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On Dispersion and Markov Constants

By

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Abstract. We proved some results on the dispersion of the real quadratic irrational numbers, and use LEO 386/25 to compute some numerical results for discriminant <200 (see the attached Table A).

1. Introduction and Results

Let $\{x_n\}$ be a sequence of numbers, $0 \leqslant x_n \leqslant 1$. In [6], H. NIEDERREITER introduced a measure of denseness of such a sequence as follows. For each N let

$$d_N = \sup_{0 \leqslant x \leqslant 1} \min_{1 \leqslant n \leqslant N} |x - x_n|$$

and define

$$D(\{x_n\}) = \limsup_{N \to \infty} N\, d_N.$$

In particular, for irrational α, the *dispersion constant* $D(\alpha)$ is defined by $D(\{n\alpha \bmod 1\})$. It turns out that $D(\alpha) < \infty$ if and only if the continued fraction expansion of α has bounded partial quotients. It is well known that, for irrational α, the *Markov constant* $M(\alpha)$ is defined by

$$M(\alpha)^{-1} = \liminf_{n \to \infty} n \, \| n\alpha \|,$$

where $\| x \|$ denotes the distance from x to the nearest integer.

Moreover if α_1 and α_2 are equivalent, then $D(\alpha_1) = D(\alpha_2)$ and $M(\alpha_1) = M(\alpha_2)$. (Two numbers α and β are called equivalent if their

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continued fraction expansions coincide from some point on or, equivalently, if there exist four rational integers r, s, t, and u such that

$$ru - ts = \pm 1, \quad \text{and} \quad \beta = \frac{r\alpha + s}{t\alpha + u},$$

cf. [2].) H. NIEDERREITER also shows that if α is equivalent to $(1+\sqrt{5})/2$ then $D(\alpha) = (5 + 3\sqrt{5})/10$, and if α is equivalent to $\sqrt{2}$ then $D(\alpha) = (1+\sqrt{2})/2$, and so on.

The results of this note are Theorem 1, Theorem 2, and Corollary 3.

Theorem 1. *Let $d \equiv b \pmod 4$ be a positive discriminant which is not a perfect square of a rational integer, and $b = 0, 1$. Then we have*

$$D\left(\frac{b+\sqrt{d}}{2}\right) = \frac{2d + (d+b)\sqrt{d}}{4d}.$$

Theorem 1 is a consequence of the following

Theorem 2. *Let $d \equiv 0, 1 \pmod 4$ be a positive discriminant which is not a perfect square of a rational integer. Let three integers a, b and c satisfy*

$$|b| \leqslant a \leqslant c, \gcd(a,b,c) = 1, \textit{ and } d = b^2 + 4ac.$$

Let $\alpha = \dfrac{b+\sqrt{d}}{2a}$ have a simple continued fraction expansion

$$\alpha = \frac{b+\sqrt{d}}{2a} = [a_0, \overline{a_1, \ldots, a_k}],$$

with the basic period $\overline{a_1, \ldots, a_k}$ and n-th complete quotient

$$\alpha_n = \frac{P_n + \sqrt{d}}{2Q_n} = [a_n, a_{n+1}, \ldots] (n \geqslant 0, \alpha_0 = \alpha).$$

Then we have

$$D(\alpha) = \sup_{1 \leqslant i \leqslant k} \frac{2d + \dfrac{d+R_i}{Q_i}\sqrt{d}}{4d}, \tag{1}$$

where

$$R_i=\begin{cases} Q_i^2-\left(\dfrac{P_{i+1}-P_i}{2}\right)^2=Q_i^2-(a_iQ_i-P_i)^2, \text{ if } a_i \text{ odd;} \\ Q_i|P_{i+1}-P_i|-\left(\dfrac{P_{i+1}-P_i}{2}\right)^2=2Q_i|a_iQ_i-P_i|-(a_iQ_i-P_i)^2, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{if } a_i \text{ even.}\end{cases} \tag{2}$$

Corollary 3. *With the same notations as in Theorem 2, we have*

$$D(\beta)=D(\alpha), \text{ where } \beta=\frac{-b+\sqrt{d}}{2a}.$$

It is well known (cf. [3]) that for *Markov constants*, we have

Theorem 4. *With the same notations as in Theorem 2, we have*

$$M(\alpha)=\frac{\sqrt{d}}{\min\limits_{1\leqslant i\leqslant k} Q_i}.$$

Theorem 5. *With the notations as in Theorem 1, for* $\alpha=\dfrac{b+\sqrt{d}}{2}$, *we have*

$$M(\alpha)=\sqrt{d}.$$

From Theorems 1 and 5, it is easy to see that

Theorem 6. *Let* $\alpha_1=\dfrac{b_1+\sqrt{d_1}}{2}$ *and* $\alpha_2=\dfrac{b_2+\sqrt{d_2}}{2}$, *where* $d_i\equiv$ $\equiv b_i(\text{mod}\,4)(i=1,2)$ *are two positive discriminants which both are not the perfect square of the rational integers, and* $b_i=0,1$ $(i=1,2)$. *Then we have*

$$M(\alpha_1)\leqslant M(\alpha_2) \quad \text{if and only if} \quad D(\alpha_1)\leqslant D(\alpha_2).$$

Remark. H. Niederreiter [6] asked whether

$$M(\alpha_1)\leqslant M(\alpha_2) \text{ if and only if } D(\alpha_1)\leqslant D(\alpha_2).$$

A. Tripathi [7] gave some negative answers for this question. And our Theorem 6 gives some positive answers for this question.

II. Proof of the Results in I

Theorems 4 and 5 are clear, cf. [3].

By [1] and [7], for a real quadratic irrational number α as in Theorem 2, we have

$$D(\alpha)=\max_{1\leqslant i\leqslant k}\psi_i(n_i), \tag{3}$$

where

$$\psi_i(x)=:\frac{-x^2+(\Lambda_i-\lambda_i-1)x+\Lambda_i(1+\lambda_i)}{M_i},$$

and n_i is the rational integer closest to

$$x_i=:(\Lambda_i-\lambda_i-1)/2,$$

with

$$M_i=:\lambda_i+\Lambda_i,$$

and

$$\Lambda_i=:[a_{i+1},a_{i+2},\dots],\lambda_i=:[0,a_i,a_{i-1},\dots,a_1,\overline{a_k,a_{k-1},\dots,a_2,a_1}].$$

By the assumption of Theorem 2, [4] and [5], we have that

$$\lambda_i=\frac{2Q_i}{P_{i+1}+\sqrt{d}}=\frac{-P_{i+1}+\sqrt{d}}{2Q_{i+1}},\quad \text{for } 1\leqslant i\leqslant k,$$

$$\Lambda_i=\frac{P_{i+1}+\sqrt{d}}{2Q_{i+1}}=\alpha_{i+1},\quad \text{for } 1\leqslant i\leqslant k,$$

and

$$M_i=\frac{\sqrt{d}}{Q_{i+1}},\quad x_i=\frac{P_{i+1}-Q_{i+1}}{2Q_{i+1}},\quad \text{for } 1\leqslant i\leqslant k.$$

Since

$$[\Lambda_i]=[\alpha_{i+1}]=\left[\frac{\sqrt{d}+P_{i+1}}{2Q_{i+1}}\right]=a_{i+1},$$

and

$$0<\lambda_i=\frac{-P_{i+1}+\sqrt{d}}{2Q_{i+1}}<1,$$

we get

$$x_i = \frac{a_{i+1} + \{\Lambda_i\} - \lambda_i - 1}{2},$$

which yields that we have, for $1 \leqslant i \leqslant k$,

$$n_i = \begin{cases} \dfrac{a_{i+1} - 1}{2}, & \text{if } a_{i+1} \text{ is odd;} \\[2mm] \dfrac{a_{i+1}}{2}, & \text{if } a_{i+1} \text{ is even and } P_{i+1} \geqslant P_{i+2}; \\[2mm] \dfrac{a_{i+1}}{2} - 1, & \text{if } a_{i+1} \text{ is even and } P_{i+1} \leqslant P_{i+2}, \end{cases}$$

since

$$\{\Lambda_i\} - \lambda_i = \frac{P_{i+1} - P_{i+2}}{2Q_{i+1}}.$$

Therefore we get, for $1 \leqslant i \leqslant k$,

$$\psi_i(n_i) = \frac{-n_i^2 + (\Lambda_i - \lambda_i - 1)n_i + \Lambda_i(1 + \lambda_i)}{M_i}$$

$$= \frac{-Q_{i+1}(2n_i + 1)^2 + 2P_{i+1}(2n_i + 1) + 2\sqrt{d} + Q_{i+1} + 4Q_i}{4\sqrt{d}}$$

$$= \begin{cases} \dfrac{-Q_{i+1}a_{i+1}^2 + 2P_{i+1}a_{i+1} + 2\sqrt{d} + Q_{i+1} + 4Q_i}{4\sqrt{d}}; \text{ if } a_{i+1} \text{ is odd;} \\[2mm] \dfrac{-Q_{i+1}(a_{i+1} \pm 1)^2 + 2P_{i+1}(a_{i+1} \pm 1) + 2\sqrt{d} + Q_{i+1} + 4Q_i}{4\sqrt{d}}, \\[2mm] \text{if } a_{i+1} \text{ is even and } P_{i+1} - P_{i+2} \gtrless 0, \end{cases}$$

$$= \begin{cases} \dfrac{2dQ_{i+1} + \sqrt{d}(-Q_{i+1}^2 a_{i+1}^2 + 2P_{i+1}a_{i+1}Q_{i+1} + Q_{i+1}^2 + 4Q_iQ_{i+1})}{4dQ_{i+1}}, \text{ if } a_{i+1} \text{ is odd;} \\[2mm] \dfrac{2dQ_{i+1} + \sqrt{d}(-Q_{i+1}^2 a_{i+1}^2 + 2P_{i+1}a_{i+1}Q_{i+1} \mp 2Q_{i+1}^2 a_{i+1} \pm 2P_{i+1}Q_{i+1} + 4Q_iQ_{i+1})}{4dQ_{i+1}}, \\[2mm] \text{if } a_{i+1} \text{ is even and } P_{i+1} - P_{i+2} \gtrless 0, \end{cases}$$

$$= \begin{cases} \dfrac{2dQ_{i+1} + \sqrt{d}(-(a_{i+1}Q_{i+1} - P_{i+1})^2 + Q_{i+1}^2 + d)}{4dQ_{i+1}}, \text{if } a_{i+1} \text{ is odd;} \\ \dfrac{2dQ_{i+1} + \sqrt{d}(-(a_{i+1}Q_{i+1} - P_{i+1})^2 \pm Q_{i+1}(P_{i+1} - P_{i+2}) + d)}{4dQ_{i+1}}, \\ \text{if } a_{i+1} \text{ is even and } P_{i+1} - P_{i+2} \gtrless 0, \end{cases}$$

$$= \frac{2d + \dfrac{d + R_{i+1}}{Q_{i+1}}\sqrt{d}}{4d}, \tag{4}$$

using the definition of R_i and

$$d = P_{i+1}^2 + 4Q_iQ_{i+1}, P_{i+2} + P_{i+1} = 2a_{i+1}Q_{i+1},$$

which can be seen in [4].

This finishes the proof of Theorem 2, by using (3) (4), and the period.

It is easy to see that under the assumption of Theorem 1, the maximum (1) is reached when $i = k$, by using Lemma 5 in [4]. The details are as follows:

(A) We have (cf. [4] and [5])

$$a_k = \begin{cases} 2a_0, \text{if } b = 0 \text{ and } a = 1; \\ 2a_0 - 1, \text{if } b = a = 1, \end{cases}$$

$$Q_k = a, P_k = 2a_kQ_k - P_{k+1} = 2a_kQ_k - P_1 = 2a_ka - (2a_0a - b),$$

so that

$$Q_k = 1, R_k = b, \text{ if } a = 1 \text{ and } b = 0 \text{ or } 1.$$

(B) From (A), we have

$$\frac{d + R_k}{Q_k} = d + b, \text{ if } a = 1 \text{ and } b = 0 \text{ or } 1.$$

(C) Then by using (2) and Lemma 5 in [4], for $1 \leqslant i < k$, we have

$$\frac{d + R_i}{Q_i} \leqslant \frac{d}{Q_i} + \begin{cases} Q_i, \text{if } a_i \text{ odd;} \\ |P_{i+1} - P_i|, \text{if } a_i \text{ even,} \end{cases}$$

$$< \frac{d}{2} + \sqrt{d} < \frac{d + R_k}{Q_k} = d + b, \text{ if } a = 1 \text{ and } b = 0 \text{ or } 1.$$

In the above, the following facts are used:

$$2 \leqslant Q_i < \sqrt{d}, \text{ for } 1 \leqslant i < k, \text{ when } a = 1 \text{ and } b = 0 \text{ or } 1,$$
$$1 \leqslant P_i < \sqrt{d}, \text{ for } 1 \leqslant i \leqslant k.$$

Therefore we have finished the proof of Theorem 1.

For the proof of Corollary 3, we point out that, with the notations as in Theorem 2, the simple continued fraction expansion of β is

$$\beta = [a_k - a_0, \overline{a_{k-1}, a_{k-2}, \ldots, a_2, a_1, a_k}],$$

with the basic period $\overline{a_{k-1}, a_{k-2}, \ldots, a_2, a_1, a_k}$, and the n-th complete quotient

$$\beta_n = \frac{P_{k+1-n} + \sqrt{d}}{2Q_{k-n}} (1 \leqslant n \leqslant k).$$

According to these facts and Theorem 2, let the corresponding R_i and Q_i for β be R_i' and Q_i' respectively, then it is easy to show the following facts:

$$Q_i' = \begin{cases} Q_{k-i}, & \text{if } 1 \leqslant i < k; \\ Q_k, & \text{if } i = k, \end{cases} \quad \text{and} \quad R_i' = \begin{cases} R_{k-i}, & \text{if } 1 \leqslant i < k; \\ R_k, & \text{if } i = k, \end{cases}$$

which together with (1) and (2) yield Corollary 3.

III. The Algorithm of $D(\alpha)$

For $\alpha = \dfrac{b + \sqrt{d}}{2a}$, with $|b| \leqslant a \leqslant c$, $\gcd(a, b, c) = 1$, $d = b^2 + 4ac$, when $d < 200$ is not a perfect square, by using the results of [4] and Theorem 2 here, it is easy to implement a C-language program to compute the desired values of $D(\alpha)$. We ran the program in a computer LEO 386/25, and the results are recorded in the attached Table A. According to Theorem 1 and Corollary 3, we need to consider only $a > 1$, and $b \geqslant 0$. So we get $D\left(\dfrac{b + \sqrt{d}}{2a}\right) = \dfrac{1}{2} + \dfrac{(R + d)\sqrt{d}}{4Qd}$, where the values of d, a, b, R and Q are as shown in the following Table A:

Table A

d	a	b	R	Q	d	a	b	R	Q	d	a	b	R	Q
17	2	1	1	1	105	3	3	3	2	156	3	0	0	1
24	2	0	0	1	105	4	3	1	1	156	5	4	4	2
28	2	2	0	1	108	2	2	0	1	157	3	1	1	1
33	2	1	1	1	109	3	1	1	1	160	3	2	8	3
37	3	1	1	1	109	5	3	1	1	160	4	4	0	1
40	2	0	0	2	112	3	2	0	1	160	5	0	8	3
40	3	2	0	2	112	4	0	0	1	161	2	1	1	1
41	2	1	1	1	113	2	1	1	1	161	4	1	1	1
44	2	2	0	1	113	4	1	1	1	161	5	1	1	1
48	3	0	0	1	116	4	2	0	1	164	5	2	0	1
52	3	2	0	1	116	5	4	0	1	165	3	3	9	3
56	2	0	0	1	120	2	0	0	2	165	5	5	9	3
57	2	1	1	1	120	3	0	0	2	168	2	0	0	2
57	3	3	1	1	120	5	0	0	1	168	3	0	0	2
60	2	2	4	2	124	2	2	0	1	168	6	0	0	1
60	3	0	4	2	124	3	2	0	1	172	2	2	0	1
61	3	1	1	1	124	5	2	0	1	172	3	2	0	1
65	2	1	3	2	128	4	4	0	1	172	6	2	0	1
65	4	1	1	1	129	2	1	1	1	176	4	0	0	1
69	3	3	1	1	129	3	3	1	1	176	5	4	0	1
72	2	0	0	1	129	4	1	1	1	177	2	1	1	1
73	2	1	1	1	129	5	3	1	1	177	3	3	1	1
73	3	1	1	1	132	3	0	0	1	177	4	1	1	1
73	4	3	1	1	133	3	1	1	1	177	6	3	1	1
76	2	2	0	1	136	2	0	0	1	180	4	2	0	1
76	3	2	0	1	136	3	2	8	3	180	5	0	0	1
80	4	0	0	1	136	5	4	8	3	181	3	1	1	1
84	3	0	0	1	137	2	1	1	1	181	5	1	1	1
84	4	2	0	1	137	4	3	1	1	184	2	0	0	1
85	3	1	5	3	140	2	2	4	2	184	3	2	0	1
88	2	0	0	1	140	5	0	4	2	184	5	2	0	1
88	3	2	0	1	141	3	3	1	1	184	6	4	0	1
89	2	1	1	1	141	5	1	1	1	185	2	1	3	2
89	4	3	1	1	145	2	1	3	2	185	4	3	1	1
92	2	2	0	1	145	3	1	3	2	185	5	5	3	2
93	3	3	1	1	145	4	1	7	4	188	2	2	0	1
96	3	0	0	3	145	5	5	7	4	189	5	3	1	1
96	4	4	0	3	145	6	1	1	1	192	3	0	0	3
97	2	1	1	1	148	3	2	8	3	192	4	4	0	3
97	3	1	1	1	148	4	2	8	3	193	2	1	1	1
97	4	1	1	1	149	5	3	1	1	193	3	1	1	1
101	5	1	1	1	152	2	0	0	1	193	4	1	1	1
104	2	0	0	2	153	2	1	1	1	193	6	1	1	1
104	5	2	0	2	153	4	3	1	1	193	6	5	1	1
105	2	1	3	2	156	2	2	4	2	197	7	1	1	1

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