_1} \wedge d\bar{z}_{j_2} \wedge \cdots \wedge d\bar{z}_{j_q}$, we denote $|H(z)| = \Sigma |H_{I,J}(z)|$. Let $\bar{\partial} H = \sum_1^n \Sigma \, \partial H_{I,J}/\partial \bar{z}_i \, d\bar{z}_i \wedge dz_I \wedge d\bar{z}_J$, where for $1 \le i \le n$, $\partial/\partial z_i = 1/2 \cdot (\partial/\partial x_i - \sqrt{-1} \cdot \partial/\partial y_i)$, $\partial/\partial \bar{z}_i = 1/2 \cdot (\partial/\partial x_i + \sqrt{-1} \cdot \partial/\partial y_i)$. If $H_{I,J}$ are not differentiable, the derivatives should be understood in the sense of distributions.

Let $\rho(z) \in C^\infty(\mathbb{C}^n)$ be a strictly plurisubharmonic defining function of D such that $D = \{ z \in \mathbb{C}^n: \rho(z) < 0 \}$ and $\nabla \rho(z) \neq 0$ for $z \in \partial D$, where $\nabla \rho$ is the gradient of ρ.

For $z \in \partial D$, the complex tangential space of ∂D at z is defined by

$$T_z^{\mathbb{C}} = \{ \xi \in \mathbb{C}^n : \sum_{j=1}^n \partial\rho(z)/\partial z_j \cdot \xi_j = 0 \}.$$

THEOREM A *Let $f \in L^2(D)$. Then the following are equivalent:*

(1) *H_f is bounded from $H^2(D)$ to $L^2(D)$.*

(2) *For each $r > 0$, $F_r(z)$ is bounded on D.*

(3) *For some $r > 0$, $F_r(z)$ is bounded on D.*

(4) *f admits a decomposition $f = f_1 + f_2$ with $f_1 \in L^2(D)$, $f_2 \in C^1(D) \cap L^2(D)$ such that*

$$|\rho(z)|^{1/2} |\bar{\partial} f_2 \wedge \bar{\partial}\rho| + |\rho(z)|\,|\bar{\partial} f_2| \text{ is bounded on } D,$$

and

$$G_r(z) = 1/|B(z,r)| \cdot \int_{B(z,r)} |f_1|^2 \, dv \text{ is bounded on } D \text{ for some } r > 0.$$

THEOREM B *Let $f \in L^2(D)$. Then the following are equivalent.*

(1) *H_f is compact from $H^2(D)$ into $L^2(D)$.*

(2) *For each* $r > 0$, $F_r(z) \to 0$ *as* $\rho(z) \to 0$.
(3) *For some* $r > 0$, $F_r(z) \to 0$ *as* $\rho(z) \to 0$.
(4) f *admits a decomposition* $f = f_1 + f_2$ *with* $f_1 \in L^2(D)$, $f_2 \in C^1(D) \cap L^2(D)$ *such that*

$$|\rho(z)|^{1/2} |\bar\partial f_2 \wedge \bar\partial\rho| + |\rho(z)| |\bar\partial f_2| \to 0 \quad as \quad \rho(z) \to 0 ,$$

and $G_r(z) \to 0$ *as* $\rho(z) \to 0$ *for some* $r > 0$, *where* $G_r(z)$ *is the same as in Theorem A* .

We also obtain L^p- versions of Theorem A and Theorem B, establish relations between the Bergman metric *BMO* (*VMO*) and the function spaces in Theorem A (B), and prove the conjecture posed at the end of [3].

In section 2, we shall give some results about the geometry of a bounded strongly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$. In section 3, we discuss the Hankel operators H_f with $f \in C^1(D)$ by using the integral representations of solutions to the $\bar\partial-$ equations. Section 4 is devoted to dealing with the Bergman space Carleson measures. The main theorems are proved in section 5. In section 6, we establish the relation between the Bergman metric *BMO* and the function space in Theorem A . In section 7, we will discuss the Hankel operators on the Fock space $H^2(\mathbb{C}^n, d\mu)$. Throughout this paper, we shall use the letter C to denote constants, and they may change from line to line.

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2. Geometry of strongly pseudoconvex domains In this section, we give some facts about the Bergman metric balls in a bounded strongly pseudoconvex domain D with smooth boundary in $\mathbb{C}^n$. From now on, we will fix a bounded strongly pseudoconvex domain D with smooth boundary and let $\rho(z) \in C^\infty(\mathbb{C}^n)$ be a strictly plurisubharmonic defining function of D. To simplify notations, we shall write $\rho_i(z) = \partial\rho(z)/\partial z_i$, $\rho_{ij}(z) = \partial^2\rho(z)/\partial z_i\partial z_j$, where $1 \le i,j \le n$. Let $F(z,w)$ denote the Levi polynomial

$$F(z,w) = \sum_{i=1}^{n} \rho_i(w)(w_i - z_i) - 1/2 \cdot \sum_{ij=1}^{n} \rho_{ij}(w)(w_i - z_i)(w_j - z_j) .$$

It is well known [14] that if D is a bounded strongly pseudoconvex domain with smooth boundary in $\mathbb{C}^n$, then there exist constants δ and C such that for $z, w \in \bar{D}$ with $|z - w| \le \delta$, we have

$$Re(\ F(z,w) - \rho(w)) \geq C\cdot(\ -\rho(z) - \rho(w) + |z-w|^2\ ).$$

LEMMA 2.1 *Let* ρ *and* δ *be the same as above. There exist functions* $h_i(z,w)$, $1 \leq i \leq n$, *and* $\Psi(z,w)$ *in* $C^\infty(\bar{D}\times\bar{D})$ *such that*

(1) *for each fixed* $w \in \bar{D}$, $h_i(z,w)$ *and* $\Psi(z,w)$ *are holomorphic in* $z \in \bar{D}$ *;*

(2) $\Psi(w,w) = -\rho(w)$ *and there is a non-vanishing smooth function* $g(z,w)$ *in* $\bar{D}\times\bar{D}$ *such that if* $|z-w| \leq \delta/2$, *then* $\Psi(z,w) = g(z,w)\cdot(F(z,w) - \rho(w))$; *if* $|z-w| \geq \delta/2$, *then* $|\Psi(z,w)| \geq 1/C$.

(3) $\Psi(z,w) = \sum_{i=1}^{n} h_i(z,w)(w_i - z_i) - \rho(w)$.

Proof. (1) and (2) are contained in [2, p.363]. An application of Proposition 3.3 in [14, p.285] to $\Psi(z,w)$ yields (3). QED.

For each $\delta > 0$, write $D_\delta = \{\ z \in D\colon\ |\rho(z)| < \delta\ \}$. It is well known [8] that when δ is small, if $z \in D_\delta$, then there is an unique point $\pi(z) \in \partial D$ such that $\pi(z)$ is the point on ∂D closest to z . We will use $P(z,r_1,r_2)$ to denote the polydisc centered at z with radius r_1 in the complex normal direction N at $\pi(z)$ and radius r_2 in each complex tangential direction T_i , $2 \leq i \leq n$, at $\pi(z)$, where $\{\ T_i : 2 \leq i \leq n\ \}$ form an orthonormal basis of the complex tangential space at $\pi(z)$. If $z \in D\backslash D_\delta$, we will simply let $\{\ N\ ,\ T_i\}$, $2 \leq i \leq n$, be the usual basis of $\mathbb{C}^n$, and still call N the complex normal direction and T_i , $2 \leq i \leq n$, the complex tangential directions. For details of the complex normal and complex tangential directions see [6, 8] .

LEMMA 2.2 *For each* $r > 0$, *there are positive constants* A, B *and* C *only depending on* r *such that*

(1) $P(z,\ A\cdot|\rho(z)|,\ A\cdot|\rho(z)|^{1/2}) \subset B(z,r) \subset P(z,\ B\cdot|\rho(z)|,\ B\cdot|\rho(z)|^{1/2})$.

(2) $|\rho(z)|^{n+1}/C \leq |B(z,r)| \leq C\cdot|\rho(z)|^{n+1}$ *for all* $z \in D$.

(3) *If* $r < 1$, *then there is a constant* $c > 0$ *independent of* r *and* $z \in D$ *such that*

$$B(z,r) \subset P(z,\ c\cdot r\cdot|\rho(z)|,\ c\cdot r\cdot|\rho(z)|^{1/2}\ ).$$

(4) *There is a constant* $1 > \epsilon_0 > 0$ *such that if* $r \leq \epsilon_0$ *and* $w \in B(z,r)$, *then*

$$|\rho(w)| \geq |\rho(z)|/2\ .$$

Proof. All those results were proved for the Kobayashi metric in [9] . Since the Bergman metric and the Kobayashi metric are equivalent on a bounded strongly pseudoconvex domain with smooth boundary, it follows that the results are true for the Bergman metric.

LEMMA 2.3 *For each* $0 < r < \epsilon_0/6$ *and each given integer* $L > 2$, *there* *is a sequence* $\{z_i\} \subset D$ *and an integer* $M(r) > 0$ *such that* $\beta(z_i, z_j) \geq r/(2L)$ *if* $i \neq j$, *and* $\{B(z_i, r/L)\}$ *form a cover of D. Moreover, for any point* $z \in D$, *there are at most* $M(r)$ *of the balls* $\{B(z_i, 3r)\}$ *containing* z.

Proof. Fix a point $p \in D$. Let $S_i = \{ w \in D : \beta(p, w) = i \cdot r/(2L) \}$, $i = 1, 2, \cdots$. We shall construct the sequence by taking finite points from each S_i as following: For each $i \geq 1$, take any one point $z_{i,1} \in S_i$. Then pick up a point $z_{i,2} \in S_i$ with $\beta(z_{i,1}, z_{i,2}) = r/(2L)$. After we have taken points $z_{i,1}, \cdots, z_{i,j}$ from S_i, if there is no point $w \in S_i$ such that $\beta(w, z_{i,k}) \geq r/(2L)$ for all $k = 1, \cdots, j$, then stop this process. Otherwise, take any one point $z_{i,j+1} \in S_i$ with $\beta(z_{i,k}, z_{i,j+1}) \geq r/(2L)$ for $k = 1, 2, \cdots, j$. We claim that this process will stop after finitely many, say m_i, steps. For otherwise, there will be infinitely many disjoint balls $\{ B(z_{i,j}, r/(4L))\}$, $j \geq 1$, contained in $D_i = B(p, (i+1) \cdot r/(2L))$. Note that $\overline{D}_i$ is a compact subset of D, then there is a constant $s > 0$ such that $|\rho(z)| \geq s$ on $\overline{D}_i$. By Lemma 2.2, $|B(z_{i,j}, r/(4L))| \geq s^{n+1}/C$, where C is a constant only depending on r and L. Thus, we get a contradiction

$\infty = \Sigma\, |B(z_{i,j}, r/(4L))| \leq |\overline{D}_i| < \infty$.

Now we prove that $B(z_{i,j}, r/L)$, $i \geq 1$, $1 \leq j \leq m_i$, form an open cover of D. In fact, for any $z \in D$, since $\beta(z,w)$ is a complete Riemannian metric on D, it follows that there is an integer k such that $k \cdot r/(2L) \leq \beta(p,z) < (k+1) \cdot r/(2L)$, and there exists a point $w \in S_k$ such that $\beta(z,w) < r/(2L)$. By the construction of $\{z_{k,j}\}$, we must have $\beta(w, z_{k,j}) < r/(2L)$ for some $j \leq m_k$. An application of the triangle inequality yields that $\beta(z_{k,j}, z) < r/L$, i.e. $z \in B(z_{k,j}, r/L)$. If we rearrange $\{z_{i,j}\}$, $i \geq 1$, $1 \leq j \leq m_i$, then we get a sequence $\{z_i\} \subset D$ such that $\beta(z_i, z_j) \geq r/(2L)$ if $i \neq j$ and $B(z_i, r/L)$ form a cover of D.

Next, we prove that $\{z_i\}$ has the last property in the Lemma. For any $z \in D$, let J be the index set such that $j \in J$ implies $z \in B(z_j, 3r)$. Then $\bigcup_{j \in J} B(z_j, 3r) \subset B(z, 6r)$. Thus $\bigcup_{j \in J} B(z_j, r/(4L)) \subset B(z, 6r)$. Note that $B(z_j, r/(4L))$ are disjoint. Then $\sum_{j \in J} |B(z_j, r/(4L))| \leq |B(z,6r)|$. Since $r < \epsilon_0/6$, by Lemma 2.2, we have $|\rho(z_j)| \geq |\rho(z)|/2$, and $|B(z_j, r/(4L))| \geq |\rho(z_j)|^{n+1}/C_1$, $|B(z, 6r)| \leq C_1 \cdot |\rho(z)|^{n+1}$ for $j \in J$, where C_1 is a constant only depending on r and L. Let $M(r) = [2^{n+1} \cdot C_1^2] + 2$, where $[2^{n+1} \cdot C_1^2]$ is the biggest integer less than or equal to $2^{n+1} \cdot C_1^2$. It is obvious that $M(r)$ depends only on r and L, and there are at most $M(r)$ integers contained in J. Therefore, $\{z_i\}$ is the desired sequence. QED.

LEMMA 2.4 *For each* $0 < r < \epsilon_0/6$, *there exist an integer* $L > 2$ *and a* *constant* $R > 0$ *such that*

$$B(z, r/L) \subset P(z, R\cdot|\rho(z)|, R\cdot|\rho(z)|^{1/2}) \subset P(z, 2R|\rho(z)|, 2R|\rho(z)|^{1/2}) \subset\subset B(z, r/2)\,.$$

Proof. The result follows easily from Lemma 2.2 (1) and (3) . QED

LEMMA 2.5 *Let* r *and* L *be the same as those in Lemma* 2.4, *and let* $\{z_i\}$ *be the same as those in Lemma* 2.3 *corresponding to the* r *and* L . *Then there is a sequence of real valued smooth functions* $\{\gamma_i\}$ *such that for each* $i \geq 1$, γ_i *has compact support in* $B(z_i, r/2)$ *and* $\gamma_i = 1$ *on* $B(z, r/L)$. *Moreover,*

$$|\rho(z)|^{1/2}|\bar\partial\gamma_i(z)\wedge \bar\partial\rho(z)| + |\rho(z)||\bar\partial\gamma_i(z)| \leq C\,. \tag{2.1}$$

Proof. Let $\varphi \in C^{\infty}(\mathbb{C}^n)$ be a real valued function which has compact support in the polydisc $D^n(2) = \{ w \in \mathbb{C}^n\colon |w_i| < 2, 1 \leq i \leq n \}$, and $\varphi(w) = 1$ for w in the unit polydisc $D^n(1) \subset \mathbb{C}^n$. For each $j \geq 1$, define a mapping $F_j = (F_j^1, \cdots, F_j^n) : \mathbb{C}^n \to \mathbb{C}^n$ by

$$F_j^1(z) = (z - z_j)_N/(R\cdot|\rho(z_j)|),\ F_j^k(z) = (z-z_j)_{T_k}/(R\cdot|\rho(z_j)|^{1/2}),\ 2 \leq k \leq n,$$

where $(z-z_j)_N$ and $(z-z_j)_{T_k}$ are components of $(z-z_j)$ in the complex normal direction N and complex tangential directions T_k, $2 \leq k \leq n$, at $\pi(z_j)$, respectively. By Lemma 2.4, it follows that $F_j(B(z_j, r/L)) \subset D^n(1) \subset D^n(2) \subset\subset F_j(B(z_j, r/2))$. Let $\gamma_j(z) = \varphi(F_j(z))$. Then γ_j has compact support in $B(z_j, r/2)$ and $\gamma_j = 1$ on $B(z_j, r/L)$. To prove that γ_i satisfy (2.1), it suffices to prove (2.1) for $z \in D_\delta$. In this case, by Lemma 2.2 and the triangle inequality, it follows that for $w \in B(z_j, r)$, $|w - \pi(z_j)| \leq C\cdot|\rho(z_j)|^{1/2}$. Thus, the coefficients of $\bar\partial\gamma_j \wedge \bar\partial\rho$ are of the forms $\sum_{k=1}^{n} \overline{\partial\gamma_j/\partial w_k\cdot\xi_k} + O(|\rho(z_j)|^{1/2}\cdot|\nabla\gamma_j|)$, where $\nabla\gamma_j$ is the gradient of γ_j, $\xi = (\xi_1, \cdots, \xi_n)$ are vectors in the complex tangential space at $\pi(z_j)$ and $|\xi| \leq C$. By using the chain rule and the very definitions of γ_j and F_j, one has $|\bar\partial\gamma_j(w)\wedge\bar\partial\rho(w)||\rho(z_j)|^{1/2} + |\bar\partial\gamma_j(w)||\rho(z_j)| \leq C$. By Lemma 2.2, $|\rho(z_j)| \geq |\rho(w)|/2$ for $w \in B(z_j, r)$. It follows that γ_j satisfy (2.1). QED.

Remark: We have used the fact [8] that $|\rho(z)|/C \leq d(z,\partial D) \leq C\cdot|\rho(z)|$ in the proof of Lemma 2.5 , where $d(z,\partial D)$ is the usual distance from $z \in D$ to ∂D .

LEMMA 2.6 *Let* r, L, $\{z_i\}$ *and* $\{\gamma_i\}$ *be the same as in Lemma* 2.5. *Then* $\psi_i = \gamma_i/(\Sigma\,\gamma_i)$ *is a partition of unity subordinate to the cover* $\{B(z_i, r/2)\}$, *and*

$$|\bar\partial\psi_i(w)\wedge\bar\partial\rho(w)|\cdot|\rho(w)|^{1/2} + |\rho(w)\bar\partial\psi_i(w)| \leq C\,. \tag{2.2}$$

Proof. Since $\{B(z_i, r/L)\}$ is an open cover of D, it follows that $\Sigma\,\gamma_i$ is bounded away from zero. Note that $\{B(z_i, 3r)\}$ is locally finite, so is $\{B(z_j, r/2)\}$.

Then $\{\gamma_j\} \subset C^\infty$ implies that $\{\psi_i\} \subset C^\infty$. For any $z \in D$, since there are at most $M(r)$ balls $B(z_j, r/2)$ containing z and each γ_j satisfies (2.1), it follows that ψ_i, $i \geq 1$, satisfy (2.2) by a straightforward calculation. QED.

3. HANKEL OPERATORS WITH C^1 SYMBOLS In this section, we discuss the boundedness and compactness of the Hankel operators H_f with $f \in C^1(D)$. The main tool used here is the integral representations of solutions to the $\bar\partial$– equation.

LEMMA 3.1 *Let D, ρ and $F(z,w)$ be the same as in section 2. There exist constants δ and c such that for any $z \in D$ with $|\rho(z)| \leq \delta$, in the Euclidean ball $B_n(z, \delta)$ we can perform a smooth change of variables $\tau = \tau(w)$ with the properties*

(1) $\tau_1(w) = \rho(w) - \rho(z)$, $\tau_2(w) = Im\, F(z,w)$;

(2) $|z-w|/c \leq |\tau(w)| \leq c\cdot|z-w|$ *for* $w \in B_n(z,\delta)$ *and* $\tau(z) = 0$;

(3) $1/c \leq |\partial\tau/\partial w| \leq c$ *for* $w \in B_n(z,\delta)$, *where* $\partial\tau/\partial w$ *denotes the Jacobian of* τ.

For any $w \in D$ with $|\rho(w)| \leq \delta$, in the ball $B_n(w,\delta)$ we can perform a smooth change of variables $\lambda = \lambda(z)$ with $\lambda_1(z) = \rho(z) - \rho(w)$ and $\lambda_2 = Im\, F(z,w)$ such that (2) and (3) hold for $\lambda(z)$.

Proof. See [1, p. 125] or [14, p.208].

LEMMA 3.2 [1, 5] *For $\epsilon > 0$, let*

$$I_{\alpha,k,s}(\epsilon) = \int_{\substack{|t|\leq 1 \\ t_1+\epsilon>0}} \frac{(t_1+\epsilon)^\alpha\, dt_1\cdots dt_{2n}}{(|t_1|+|t_2|+\epsilon+|t|^2)^k\cdot|t|^s}, \tag{3.1}$$

where α, k, s are real, and $\alpha > -1$. Then

(*a*) $I_{\alpha,k,2n-1} = O(\epsilon^{1-k+\alpha})$ *if* $k-\alpha > 1$;

(*b*) $I_{\alpha,k,2n-4} = O(\epsilon^{3-k+\alpha})$ *if* $k-\alpha > 3$;

(*c*) $I_{\alpha,k,2n-3} = O(\epsilon^{5/2-k+\alpha})$ *if* $k-\alpha > 5/2$.

(*d*) $I_{\alpha,k,0} = O(\epsilon^{\alpha-k+n+1})$ *if* $k-\alpha > n+1$.

Write $s_i(z,w) = \overline{(w_i-z_i)}$. It is obvious that $|z-w|^2 = \Sigma\, s_i(z,w)\cdot(w_i-z_i)$. Let $h_i(z,w)$ be the functions given in Lemma 2.1.

We define

$$s(z,w) = \sum_1^n s_i(z,w)\, dw_i;$$

$$h(z,w) = \sum_1^n h_i(z,w)\, dw_i;$$

$$q(z,w) = \bar\partial_w h(z,w)/\rho(w) - \bar\partial\, \rho(w) \wedge h(z,w)/\rho(w)^2;$$

$$L(z,w) = C_n \cdot \sum_0^{n-1} a_k \cdot [-\rho(w)/\Psi(z,w)]^{k+2} \cdot s(z,w) \wedge q(z,w)^k \wedge (\bar\partial_w s)^{n-k-1}/|z-w|^{2(n-k)},$$

where a_k are the constants defined in the equation (20) of [4, p.102] by letting $N = 2$, and $C_n = (-1)^{n(n-1)/2}/(n-1)!$.

LEMMA 3.3 [4] *If* u *is a* $\bar\partial$*-closed* (0,1) *form such that* $|\rho\cdot u| + |\rho|^{1/2}\cdot|u \wedge \bar\partial\rho| \in L^1(D)$, *then*

$$U(z) = T(u)(z) = \int_D u(w) \wedge L(z,w) \tag{3.2}$$

is a solution to the equation $\bar\partial U = u$ *and* $U \in L^1(D)$.

Remark: In [4], Berndtsson and Andersson only proved the results for the $\bar\partial$–closed (0,1)–forms with coefficients in $C^1(D)$ on bounded strongly convex domains D by letting $h_i(z,w) = \rho_i(w)$ in the definition of $L(z,w)$. As indicated in [4, p.104] , an application of the same process as in [4, p.101–103] yields our lemma for the $\bar\partial$ – closed (0,1)–forms with coefficients in $C^1(\bar D)$ on bounded strongly pseudoconvex domains D . Finally, by the same arguments as those given in [5, p.455–456] , one obtains the results in the Lemma.

LEMMA 3.4 *If* $f \in C^1(D)$ *and* $|\bar\partial f \wedge \bar\partial\rho|\,|\rho|^{1/2} + |\rho\cdot\bar\partial f| \le C$, *then, for* $0 < \epsilon < 1$, *it follows that*

$$\sup_{z\in D} |\rho(z)|^{\epsilon}\cdot\int_D |\bar\partial f \wedge L(z,w)|\cdot|\rho(w)|^{-\epsilon}\, dw < \infty , \tag{3.3}$$

$$\sup_{w\in D} |\rho(w)|^{\epsilon}\cdot\int_D |\bar\partial f \wedge L(z,w)|\cdot|\rho(z)|^{-\epsilon}\, dz < \infty , \tag{3.4}$$

Proof: Note that $|\Psi(z,w)| \ge (|\rho(z)| + |\rho(w)| + |\mathrm{Im}\, F(z,w)| + |z-w|^2)/C$. By direct computation, it follows that the coefficient of $\bar\partial f(w) \wedge L(z,w)$ is dominated by a linear combination of functions of the forms

$$A_0 = |\rho(w)\cdot\bar\partial f(w)|/(|\Psi(z,w)|\,|z-w|^{2n-1}) ; \tag{3.5}$$

$$A_k = (|\rho(w)\bar\partial f(w)| + |\bar\partial f \wedge \bar\partial\rho|)\cdot|\rho(w)|/(|\Psi(z,w)|^{k+2}\cdot|z-w|^{2(n-k)-1}) ,$$

where $1 \le k \le n-1$. Again, from $|\Psi(z,w)| \ge (|\rho(z)| + |\rho(w)| + |\mathrm{Im}\, F(z,w)| + |z-w|^2)/C$ it follows that

$$A_k \le C\cdot(|\rho(w)\bar\partial f(w)| + |\bar\partial f \wedge \bar\partial\rho|)|\rho(w)|^{1/2}/(|\Psi(z,w)|^{5/2}\cdot|z-w|^{2n-3}) ,$$

where $1 \le k \le n-1$. By using the estimates in Lemma 3.2 and the coordinate system in

Lemma 3.1, one can easily check that A_i , $0 \le i \le n-1$, satisfy (3.3) and (3.4). Consequently, $|\bar\partial f \wedge L(z,w)|$ satisfies (3.3) and (3.4). QED.

THEOREM 3.5 *Let $f \in L^2(D) \cap C^1(D)$ satisfy the conditions in Lemma 3.4 . Then the Hankel operator H_f is bounded from $H^2(D)$ to $L^2(D)$.*

Proof. For $g \in H^2(D)$, considering the equation

$$\bar\partial u = \bar\partial(f \cdot g) = g \cdot \bar\partial f. \qquad (*)$$

It's obvious that $g \cdot \bar\partial f$ is $\bar\partial$ – closed and $|g \cdot \bar\partial f \wedge \bar\partial \rho| \cdot |\rho|^{1/2} + |\rho \cdot g \cdot \bar\partial f| \in L^2(D)$. By Lemma 3.3, $u = T(g \cdot \bar\partial f)$ is a solution to (*). Let $T_0(g) = T(g \cdot \bar\partial f)$. By Lemma 3.4, an application of Schur's test [7] yields that T_0 is a bounded operator from $H^2(D)$ to $L^2(D)$. Note that for $g \in H^\infty(D)$, $f \cdot g \in L^2(D)$ is a solution to (*). By the uniqueness of the solution orthogonal to $H^2(D)$, we have $H_f(g) = (I-P)(f \cdot g) = (I-P)T_0(g)$. Since $H^\infty(D)$ is dense [14] in $H^2(D)$, it follows that the boundedness of T_0 implies the boundedness of H_f . This finishes the proof. QED.

THEOREM 3.6 *For $f \in C^1(D) \cap L^2(D)$, if $|\bar\partial f \wedge \bar\partial \rho| \cdot |\rho|^{1/2} + |\rho\, \bar\partial f| \to 0$ as $|\rho(z)| \to 0$, then the Hankel operator H_f is compact.*

Proof. By the same reasoning as in the proof of Theorem 3.5, it suffices to prove the compactness of T_0 . Note that T_0 is an integral operator with the kernel $\bar\partial f(w) \wedge L(z,w)$. Write $\Omega_m = \{ z \in D: |\rho(z)| \ge 1/m \}$, $m = 1, 2, \cdots$, then $\{\Omega_m\}$ is a sequence of compact subsets of D . Let χ_m be the characteristic functions of Ω_m . Note that for each m , $|\chi_m \cdot \bar\partial f \wedge L(z,w)| \le C(m)/|z-w|^{2n-1}$. It follows from the Theorem in [14, p.360] that the operators T_m with the integral kernels $\chi_m(w) \cdot \bar\partial f(w) \wedge L(z,w)$, $m \ge 1$, are compact. Note that

$$T_0(g) - T_m(g) = \int_D (1 - \chi_m)\, \bar\partial f(w) \wedge L(z,w) \cdot g .$$

Since $|\bar\partial f \wedge \bar\partial \rho| \cdot |\rho|^{1/2} + |\rho \cdot \bar\partial f| \to 0$ as $|\rho(z)| \to 0$, it follows that $\forall\, \epsilon > 0$, there is an integer M such that when $m \ge M$, $(1 - \chi_m)(|\bar\partial f \wedge \bar\partial \rho| \cdot |\rho|^{1/2} + |\rho \cdot \bar\partial f|) < \epsilon$. By the proof of Lemma 3.4 , there is a constant C such that for $0 < \alpha < 1$,

$$\int_D |(1 - \chi_m)\, \bar\partial f(w) \wedge L(z,w)| \cdot |\rho(w)|^{-\alpha}\, dw \le C \cdot \epsilon \cdot |\rho(z)|^{-\alpha}, \qquad (3.6)$$

$$\int_D |(1 - \chi_m)\, \bar\partial f(w) \wedge L(z,w)| \cdot |\rho(z)|^{-\alpha}\, dz \le C \cdot \epsilon \cdot |\rho(w)|^{-\alpha}. \qquad (3.7)$$

Again by using Schur's test [7], we have the operator norm $\|T_0 - T_m\|_2 \le C\cdot\epsilon$. It follows that $T_m \to T_0$. Thus, the compactness of T_m implies the compactness of T_0 . Consequently, $H_f = (I-P)\, T_0$ is compact. QED.

Note that [13] the Bergman projection is bounded from $L^p(D)$ to $H^p(D)$ consisting of holomorphic L^p- functions in D , $1 < p < \infty$. By the estimates in Lemma 3.4, an application of the L^p- version of Schur's test (see [13, Lemma 5]) yields the following theorem.

THEOREM 3.7 *Let $f \in C^1(D) \cap L^p(D)$. If f satisfies the conditions in Theorem 3.5, then H_f is bounded from $H^p(D)$ to $L^p(D)$; If f satisfies the conditions in Theorem 3.6, then H_f is compact from $H^p(D)$ to $L^p(D)$.*

Remark: By using the integral representations of solutions to the $\bar\partial-$ equations constructed by Dautov and Henkin in [5] , one can also obtain the results above.

For the Schatten class Hankel operators, we have the following result.

THEOREM 3.8 [10] *For $f \in L^2(D) \cap C^1(D)$ and $p > 2n$, if both $|\bar\partial f \wedge \bar\partial\rho| \cdot |\rho|^{1/2-(n+1)/p}$ and $|\bar\partial f| \cdot |\rho|^{1-(n+1)/p}$ are in $L^p(D)$, then the Hankel operator H_f is in the Schatten class S_p .*

Remark: If $\bar f \in H^2(D)$, then the conditions in Theorem 3.5, Theorem 3.6 and Theorem 3.8 are also necessary, and they are, respectively, equivalent to that $\bar f$ is in the Bloch space, the little Bloch space and the Besov space. For details see [9] and [10] .

4. CARLESON MEASURES FOR THE BERGMAN SPACES $H^p(D)$

DEFINITION: For $1 < p < \infty$, we call a positive measure μ on D an H^p- *Carleson measure* if

$$\left(\int_D |f|^p \, d\mu \right)^{1/p} \le C\cdot \left(\int_D |f|^p \, dv \right)^{1/p} .$$

LEMMA 4.1 *Let $f \in L^p(D)$ and $r>0$. If $G_{rp}(z) = |B(z,r)|^{-1} \int_{B(z,r)} |f|^p \, dv$ is bounded on D, then the multiplication operator $M_f(g) = f\cdot g$ is bounded from H^p to L^p , and the operator norm $\|M_f\|_p \le C\cdot \|G_{rp}\|_\infty^{1/p}$.*

Proof: By Lemma 2.2, it is easy to check that G_{rp} is bounded on D implies that G_{sp} is bounded on D for all $s < r$. Without lose of generality, assume $r < \epsilon_0/3$.

Following D.Luecking [11] , associate with each point $z \in D$ an open set $E(z) = B(z,r/3)$. Let $E^2(z) = \cup \{ E(y) : E(y) \cap E(z) \neq \phi \}$. Then $E^2(z) = B(z,r)$. By Lemma 2.2, it follows that $|E^2(z)| \leq C \cdot |E(z)|$, and for $g \in H^p(D)$,

$$|g(z)|^p \leq C \cdot |E(z)|^{-1} \cdot \int_{E(z)} |g|^p \, dv ,$$

the last inequality holds because each $E(z) = B(z,r/3)$ contains a polydisc centered at z and the volume of the polydisc is comparable with $|E(z)|$.

Let $d\mu = |f|^p \, dv$. By the assumption, $\mu(E^2(z)) \leq \|G_{rp}\|_\infty \cdot |E^2(z)|$. An application of Luecking's criterion [11, Lemma 1] yields that μ is an H^p- Carleson measure. Thus, M_f is bounded from H^p into L^p. From the proof of Lemma 1 in [11] we have $\|M_f\|_p \leq C \cdot \|G_{rp}\|_\infty^{1/p}$. QED.

LEMMA 4.2 *Let $f \in L^p(D)$. If $G_{rp}(z) \to 0$ as $\rho(z) \to 0$, then the multiplication operator M_f is compact from $H^p(D)$ into $L^p(D)$.*

Proof. Let $K_m = \{ z \in D : |\rho(z)| \geq 1/m \}$, $m \geq 1$. Then K_m are compact subsets of D . Let χ_m be the characteristic function of K_m . It is easy to check that $M_{\chi_m \cdot f}$ are compact operators from H^p to L^p because each $\chi_m \cdot f$ has compact support in D. Note that $M_f - M_{\chi_m \cdot f} = M_{(1-\chi_m) \cdot f}$, and $G_{rp}(z) \to 0$ as $\rho(z) \to 0$ implies that

$$\sup_{z \in D} \{ |B(z,r)|^{-1} \int_{B(z,r)} |(1-\chi_m) \cdot f|^p \, dv \} \to 0 \text{ as } m \to \infty .$$

By Lemma 4.1, we have $\|M_f - M_{\chi_m \cdot f}\|_p \to 0$ as $m \to \infty$. Therefore, M_f is compact. QED.

5. MAIN THEOREMS

In this section, we prove the main theorems.

THEOREM A *Let $f \in L^2(D)$. Then the following are equivalent:*

(1) *H_f is bounded from $H^2(D)$ to $L^2(D)$.*

(2) *For each $r > 0$, $F_r(z)$ is bounded on D .*

(3) *For some $r > 0$, $F_r(z)$ is bounded on D .*

(4) *f admits a decomposition $f = f_1 + f_2$ with $f_1 \in L^2$ and $f_2 \in C^1(D) \cap L^2$ such that*

$$|\rho(z)|^{1/2} |\bar\partial f_2 \wedge \bar\partial \rho| + |\rho(z)| |\bar\partial f_2| \text{ is bounded on } D ,$$

and

$$G_r(z) = 1/|B(z,r)| \cdot \int_{B(z,r)} |f_1|^2 \, dv \text{ is bounded for some } r > 0 .$$

Proof. (1)=>(2) . For each $\lambda \in D$, let $S_\lambda(z) = |\rho(\lambda)|^{(n+1)/2} / \Psi(z,\lambda)^{n+1}$. By Lemma 2.1, it follows that both $S_\lambda(z)$ and $1/S_\lambda(z)$ are in $H^\infty(D)$. By using Lemma

2.1, Lemma 3.1 and Lemma 3.2, one can easily check that $\|S_\lambda\|_2 \le C$, where C is a constant independent of $\lambda \in D$. (See also Lemma 7.3 in [14, p.310]). If H_f is bounded, then

$$\|H_f(S_\lambda)\|_2 = \| f\cdot S_\lambda - P(f\cdot S_\lambda)\|_2 = \|(f - S_\lambda^{-1}\cdot P(f\cdot S_\lambda))\cdot S_\lambda\|_2 \le C\cdot\|H_f\|_2 .$$

For each $r > 0$, by using the estimates in Lemma 2.1 and Lemma 2.2, one can easily check that for $z \in B(\lambda,r)$, $|S_\lambda(z)|^2 \ge C/|\rho(\lambda)|^{n+1}$ (For details see [9]) . Again by Lemma 2.2, we have $|S_\lambda(z)|^2 \ge C/|B(\lambda,r)|$ for $z \in B(\lambda, r)$. Thus,

$$\{|B(\lambda,r)|^{-1}\cdot\int_{B(\lambda,r)}|f - S_\lambda^{-1}\cdot P(f\cdot S_\lambda)|^2\, dv\}^{1/2}$$

$$\le C_1\cdot\|(f - S_\lambda^{-1}\cdot P(f\cdot S_\lambda))\cdot S_\lambda\|_2 \le C_2\cdot\|H_f\|_2 .$$

Obviously, $S_\lambda^{-1}\cdot P(S_\lambda\cdot f) \in H^2(D)$. It follow that $F_r(\lambda)$ is bounded on D.

(2) => (3) is trivial .

Now we assume (3). By Lemma 2.2 (2), one can easily check that $F_r(z)$ is bounded in D implies that $F_s(z)$ is bounded in D for all $s < r$. Without lose of generality, we can let $r < \epsilon_0/6$, where ϵ_0 is the same constant as in Lemma 2.2(4) . Let L, $\{z_j\}$ and $\{\psi_j\}$ be the same as those in Lemma 2.5 and Lemma 2.6. By (3), for each $j \ge 1$, there is a function $h_j \in H^2(D)$ such that

$$|B(z_j, r)|^{-1}\cdot\int_{B(z_j, r)}|f - h_j|^2\, dv \le 4F_r(z_j)^2 .$$

We define $f_2 = \Sigma\, h_j\cdot\psi_j$ and $f_1 = f - f_2$. To verify that f_1 and f_2 satisfy (4), we use the arguments given by Luecking [12] . First of all, we show that if $z \in B(z_j, r/2)\cap B(z_k, r/2)$, then $|h_j(z) - h_k(z)|$ is bounded. As we did in the proof of Lemma 4.1 ,

$$|h_j(z) - h_k(z)|\,|B(z,r/2)|^{1/2} \le (C\cdot\int_{B(z,r/2)}|h_j - h_k|^2\, dv)^{1/2}$$

$$\le (C\cdot\int_{B(z_j, r)}|h_j - f|^2\, dv)^{1/2} + (C\cdot\int_{B(z_k,r)}|h_k - f|^2\, dv)^{1/2}$$

$$\le C\cdot[\, F_r(z_j)\cdot|B(z_j, r)|^{1/2} + F_r(z_k)\cdot|B(z_k,r)|^{1/2}\,] .$$

Since z_j, $z_k \subset B(z,r/2)$, by Lemma 2.2 we have $|B(z_j, r)| \le C\cdot|B(z,r/2)|$ and $|B(z_k,r)| \le C\cdot|B(z,r/2)|$. Thus,

$$|h_j(z) - h_k(z)| \le C\cdot \sup_{w\in B(z,2r)}\{\, F_r(w)\,\} . \qquad (5.1)$$

Now we estimate f_1. To simplify notations, we write $B(z) = B(z,r)$. Note that if $B(z) \cap B(z_j, r/2) \neq \phi$, then $z \in B(z_j, 3r)$. By our construction of $\{z_i\}$, one can easily see that for each $z \in D$, there are at most $M(r)$ balls $B(z_j, r/2)$ intersect $B(z, r)$. Thus,

$$\left(\int_{B(z)} |f_1|^2\, dv\right)^{1/2} = \left(\int_{B(z)} \left| \Sigma\, (f - h_j)\cdot \psi_j \right|^2 dv\right)^{1/2}$$

$$\leq \Sigma \left(\int_{B(z)} |f-h_j|^2 \cdot |\psi_j|^2\, dv\right)^{1/2} \leq \Sigma \left(\int_{B(z_j,\, r/2) \cap B(z)} |f - h_j|^2\, dv \right)^{1/2}$$

$$\leq M(r)\cdot \sup_{w \in B(z,2r)} \{ F_r(w) \} \cdot |B(z, 3r)|^{1/2}.$$

The last inequality is because $B(z_j, r/2) \subset B(z_j) \subset B(z,3r)$ *and* $\beta(z,z_j) < 2r$ if $B(z_j, r/2) \cap B(z) \neq \phi$, and there are at most $M(r)$ of the $B(z_j, r/2)$ intersect $B(z)$. Since $|B(z,3r)| \leq C\cdot |B(z,r)|$, it follows that

$$\left(|B(z,r)|^{-1} \cdot \int_{B(z)} |f_1|^2\, dv\right)^{1/2} \leq C\cdot \sup_{w \in B(z,2r)} \{ F_r(w) \}. \tag{5.2}$$

Thus, $|G_r(z)| \leq C\cdot \|F_r\|_\infty^2$. Note that $\{B(z_j, r)\}$ is an open cover of D and $\Sigma\, \chi_j \leq M(r)$, where χ_j are characteristic functions of $B(z_j, r)$. Then

$$\int_D |f_1|^2\, dv \leq \sum_{j=1}^{\infty} \int_{B(z_j,\, r)} |f_1|^2\, dv \leq \sum_{j=1}^{\infty} G_r(z_j)\cdot |B(z_j, r)|$$

$$\leq C\cdot \|F_r\|_\infty^2 \cdot \sum_{j=1}^{\infty} |B(z_j, r)| \leq C\cdot \|F_r\|_\infty^2 \cdot M(r)\cdot |D| < \infty.$$

Consequently, we have $f_1 \in L^2(D)$.

Now we estimate f_2. Fix a point $z \in D$ and let J be the set of integers j such that $z \in B(z_j, r/2)$. Then $f_2(z) = \sum_{j \in J} h_j(z)\cdot \psi_j(z)$. Let us suppose for convenience that $1 \in J$ and write

$$f_2(z) = h_1(z) + \sum_{j \in J} (h_j - h_1)\cdot \psi_j.$$

Note that $\{ h_j \}$ are holomorphic functions in D, then

$$|\bar\partial f_2(z) \wedge \bar\partial \rho|\, |\rho|^{1/2} + |\rho\cdot \bar\partial f_2| = \left| \sum_{j \in J} (h_j - h_1)\cdot \bar\partial \psi_j \wedge \bar\partial \rho \right| |\rho|^{1/2} + \left|\rho\cdot \sum_{j \in J} (h_j - h_1)\cdot \bar\partial \psi_j\right|$$
$$\leq C\cdot M(r)\cdot \sup_{w \in B(z,2r)} \{ F_r(w) \},$$

because J contains at most $M(r)$ integers, $|h_1 - h_j| \leq C\cdot \sup_{w \in B(z,2r)} \{ F_r(w) \}$ from (5.1) and ψ_j satisfy the estimates in Lemma 2.6. Since $f_2 = f - f_1$ and $f, f_1 \in L^2(D)$, it follows that $f_2 \in L^2(D)$. This finishes the proof of (3)=>(4).

Assume (4). By Theorem 3.5, H_{f_2} is bounded. By Lemma 4.1, $M_{f_1}(g) = f_1 \cdot g$ is bounded from $H^2(D)$ into $L^2(D)$. Therefore, $H_{f_1} = (I-P)M_{f_1}$ is bounded. Note that $H_f = H_{f_1} + H_{f_2}$, then H_f is bounded. QED.

THEOREM B *Let $f \in L^2(D)$. Then the following are equivalent:*

(1) *H_f is compact from $H^2(D)$ into $L^2(D)$.*

(2) *For each $r > 0$, $F_r(z) \to 0$ as $\rho(z) \to 0$.*

(3) *For some $r > 0$, $F_r(z) \to 0$ as $\rho(z) \to 0$.*

(4) *f admits a decomposition $f = f_1 + f_2$ with $f_1 \in L^2$ and $f_2 \in C^1(D) \cap L^2(D)$ such that*

$$|\rho(z)|^{1/2} |\bar{\partial} f_2 \wedge \bar{\partial}\rho| + |\rho(z)| |\bar{\partial} f_2| \to 0 \quad as \quad \rho(z) \to 0 ,$$

and $G_r(z) \to 0$ as $\rho(z) \to 0$ for some $r > 0$, where $G_r(z)$ is the same as in Theorem A .

Proof. Let the notations be the same as those in the proof of Theorem A.

Assume (1). Note that $S_\lambda(z) \to 0$ at every point $z \in D$ as $|\rho(\lambda)| \to 0$ and $\|S_\lambda\| \le C$. By a standard argument it follows that $S_\lambda \to 0$ weakly in $H^2(D)$ as $|\rho(\lambda)| \to 0$. Thus, H_f is compact implies that

$$\|H_f(S_\lambda)\|_2 \to 0 \quad \text{as} \quad |\rho(\lambda)| \to 0 .$$

By the estimates in the proof of Theorem A , we have $F_r(z) \to 0$ as $\rho(z) \to 0$.

(2) => (3) is trivial.

Assume (3). As we did before, without lose of generality, we can assume $r < \epsilon_0/6$. Note that (by Lemma 2.2) $\sup_{w \in B(z,2r)} \{ |\rho(w)| \} \to 0$ as $\rho(z) \to 0$. From the estimates given in the proof of Theorem A , it follows that f_1 and f_2 satisfy the conditions in (4).

Finally, we prove (4)=>(1). By Theorem 3.6, H_{f_2} is compact. By lemma 4.2, M_{f_1} is compact, and then $H_{f_1} = (I-P)M_{f_1}$ is compact. Therefore, H_f is compact. QED.

For $1 < p < \infty$, we write

$$F_{r,p}(z) = inf\{ |B(z,r)|^{-1} \cdot \int_{B(z,r)} |f - h|^p \, dv : h \in H^p(D) \} .$$

THEOREM C. *Let $f \in L^p(D)$. Then the following are equivalent:*

(1) *H_f is bounded from $H^p(D)$ to $L^p(D)$.*

(2) *For each $r > 0$, $F_{r,p}(z)$ is bounded on D .*

(3) *For some $r > 0$, $F_{r,p}(z)$ is bounded on D .*

(4) *f admits a decomposition* $f = f_1 + f_2$ *with* $f_1 \in L^p(D)$ *and* $f_2 \in C^1(D) \cap L^p$ *such that*

$$|\rho(z)|^{1/2} |\bar\partial f_2 \wedge \bar\partial \rho| + |\rho(z)| |\bar\partial f_2| \quad \textit{is bounded on } D,$$

and

$$G_{r,p}(z) = 1/|B(z,r)| \cdot \int_{B(z,r)} |f_1|^p \, dv \ \textit{is bounded for some } r > 0 .$$

Proof. (1) =>(2). For each $\lambda \in D$, let $S_{\lambda,p}(z) = |\rho(\lambda)|^{(1-1/p)\cdot(n+1)}/\Psi(z,\lambda)^{n+1}$. By Lemma 2.1, both $S_{\lambda,p}(z)$ and $S^{-1}_{\lambda,p}(z)$ are in $H^\infty(D)$. By the same arguments as those in the proof of Theorem A, it follows that $\|S_{\lambda,p}\|_p \le C$ and $|S_{\lambda,p}(z)|^p \ge 1/(C \cdot |B(\lambda,r)|)$ for $z \in B(\lambda,r)$, where C is a constant independent of $z, \lambda \in D$. Thus

$$\{|B(\lambda,r)|^{-1} \cdot \int_{B(\lambda,r)} |f - S^{-1}_{\lambda,p} \cdot P(f \cdot S_{\lambda,p})|^p \, dv\}^{1/p} \le C \cdot \|H_f(S_{\lambda,p})\|_p \le C \cdot \|H_f\|_p .$$

It is known [13] that the Bergman projection is bounded from $L^p(D)$ to $H^p(D)$. Thus, $S^{-1}_{\lambda,p} \cdot P(f \cdot S_{\lambda,p}) \in H^p(D)$. Therefore, $F_{r,p}$ is bounded.

(2) => (3) is trivial.

Assume (3). The arguments in the proof of (3)=>(4) in Theorem A can be carried over word by word, except that L^p integrals appear everywhere in place of L^2 integrals, to give assertion (4) .

Finally, assume (4). By Theorem 3.7, H_{f_2} is bounded from $H^p(D)$ to $L^p(D)$. By Lemma 4.1 , one can easily show that M_{f_1} is bounded from $H^p(D)$ into $L^p(D)$ and so is $H_{f_1} = (I-P)M_{f_1}$. This finishes the proof of the theorem. QED.

Similarly, we have

THEOREM D *Let* $f \in L^p(D)$. *Then the following are equivalent:*

(1) H_f *is compact from* $H^p(D)$ *to* $L^p(D)$.
(2) *For each* $r > 0$, $F_{r,p}(z) \to 0$ *as* $\rho(z) \to 0$.
(3) *For some* $r > 0$, $F_{r,p}(z) \to 0$ *as* $\rho(z) \to 0$.
(4) *f admits a decomposition* $f = f_1 + f_2$ *with* $f_1 \in L^p(D)$ *and* $f_2 \in C^1(D) \cap L^p$ *such that*

$$|\rho(z)|^{1/2} |\bar\partial f_2 \wedge \bar\partial \rho| + |\rho(z)| |\bar\partial f_2| \to 0 \quad \textit{as } \rho(z) \to 0 ,$$

and $G_{r,p}(z) \to 0$ *as* $\rho(z) \to 0$ *for some* $r > 0$.

6. THE BERGMAN METRIC BMO AND VMO For $f \in L^2(D)$ and $r > 0$, write

$$MO_r(f,z) = |B(z,r)|^{-1}\int_{B(z,r)} |f - |B(z,r)|^{-1}\int_{B(z,r)} f\,dv\,|^2\,dv\,.$$

DEFINITION 6.1. We say that $f \in BMO(D)$ provided that $MO_r(f,z)$ is bounded on D for some $r > 0$; we say that $f \in VMO(D)$ provided that $MO_r(f,z) \to 0$ as $z \to \partial D$ for some $r > 0$. It was proved in [9] that the definitions of BMO and VMO don't rely on the choice of r.

Following D.Luecking, if f satisfies the condition in Theorem A (3), then we say that $f \in BDA$; if f satisfies the condition in Theorem B (3) , then we say that $f \in VDA$.

THEOREM 6.1. $BMO(D) = BDA(D) \cap \overline{BDA(D)}$; $VMO(D) = VDA(D) \cap \overline{VDA(D)}$.

Proof. It is obvious that $BMO(D) \subset BDA(D)\cap\overline{BDA(D)}$; $VMO(D) \subset VDA(D)\cap\overline{VDA(D)}$. To prove other inclusions, it suffices to prove that if a real function $f \in BDA$ (VDA) , then $f \in BMO$ (VMO) . By Lemma 2.2, there are constants $c, s, r > 0$ such that

$$B(z,s) \subset P(z,\, c\cdot|\rho(z)|,\, c\cdot|\rho(z)|^{1/2}) \subset B(z,r)\,. \tag{6.1}$$

To simplify notations, we shall write $P(z) = P(z,\, c\cdot|\rho(z)|,\, c\cdot|\rho(z)|^{1/2})$. By the definition of $F_r(z)$, for each $\lambda \in D$, there is a $h \in H^2(D)$ such that

$$|B(\lambda,r)|^{-1}\int_{B(\lambda,r)} |f - h|^2\,dv \le 2\cdot F_r(\lambda)^2\,.$$

By Lemma 2.2 and (6.1), one has

$$|P(\lambda)|^{-1}\int_{P(\lambda)} |f - h|^2\,dv \le C\cdot F_r(\lambda)^2\,. \tag{6.2}$$

Note that $P(\lambda)$ is a polydisc centered at λ and $h \in H^2(D)$, it follows that

$$|P(\lambda)|^{-1}\int_{P(\lambda)} |h - h(\lambda)|^2\,dv \le |P(\lambda)|^{-1}\int_{P(\lambda)} |Im\,h|^2\,dv\,,$$

where $Im\,h$ is the imaginary part of h . Thus, for real valued f, we have

$$|P(\lambda)|^{-1}\int_{P(\lambda)} |h - h(\lambda)|^2\,dv \le |P(\lambda)|^{-1}\int_{P(\lambda)} |Im\,(h-f)|^2\,dv \le C\cdot F_r(\lambda)^2. \tag{6.3}$$

By Lemma 2.2 , (6.1) , (6.2) and (6.3) , it follows that

$$[|B(\lambda,s)|^{-1}\int_{B(\lambda,s)} |f - h(\lambda)|^2\,dv\,]^{1/2} \le C\cdot[|P(\lambda)|^{-1}\int_{P(\lambda)} |f - h(\lambda)|^2\,dv\,]^{1/2}$$

$$\le C\cdot[|P(\lambda)|^{-1}\int_{P(\lambda)} |h - h(\lambda)|^2\,dv\,]^{1/2} + C\cdot[|P(\lambda)|^{-1}\int_{P(\lambda)} |f - h|^2\,dv\,]^{1/2}$$

$\leq C \cdot F_r(\lambda)$.

By a standard argument, we have $MO_s(f, \lambda) \leq C \cdot F_r(\lambda)^2$. Consequently, $f \in BDA$ (VDA) implies that $f \in BMO$ (VMO). This finishes the proof of the theorem. QED.

COROLLARY *For* $f \in L^2(D)$, H_f *and* $H_{\bar{f}}$ *are bounded if and only if* $f \in BMO$; H_f *and* $H_{\bar{f}}$ *are compact if and only if* $f \in VMO$.

Remark: Note that [3] H_f and $H_{\bar{f}}$ are bounded (compact) if and only if the commutator $[M_f , P]$ is bounded (compact) on $L^2(D)$. The Corollary gives us function–theoretic characterizations of the boundedness and compactness of $[M_f, P]$.

7. FURTHER DISCUSSION Let P be the orthogonal projection from $L^2(\mathbb{C}^n, d\mu)$ to the Fock space $H^2(\mathbb{C}^n, d\mu)$ with $d\mu = (2\pi)^{-n}\exp(-|z|^2/2)\, dv$ and dv the usual Lebesgue measure on $\mathbb{C}^n$. For $f \in L^2(\mathbb{C}^n, d\mu)$, consider the Hankel operator $H_f(g) = (I-P)(f \cdot g)$. If $f \in C^1(\mathbb{C}^n)$ and $|\bar{\partial} f| \in L^\infty(\mathbb{C}^n)$, for any polynomial g , consider the $\bar{\partial}$–equation

$$\bar{\partial} u = \bar{\partial}(f \cdot g) = g \cdot \bar{\partial} f. \qquad (**)$$

By Proposition 10 in [4],

$$u(z) = T(g \cdot \bar{\partial} f) = C_n \cdot \int_{\mathbb{C}^n} e^{(z \cdot \bar{w} - |w|^2)/2} \cdot g \cdot \bar{\partial} f \wedge \sum_{k<n} 2^{-k} \frac{\partial |z-w|^2 \wedge (\bar{\partial} \partial |w|^2)^{n-1}}{k! \cdot |z - w|^{2n-2k}}$$

is a solution to (**) and $u(z)$ is orthogonal to $H^2(\mathbb{C}^n, d\mu)$. Let $T_0(g) = T(g \cdot \bar{\partial} f)$. Then

$$H_f(g) = u = T(\bar{\partial} f \cdot g) = T_0(g).$$

Let $Q(z,w)$ denote the integral kernel of T_0 . By direct computation it follows that

$$|Q(z,w)| \leq C \cdot e^{(Re z \cdot \bar{w} - |w|^2)/2} \cdot (|z-w|^{1-2n} + |z-w|^{-1}) .$$

Thus

$$e^{-|z|^2/4} \cdot |Q(z,w)| \cdot e^{|w|^2/4} \leq C \cdot e^{-|z-w|^2/4} (|z-w|^{1-2n} + |z-w|^{-1}). \qquad (7.1)$$

Denote the right side of (7.1) by $Q_0(z,w)$. Note that $e^{-|z|^2/4}(|z|^{1-2n} + |z|^{-1})$ is in $L^1(\mathbb{C}^n, dv)$. It follows that the integral operator with kernel Q_0 is a bounded convolution operator on $L^2(\mathbb{C}^n, dv)$. Thus $T_0 = H_f$ is bounded from $H^2(\mathbb{C}^n, d\mu)$ to $L^2(\mathbb{C}^n, d\mu)$. Similarly, we can prove that if $\bar{\partial} f \to 0$ as $|z| \to \infty$, then the Hankel operator H_f is compact.

Finally, we note that our methods and results can be extended to the weighted Bergman spaces $H^p(D, |\rho|^{\alpha})$, $\alpha > -1$, without essential difficulties on bounded strongly pseudoconvex domains D with smooth boundary.

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Department of Mathematics
SUNY at Buffalo
Buffalo, NY 14214
USA

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