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Numerical Schemes for Investment Models with Singular Transactions

AGNÈS TOURIN[1] and THALEIA ZARIPHOPOULOU[2]
[1]*CEREMADE, Université de Paris IX Dauphine, Place de Lattre-De-Tassigny, 75775 Paris Cedex 16, France;* [2]*Departments of Finance and Mathematics, University of Wisconsin, Madison, WI 53706, U.S.A.*

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Abstract. This paper considers an infinite horizon investment-consumption model in which a single agent consumes and distributes his wealth between two assets, a bond and a stock. The problem of maximization of the total utility from consumption is treated, when state (amount allocated in assets) and control (consumption, rates of trading) constraints are present. The value function is characterized as the unique viscosity solution of the Hamilton-Jacobi-Bellman equation which, actually, is a Variational Inequality with gradient constraints. Numerical schemes are then constructed in order to compute the value function and the location of the free boundaries of the so-called transaction regions. These schemes are a combination of implicit and explicit schemes; their convergence is obtained from the uniqueness of viscosity solutions to the HJB equation.

1. **Introduction**

In this paper we examine a general investment and consumption decision problem for a single agent. The investor consumes at a nonnegative rate and he distributes his current wealth between two assets. One asset is a *bond*, i.e. a riskless security with instantaneous rate of return r. The other asset is a *stock*, whose price is driven by a Wiener process.

When the investor makes a transaction, he pays transaction fees which are assumed to be proportional to the amount transacted. More specifically, let x_t and y_t be the investor's holdings in the riskless and the risky security prior to a transaction at time t. If the investor increases (or decreases) the amount invested in the risky asset to $y_t + h_t$ (or $y_t - h_t$), the holding of the riskless asset decreases (increases) to $x_t - h_t - \lambda h_t$ (or $x_t + h_t - \mu h_t$). The numbers λ and μ are assumed to be nonnegative and one of them must always be positive. The control objective is to maximize, in an infinite horizon, the expected discounted utility which comes only from consumption. Due to the presence of the transaction fees, this is a singular control problem.

Our goals are to derive the Hamilton-Jacobi-Bellman (HJB) equation that the value function solves and to characterize the latter as its unique weak solution, to come up with numerical schemes which converge to the value function as well as the optimal investment and consumption rules and to perform actual numerical computations and compare some of the results to the ones obtained in closed form by Davis and Norman.

We continue with the description of the model. The price P_t^0 of the bond is given by

$$\begin{cases} dP_t^0 = rP_t^0\,dt \quad (t \geqslant 0), \\ P_0^0 = p_0, \end{cases} \tag{1.1}$$

where $r > 0$ is the *interest rate*. The price P_t of the stock satisfies

$$\begin{cases} dP_t = bP_t\,dt + \sigma P_t\,dW_t \quad (t \geqslant 0), \\ P_0 = p, \end{cases} \tag{1.2}$$

where b is the *mean rate of return*, σ is the *dispersion coefficient* and the process W_t, which represents the source of uncertainty in the market, is a standard Brownian motion defined on the underlying probability space (Ω, F, P). As usual, F_t is the augmentation under P of $F_t^w = \sigma(W_s : 0 < s \leqslant t)$ for $t > 0$. The market coefficients r, b and σ are assumed to be constant with $\sigma \neq 0$ and $b > r > 0$.

The amounts x_t and y_t, invested at time t in bond and stock, respectively, are the *state variables* and they evolve according to the equations

$$\begin{cases} dx_t = (rx_t - C_t)\,dt - (1+\lambda)\,dM_t + (1-\mu)\,dN_t, \\ dy_t = by_t + \sigma y_t\,dw_t + dM_t - dN_t, \\ x_0 = x, \quad y_0 = y, \end{cases} \tag{1.3}$$

where (x, y) is the *endowment* of the investor. For simplicity, we assume here that all financial charges are paid from the holdings in the bond.

The *control processes* are the *consumption rate* $C_\bullet$ and the processes $M_\bullet$ and $N_\bullet$ which represent, respectively, the *cumulative purchases* and *sales of stock*. We say that the controls $(C_\bullet, M_\bullet, N_\bullet)$ are admissible if:

(i) C_t is F_t-measurable and $C_t \geqslant 0$ a.s. and $\int_0^t e^{-rs} C_s\,ds < +\infty$ a.s., $\forall t \geqslant 0$.

(ii) M_t, N_t are F_t-measurable, right continuous and non-decreasing processes.

(iii) If x_t, y_t are the state trajectories given by (1.3), when controls M_t, N_t are used, then, for all $t \geqslant 0$,

$$\begin{cases} x_t + (1+\lambda)y_t \geqslant 0 \ \text{ a.s. if } y_t \leqslant 0, \\ x_t + (1-\mu)y_t \geqslant 0 \ \text{ a.s. if } y_t \geqslant 0, \end{cases} \tag{1.4}$$

and we denote by $\mathcal{A}(x, y)$ the set of admissible policies.

The total *expected discounted utility* J from consumption, is given by

$$J(x, y, C, M, N) = E \int_0^{+\infty} e^{-\beta t} U(C_t)\,dt \tag{1.5}$$

with $(C, M, N) \in \mathcal{A}(x, y)$ and $(x, y) \in \bar{\Omega}$ where

$$\Omega = \{(x, y) \in \mathcal{R} \times \mathcal{R} : x + (1+\lambda)y > 0 \text{ if } y < 0 \\ \text{and } x + (1-\mu)y > 0 \text{ if } y \geqslant 0\}.$$

The utility function $U : [0,+\infty) \to [0,+\infty)$ is assumed to have the following properties:

$$\begin{cases} \text{(i)} & U \in C^2((0,\infty)) \text{ and is strictly increasing, nonnegative and concave in } [0,+\infty). \\ \text{(ii)} & \text{There exists } K > 0 \text{ and } \gamma \in (0,1) \text{ such that, for all } c \geqslant 0, U(c) \leqslant K(1+c)^{\gamma}. \\ \text{(iii)} & \lim_{c \to 0} U'(c) = +\infty \text{ and } \lim_{c \to +\infty} U'(c) = 0. \end{cases}$$

The *discount factor* $\beta > 0$ weights consumption now versus consumption later. Note that the controls $M_\bullet$ and $N_\bullet$ are acting implicitly through the state constraints given by (1.4).

The *value function* u is given by

$$\begin{aligned} u(x,y) &= \sup_{(C,M,N)\in A(x,y)} J(x,y,C,M,N) \\ &= \sup_{\mathcal{A}(x,y)} E\int_0^{+\infty} e^{-\beta t} U(C_t)dt. \end{aligned} \tag{1.6}$$

To guarantee that the value function is well defined when U is unbounded, we assume that

$$\beta > r\gamma + \Big(\gamma(b-r)/\sigma^2(1-\gamma)\Big). \tag{1.7}$$

This condition yields that the value function which corresponds to $\lambda = \mu = 0$ and $U(c) = K(1+c)^\gamma$, and thereby all value functions for $0 < \lambda, \mu < 1$, are finite (see Karatzas et al. (1987) or Zariphopoulou (in press)).

Our goal is first to derive the Hamilton-Jacobi-Bellman equation associated with the above singular stochastic control problem and to characterize u as its unique weak solution. It turns out that the Bellman equation here is a Variational Inequality with gradient constraints.

Due to the nature of our goals, in the Introduction we only state our result regarding the characterization of the value function. The rest of our results are far more complicated to state here. We hence choose to present them in the main body of the paper.

THEOREM. *The value function u is the unique constrained viscosity solution of*

$$\begin{aligned} \min\Big[& \beta u - \frac{1}{2}\sigma^2 y^2 u_{yy} - byu_y - rxu_x - \max_{c\geqslant 0}[-cu_x + U(c)], \\ & (1+\lambda)u_x - u_y, -(1-\mu)u_x + u_y\Big] = 0 \textit{ in } \Omega, \end{aligned} \tag{1.8}$$

in the class of concave and uniformly continuous functions.

The fact that the value function turns out to be the *unique* viscosity solution of (1.8) plays a very crucial role for the convergence of the numerical schemes proposed in Section 5.

We continue with a short discussion about the history of the model: Transaction costs are an essential feature of some economic theories and many times are incorporated in the two-asset portfolio selection model. Constantinides (1979, 1986) assumes that the transaction costs deplete only the riskless asset and that the stock price is a logarithmic Brownian motion. In the continuous time framework, Taksar, Klass and Assaf (1986) assume that the investor does not consume but maximizes the long term expected rate of growth of wealth. In the same framework, but under more general assumptions, Fleming, Grossman, Vila and Zariphopoulou (preprint) study the finite horizon problem, the average cost per unit time problem and an asymptotic growth problem.

Davis and Norman (1990) relax the assumption that the transaction costs are charged only to the nonrisky asset. They consider a particular class of utility functions of the form $U(c) = c^p (0 < p < 1)$ and they get an explicit form for the value function. They also prove that the optimal strategy confines the investor's portfolio to a certain wedge-shaped region in the portfolio plane. Their results discussed are presented in detail in Section 5. In a paper which appeared after a preliminary version of this paper was circulated, Shreve and Soner (preprint) examine the above class of utility functions and they relax some assumptions on the market parameters in order for the value function to be finite. Moreover, they prove that the value function is a smooth solution of the HJB equation and that the boundary of the aforementioned wedge-shaped region is also smooth.

Finally, there are several directions in which the two-asset problem with transaction costs can be extended. Firstly, more than one risky asset can be allowed. Although this extension is straightforward, the computational requirements are enormous. Secondly, fixed transaction costs can be introduced. Some single-period models with fixed transaction costs are discussed in Leland (1985), Brennan (1975) and Goldsmith (1976). Kandel and Ross (1983) introduce quasi-fixed transaction costs and portfolio management fees. In a different direction, a model with proportional fees when the rate of return of the risky asset is a continuous time Markov chain is examined by Zariphopoulou (in press).

The paper is organized as follows: Section 2 is about some basic properties of the value function. In Section 3, we study the solutions of the Variational Inequality (1.8) and we characterize the value function as its unique solution. Section 4 reviews the results of Davis and Norman for the H.A.R.A. utility functions. In Section 5, we present the numerical algorithms and we study the behavior of the transaction regions. Finally, in section 6, we summarize the main conclusions of the paper.

2. Basic properties of the value function

In this section we derive some basic properties of the value function.

PROPOSITION 2.1. *The value function u is jointly concave in x and y, strictly increasing in x and increasing in y.*

Sketch of the proof. The joint concavity of the value function comes from the concavity of the utility function and the linearity of the dynamics. Indeed, if (C_1, L_1, M_1) and (C_2, L_2, M_2) are optimal policies for the points (x_1, y_1) and (x_2, y_2), then $(\lambda C_1+(1-\lambda)C_2, \lambda M_1+(1-\lambda)M_2, \lambda N_1+(1-\lambda)N_2)$ is admissible for $(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2)$.

The monotonicity of u follows from the same monotonicity of the utility function and the linear dynamics of the state trajectories. The strict monotonicity in x, comes from the fact that u is also concave. (For a detailed proof of a similar question, see Proposition 2.1 in Zariphopoulou (in press).)

PROPOSITION 2.2. *The value function u is uniformly continuous on $\bar{\Omega}$.*

Proof. Since u is concave, it is obviously continuous in Ω.

We next show that u is continuous on the boundary. The continuity at the point $(0,0)$ follows as in Proposition 2.2 in Zariphopoulou (in press).

We now show that

$$\lim_{(x_n,y_n)\to(x_0,y_0)} u(x_n, y_n) = u(x_0, y_0)$$

where

$$(x_0, y_0) \in l_1 = \{(x,y) \in \mathcal{R}^+ \times \mathcal{R}^- : x + (1+\lambda)y = 0\}$$

or

$$(x_0, y_0) \in l_2 = \{(x,y) \in \mathcal{R}^- \times \mathcal{R}^+ : x + (1-\mu)y = 0\}.$$

We only examine the case $(x_0, y_0) \in l_1$ since the other is treated similarly. To this end, consider a point $(x_0, y_0) \in l_1$ and a sequence

$$(x_n, y_n) \in l_1^+ = \{(x,y) \in \mathcal{R}^+ \times \mathcal{R}^- : x + (1+\lambda)y > 0\}$$

such that

$$\lim_{n\to+\infty} (x_n, y_n) = (x_0, y_0).$$

Since u is locally Lipschitz, by concavity, it suffices to show that

$$\lim_{n\to+\infty} |u(x_0, y_n) - u(x_0, y_0)| = 0.$$

Finally, since u is increasing, we only need to show that

$$u(x_0, y_n) \leqslant u(x_0, y_0) + \epsilon$$

for any $\epsilon > 0$ and n sufficiently large.

Let (C^n, M^n, N^n) be an ϵ-optimal policy at (x_0, y_n). Then

$$u(x_0, y_n) \leqslant E \int_0^{+\infty} e^{-\beta t} U(C_t^n)\, dt + \epsilon.$$

Moreover the control $(C^n, \overline{M^n}, N^n)$, where

$$d\overline{M_t^n} = dM_t^n + \frac{(1-\mu)}{(\lambda+\mu)}(y_n - y_0)\delta_0(t)$$

is admissible for

$$(\overline{x_n}, \overline{y_n}) = \left(x_0 + \frac{(1-\mu)(1+\lambda)}{(\lambda+\mu)}(y_n - y_0), y_0 - \frac{(1-\mu)}{\lambda+\mu}(y_n - y_0)\right) \in l_1.$$

Therefore,

$$E\int_0^{+\infty} e^{-\beta t} U(C_t^n)\, dt \leqslant u(\overline{x_n}, \overline{y_n}) + \epsilon.$$

Combining the last two inequalities and using that u is continuous on $l_1 - \{0, 0\}$ we conclude.

Finally, since u is uniformly continuous on compact subsets of $\bar{\Omega}$, we remark that its uniform continuity on $\bar{\Omega}$ follows from the fact that, by concavity, u is Lipschitz continuous in $[x, +\infty) \times [y, +\infty)$ with Lipschitz constant $1/|(x, y)|$ for every $(x, y) \in \Omega$.

We conclude this section by stating (for a proof see, for example, Lions (1983)) a fundamental property of the value function known as the *Dynamic Programming Principle*.

PROPOSITION 2.3. *If θ is a stopping time (i.e. a nonnegative, F-measurable random variable), then*

$$u(x, y) = \sup_{\mathcal{A}_{(x,y)}} E\left\{\int_0^\theta e^{-\beta t} U(C_t)\, dt + e^{-\beta\theta} u(x_\theta, y_\theta)\right\}. \tag{2.1}$$

3. Viscosity solutions

In this section we characterize the value function as the *unique constrained viscosity solution* of the (HJB) equation (1.8). The characterization of u as a constrained solution is natural because of the presence of state constraints given by (1.4).

The notion of viscosity solutions was introduced by Crandall and Lions (1984) for first-order and by Lions (1983) for second order equations. For a general overview of the theory we refer to the "User's Guide" by Crandall, Ishii and Lions (1992).

Next, we recall the notion of constrained viscosity solutions, which was introduced by Soner (1986) and Capuzzo-Dolcetta and Lions (forthcoming) for first-order equations and by Lions for second-order equations (see also Ishii and Lions (1990) and Katsoulakis ((1991)). To this end, we consider a non-linear second order partial differential equation of the form

$$F(X, v, Dv, D^2v) = 0 \text{ in } \Omega, \tag{3.1}$$

where Dv and D^2v stand respectively for the gradient vector and the second derivative matrix of v; F is continuous in all its arguments and degenerate elliptic, meaning that

$$F(X,p,q,A+B) \leqslant F(X,p,q,A) \text{ if } B \geqslant 0. \tag{3.2}$$

DEFINITION 3.1. A continuous function $u : \mathbb{R} \to \mathbb{R}$ is a *constrained viscosity solution* of (3.1) if

i) u is a *viscosity subsolution* of (3.1) on $\bar{\Omega}$, that is for any $\phi \in C^2(\bar{\Omega})$ and any local maximum point $X_0 \in \bar{\Omega}$ of $u - \phi$
$$F(X_0, u(X_0), D\phi(X_0), D^2\phi(X_0)) \leqslant 0$$
and

ii) u is a *viscosity supersolution* of (3.1) in Ω, that is for any $\phi \in C^2(\bar{\Omega})$ and any local minimum point $X_0 \in \Omega$ of $u - \phi$
$$F(X_0, u(X_0), D\phi(X_0), D^2\phi(X_0)) \geqslant 0.$$

THEOREM 3.2. *The value function u is a constrained viscosity solution of (1.8) on $\bar{\Omega}$.*

Proof. i) We first show that v is a viscosity subsolution of (1.8) on $\bar{\Omega}$. Let $\phi \in C^2(\bar{\Omega})$ and $X_0 = (x_0, y_0) \in \bar{\Omega}$ be a maximum of $u - \phi$; without loss of generality we may assume that

$$v(X_0) = \phi(X_0) \quad \text{and} \quad u \leqslant \phi \text{ on } \bar{\Omega}. \tag{3.3}$$

We need to show that

$$\begin{aligned}\min\Bigg[& \beta\phi(X_0) - \frac{1}{2}\sigma^2 y_0^2 \phi_{yy}(X_0) - by_0\phi_y(X_0) - rx_0\phi_x(X_0) \\ & - \max_{c\geqslant 0}(-c\phi_x(X_0) + U(c)), (1+\lambda)\phi_x(X_0) - \phi_y(X_0), \\ & -(1-\mu)\phi_x(X_0) + \phi_y(X_0)\Bigg] \leqslant 0.\end{aligned} \tag{3.4}$$

We argue by contradiction and we assume that

$$(1+\lambda)\phi_x(X_0) - \phi_y(X_0) > 0, \tag{3.5}$$

$$-(1-\mu)\phi_x(X_0) + \phi_y(X_0) > 0, \tag{3.6}$$

and

$$\begin{aligned}&\beta\phi(X_0) - \frac{1}{2}\sigma^2 y_0^2 \phi_{yy}(X_0) - by_0\phi_y(X_0) - rx_0\phi_x(X_0) \\ &\quad - \max_{c\geqslant 0}(-c\phi_x(X_0) + U(c)) > \theta\end{aligned} \tag{3.7}$$

for some $\theta > 0$.

From the fact that ϕ is smooth, the above inequalities become

$$(1+\lambda)\phi_x(X) - \phi_y(X) > 0, \tag{3.8}$$

$$-(1-\mu)\phi_x(X) + \phi_y(X) > 0, \tag{3.9}$$

and

$$\begin{aligned}\beta\phi(X) - \frac{1}{2}\sigma^2 y^2 \phi_{yy}(X) - by\phi_y(X) - rx\phi_x(X)\\ - \max_{c\geqslant 0}(-c\phi_x(X) + U(c)) > \theta\end{aligned} \tag{3.10}$$

for some $\theta > 0$ where $X = (x, y) \in \mathcal{B}(X_0)$ a neighborhood of X_0.

We now consider the optimal trajectory $X_0^*(t) = (x_0^*(t), y_0^*(t))$ where $X_0^*(0) = (x_0, y_0)$ with optimal policies (C_t^*, M_t^*, N_t^*) begin used. (The existence of optimal policies was shown in (Th. (iv) 2).) We will need the following lemma which shows that X_0^* has no jumps a.s. at $t = 0^+$.

LEMMA 3.1. *Assume that inequality (3.5) (resp. (3.6)) holds and let A be the event that the optimal trajectory $X_0^*(t)$ has a jump at least of size ϵ at $t = 0^+$ along the direction $(-(1+\lambda), 1)$ (resp. $((1-\mu), -1)$). If*

$$\begin{aligned}(x_0 - (1+\lambda)\epsilon, y_0 + \epsilon) &\in \mathcal{B}(X)\\ (\textit{resp. } (x_0 + (1-\mu)\epsilon, y_0 - \epsilon) &\in \mathcal{B}(X_0))\end{aligned}$$

then $P(A) = 0$.

Since the proof is similar to the one of Lemma 1 in Davis, Panas and Zariphopoulou (1993), it is not presented here.

We now continue the proof of the Theorem. We define the random time τ to be $\tau(\omega) = \inf\{t \geqslant 0 : X_0^*(t) \notin \mathcal{B}(X_0)\}$. Notice that by the preceding Lemma, $\tau(\omega) > 0$ a.s. Combining (3.8), (3.9) and (3.10) we get

$$\begin{aligned} E\int_0^\tau \theta e^{-\beta s} ds &< E\int_0^\tau e^{-\beta s}[\beta\phi(X_s^*) - \frac{1}{2}\sigma^2 (y_s^*)^2 \phi_{yy}(X_s^*) - by_s^*\phi_y(X_s^*)\\ &\quad -rx_s^*\phi_x(X_s^*) - \{-C_s^*\phi_x(X_s^*) + U(C_s^*)\}]\, ds\\ &\quad +E\int_0^{\tau(\omega)} e^{-\beta s}[(1+\lambda)\phi_x(X_s^*) - \phi_y(X_s^*)]\, dM_s\\ &\quad +E\int_0^{\tau(\omega)} e^{-\beta s}[-(1-\mu)\phi_x(X_s^*) - \phi_y(X_s^*)]\, dN_s\\ &= E(I_1(\tau)) - E\int_0^\tau e^{-\beta s} U(C_s^*)\, ds\\ &\quad +E(I_2(\tau)) + E(I_3(\tau)). \end{aligned} \tag{3.11}$$

Applying Itô's formula to $e^{-\beta\tau}\phi(X_0^*(\tau))$ gives

$$E\{e^{-\beta\tau}\phi(X_0^*(\tau))\} = \phi(X_0) - [E(I_1(\tau)) + E(I_2(\tau)) + E_3(\tau))]. \quad (3.12)$$

Combining (3.3), (3.11) and (3.12) we get

$$E\{u(X_0^*(t))\} \leqslant u(X_0) - \left[E\int_0^\tau e^{-\beta s}U(C_s^*)ds + \theta\frac{1 - Ee^{-\beta\tau}}{\beta}\right], \quad (3.13)$$

which violates the dynamic programming principle, together with the optimality of (C_t^*, M_t^*, N_t^*). Therefore, at least one of the arguments of the minimum operator in (3.4) must be non-positive and hence the value function is a viscosity subsolution of (1.8).

ii) In the second part of the proof, we show that u is a viscosity supersolution of (1.8) in Ω; for this we must show that, for all smooth functions $\varphi(X)$, such that $u - \varphi$ has a local minimum at $X_0 \in \Omega$, the following holds:

$$\min\Big[\beta\varphi(X_0) - \frac{1}{2}\sigma^2 y_0^2\varphi_{yy}(X_0) - by_0\varphi_y(X_0) - rx_0\varphi_x(X_0) \\ - \max_{c\geqslant 0}(-c\varphi_x(X_0) + U(c)), (1+\lambda)\varphi_x(X_0) - \varphi_y(X_0), \\ -(1-\mu)\varphi_x(X_0) + \varphi_y(X_0)\Big] \geqslant 0,$$

where, without loss of generality, $u(X_0) = \varphi(X_0)$ and $u \geqslant \varphi$ on $\bar{\Omega}$. In this case, we prove that each argument of the above minimum operator is nonnegative.

Consider the trading strategy $L(t) = L_0 > 0$ and $M(t) = 0$ for $t \geqslant 0$. By the Dynamic Programming Principle,

$$u(x_0, y_0) \geqslant u(x_0 - (1+\lambda)L_0, y_0).$$

This inequality holds for φ as well, and, by taking the left-hand side to the right-hand side, dividing by L_0 and sending $L_0 \to 0$, we get

$$(1+\lambda)\varphi_x(X_0) - \varphi_y(X_0) \geqslant 0.$$

Similarly, by using the trading strategy $L(t) = 0$ and $M(t) = M_0 > 0$, for $t \geqslant 0$, we obtain

$$-(1-\mu)\varphi_x(X_0) + \varphi_y(X_0) \geqslant 0.$$

Finally consider the case where the investor does not trade but consumes at a constant rate $C_t = C$ for $0 < t \leqslant \tau$ where $\tau = n \wedge \tau_1 \wedge \tau_2$ with $n \in \mathbb{N}$

$$\begin{aligned} \tau_1 &= \inf\{t : x_t + (1+\lambda)y_t \geqslant 0 \text{ a.s. if } y_t \leqslant 0\}, \\ \tau_2 &= \inf\{t : x_t + (1-\mu)y_t \geqslant 0 \text{ a.s. if } y_t \geqslant 0\} \end{aligned}$$

and x_t, y_t are the state trajectories, given by (1.3), under policy $(C, 0, 0)$. The Dynamic Programming Principle yields

$$u(x_0, y_0) \leqslant E\left[\int_0^\tau e^{-\beta s} U(C)\, ds + e^{-\beta\tau} u(x_\tau, y_\tau)\right].$$

The same inequality holds for φ which, in turn, combined with the Itô's rule applied to $e^{-\beta\tau}\varphi(x_\tau, y_\tau)$ gives

$$E\int_0^\tau e^{-\beta s}\Big[-\beta u(X_s) + \frac{1}{2}\sigma^2 y_s^2 \varphi_{yy}(X_s) - b y_s \varphi_y(X_s) - r x_s \varphi_x(X_s) \\ + C\varphi_x(X_s) - U(C)\Big] ds \leqslant 0.$$

Dividing by n, and sending $n \to +\infty$ we get

$$\beta\varphi(X_0) - \frac{1}{2}\sigma^2 y_0^2 \varphi_{yy}(X_0) - b y_0 \varphi_y(X_0) - r x_0 \varphi_x(X_0) \\ - \max_{c \geqslant 0}(-c\varphi_x(X_0) + U(c)) \geqslant 0$$

(for a detailed argument, see Zariphopoulou (in press)). This completes the proof.

We conclude this section by presenting a comparison result for constrained viscosity solutions of (1.8). This result will be used later in Section 5 to obtain convergence of the numerical schemes employed for the value function and the optimal policies.

THEOREM 3.3. *Let u be an upper semi-continuous viscosity subsolution of (1.8) on $\bar{\Omega}$ with sublinear growth and v be a bounded from below uniformly continuous viscosity supersolution of (1.8) in Ω. Then, $u \leqslant v$ on $\bar{\Omega}$.*

Sketch of the proof. We first construct a positive strict supersolution of (1.8) ion Ω. To this end, let $w(x, y)$ be the value function defined as in (1.6) with U replaced by some U_1 such that $U_1(c) > U(c)$ for $c > 0$, $U_1(0) = U(0) \geqslant 0$ and $\lambda = \mu = 0$. This value function is the solution to the classical Merton consumption-portfolio problem in the absence of transaction costs and satisfies $w(x, y) = v(z)$, where $z = x + y$ and v solves

$$\begin{cases} \beta v = -\frac{(b-r)^2}{2\sigma^2}\frac{v'^2}{v''} + rzv' + \max_{c \geqslant 0}\{-cv' + U_1(c)\} & (z > 0) \\ v > 0, v' > 0 \text{ and } v'' < 0, \quad (z > 0). \end{cases} \tag{3.14}$$

We now let

$$V(x, y) = v(x + ky) \text{ with } 1 - \mu < k < 1 + \lambda((x, y) \in \bar{\Omega})$$

and claim that V is a positive supersolution of (1.8).

The choice of k implies

$$x + ky > 0 \text{ whenever } (x, y) \in \bar{\Omega}$$

which combined with (3.14) yields

$$v > 0 \text{ and } v' > 0 \text{ on } \bar{\Omega}.$$

It then follows that

$$\left\{ \begin{array}{rl} (1+\lambda)V_x(x,y) - V_y(x,y) & = (1+\lambda-k)v'(x+ky) \\ & = f_1(x,y) > 0 \\ -(1-\mu)V_x(x,y) + V_y(x,y) & = (-1+\mu+k)v'(x+ky) \\ & = f_2(x,y) > 0. \end{array} \right. \tag{3.15}$$

Moreover, using (3.14), we obtain

$$\begin{aligned} & \beta V(x,y) - \frac{1}{2}\sigma^2 y^2 V_{yy}(x,y) - byV_y(x,y) - rxV_x(x,y) \\ & \quad - \max_{c \geqslant 0}\{-cV_x(x,y) + U(c)\} \\ & = \beta v(z) - \frac{1}{2}\sigma^2(ky)^2 v''(z) - b(ky)v'(z) - rxv'(z) \\ & \quad - \max_{c \geqslant 0}\{-cv'(z) + U(c)\} \\ & = \left[-\frac{(b-r)^2}{2\sigma^2}\frac{(v'(z))^2}{v''(z)} - \frac{1}{2}\sigma^2(ky)^2 v''(z) - (b-r)kyv'(z) \right] \\ & \quad + \left[(\max_{c \geqslant 0}(cv'(z) + U_1(c)) - \max_{c \geqslant 0}(-cv'(z) + U(c)) \right] \\ & = g_1(z,y) + g_2(z). \end{aligned}$$

We now observe that the first term g_1 in the above sum is nonnegative, since the maximum value of the quadratic $\mathcal{D}(g_1) = \frac{1}{2}\sigma^2 q^2 v'' + (b-r)qv'$ is $-[(b-r)^2/2\sigma^2](v')^2/v''$, where $q = ky$. Moreover

$$\begin{aligned} g_2(z) = g_2(x+ky) & = \max_{c \geqslant 0}\{-cv'(x+ky) + U_1(c)\} \\ & \quad - \max_{c \geqslant 0}\{-cv'(x+ky) + U(c)\} > 0 \end{aligned} \tag{3.16}$$

due to the choice of U_1.

Let

$$\begin{aligned} H(X, V, DV, D^2V) & = \min\Big\{ \beta V - \frac{1}{2}\sigma^2 y^2 V_{yy} - byV_y - rxV_x \\ & \quad - \max_{c \geqslant 0}\{-cV_x + U(c)\}, (1+\lambda)V_x - V_y, \\ & \quad -(1-\mu)V_x + V_y \Big\}. \end{aligned}$$

Combining (3.15) and (3.16) yields

$$H(X,V,DV,D^2V) \geqslant \min\{f_1(x,y), f_2(x,y), g(x,y)\} = h(x,y) > 0 \text{ in } \Omega.$$

To conclude the proof of the theorem we will need the following lemma. Its proof follows along the lines of Theorem VI.5 in Ishii and Lions (1990) and therefore it is omitted.

LEMMA 3.2. *Let u be upper semi-continuous with sublinear growth viscosity subsolution of (1.8) on $\bar{\Omega}$ and v be bounded from below uniformly continuous viscosity supersolution of $H(X,u,Du,D^2u) - h(X)$, where $h > 0$ in Ω. Then $u \leqslant v$ on $\bar{\Omega}$.*

We now conclude the proof of the theorem. We define the function $w^\theta = \theta v + (1-\theta)V$ where $0 < \theta < 1$ and we observe that w^θ is a viscosity supersolution of $H - h = 0$. (See also Davis et al. (1993), Th. 2, for a similar argument.) Applying the above Lemma to u and w^θ we get

$$u \leqslant w^\theta \text{ on } \bar{\Omega};$$

sending θ to 1 concludes the proof.

4. The case of H.A.R.A. utilities

Davis and Norman (1990) solve explicitly the problem in the case of Hyperbolic Absolute Risk Aversion utility functions $U(c) = (1/\gamma)c^\gamma$, $0 < \gamma < 1$ and $U(c) = \log c$ for $\gamma = 0$.

They first remark that the solvability region has to be depleted into three regions: the so-called *sell* and *buy regions* (sales and purchases respectively take place instantaneously), and the *non-transaction region*.

In order to find the location of the free boundaries, they use the homothetic property of the value function and they reduce the problem to a one dimensional one. More precisely, they set $v(x,y) = y^\gamma \Psi(x/y)$ where the function Ψ satisfies:

$$\begin{cases} \Psi(x) = \frac{1}{2}A(x + (1-\mu)y)^\gamma \quad (x \leqslant x_1), \\ \beta_1\Psi(x) + \beta_2 x\Psi'(x) + \beta_3 x^2\Psi''(x) + \frac{1-\gamma}{\gamma}\Psi'(x)^{\frac{-\gamma}{1-\gamma}} = 0 \quad (x_1 \leqslant x \leqslant x_2), \\ \Psi(x) = \frac{1}{\gamma}B(x + (1+\lambda)y)^\gamma \quad (x \geqslant x_2), \end{cases}$$

where $\beta_1 = -\frac{1}{2}\sigma^2\gamma(1-\gamma) + b\gamma - \beta$, $\beta_2 = \sigma^2(1-\gamma) + r - \beta$, $\beta_3 = \frac{1}{2}\sigma^2$ and the points x_1, x_2 and the coefficients A and B are explicitly determined.

They prove that the existence of such a function Ψ provides a sufficient condition for the optimality of a policy (C, M, N) such that the corresponding process (x_t, y_t) is a reflecting diffusion in the non-transaction region and M and N are the local times at the lower and upper boundaries respectively. Besides, they prove the existence of such a solution Ψ.

Finally, they propose an algorithm that solves the above algorithm by integrating backwards a system of differential equations.

5. Numerical scheme

This section is devoted to the construction of a finite difference scheme in order to computer the unique viscosity solution of the Variational Inequality (1.8). The approach which relies on the theory of viscosity solutions and the Dynamic Programming Principle yields *monotone, stable and consistent schemes*. The convergence of such schemes was originally proved in some situations by Crandall and Lions (1984), Barles and Souganidis (1991) and Souganidis (1985). More recently, it was proved for parabolic equations, arising in problems of option pricing, by Barles, Daher and Romano (1991) and by Davis, Panas and Zariphopoulou (1993).

In the scheme constructed here, the first-order operators are approximated by a monotone finite difference scheme. As far as the second-order operator is concerned, the first-order part is approximated by a monotone explicit scheme based on the Dynamic Programming Principle whereas the second-order term is approximated by an implicit Cranck-Nickolson scheme. Thus splitting into two half iterations allows one to choose a time step of the same order as the mesh size. This method is known as the *time splitting* method or method of *fractional steps*.

We next present the numerical scheme we developed. To this end, we first write (1.8) in the concise form

$$\min\{L_0(x, y, u, u_x, u_y, u_{yy}), L_1(u_x, u_y), L_2(u_x, u_y)\} = 0 \tag{5.1}$$

where

$$\begin{aligned} L_0(x, y, u, u_x, u_y, u_{yy}) &= \beta u - \frac{1}{2}\sigma^2 y^2 u_{yy} \\ &\quad -byu_y - rxu_x - \max_{c \geqslant 0}\{-cu_x + U(c)\} \end{aligned}$$

and

$$L_1(u_x, u_y) = (1+\lambda)u_x - u_y, L_2(u_x, u_y) = -(1-\mu)u_x + u_y.$$

To simplify, we restrict ourselves to the rectangular domain $D \subset \mathbb{R}^2$

$$\mathcal{D} = [0, (M-1)\Delta x] \times [0, (L-1)\Delta y]$$

of $\mathbb{R}^2$, where M and L denote the number of grid points on the x and y axis and $\Delta x, \Delta y > 0$ are the mesh sizes. The value of our numerical approximation at the point $((i-1)\Delta x, (j-1)\Delta y)$, for $i = 1, \ldots, M$ and $j = 1, \ldots, L$, will be denoted by V_{ij}.

We then define the first-order differences

$$D_x^+ V_{ij} = \frac{V_{i+1,j} - V_{ij}}{\Delta x}, \quad D_y^+ V_{ij} = \frac{V_{i,j+1} - V_{ij}}{\Delta y},$$

and

$$D_x^- V_{ij} = \frac{V_{ij} - V_{i-1,j}}{\Delta x}, \quad D_y^- V_{ij} = \frac{V_{ij} - V_{i,j-1}}{\Delta y},$$

Inside the domain, the first-order operators L_1 and L_2 are approximated in a monotone way using the appropriate backward and forward finite differences

$$g_1(D_x^- V_{ij}, D_y^+ V_{ij}) = (1+\lambda) D_x^- V_{ij} - D_y^+ V_{ij} \tag{5.2}$$

$$g_2(D_x^+ V_{ij}, D_y^- V_{ij}) = -(1-\mu) D_x^+ V_{ij} + D_y^- V_{ij}. \tag{5.3}$$

Next, we consider the first-order operator L_0' obtained by eliminating the second-order term from L_0, i.e.

$$L_0'(x, y, u, u_x, u_y) = \beta u - byu_x - rxu_x - \max_{c \geqslant 0}\{-cu_x + U(c)\}.$$

The solution of the equation $L_0'(x, y, \bar{u}, \bar{u}_x, \bar{u}_y) = 0$ can be characterized (see, for example Lions (1983)) as the value function of the deterministic control problem

$$\bar{u}(x, y) = \max_{c \geqslant 0}\left\{\int_0^\infty e^{-\beta t} U(C_t)\, dt\right\} \tag{5.4}$$

where the state trajectories x_t and y_t solve

$$\begin{cases} dx_t = (rx_t - C_t)\, dt - (1+\lambda)\, dM_t + (1-\mu)\, dN_t \\ dy_t = by_t\, dt + dM_t - dN_t \\ x_0 = x, \quad y_0 = y. \end{cases}$$

We are going to construct a monotone scheme to approximate the value function $\bar{u}$. To this end, we apply the Dynamic Programming Principle to (5.4) (see more details, Alziary de Roquefort (1991), Capuzzo-Dolcetta (1983), Falcone (1985, 1987) and Rouy and Tourin (1992)), to get

$$\bar{u}(x, y) = \max_{c \geqslant 0}\left\{\int_0^T e^{-\beta t} U(C_t) dt + e^{-\beta T} \bar{u}(x_T, y_T)\right\}.$$

We choose $T = \Delta\tau$ arbitrarily small and assume that the control remains constant in the time interval $[0, T]$. The above equality then yields

$$\sup_{c \geqslant 0}\left\{ U(c) + \frac{\bar{u}(x + \Delta\tau(rx - c), y + \Delta\tau by) - \bar{u}(x, y)}{\Delta\tau} e^{-\beta\Delta\tau} \right.$$
$$\left. + \bar{u}(x, y)\frac{e^{-\beta\Delta\tau} - 1}{\Delta\tau}\right\} = 0.$$

In order to approximate the operator L_0', one has to find an explicit formulation for the following optimum:

$$\max\left\{ \sup_{0 \leqslant c \leqslant r(i-1)\Delta x} (U(c) - \beta V_{ij} + D_x^+ V_{ij}(r(i-1)\Delta x - c) \right.$$
$$+ D_y^+ V_{ij} b(j-1)\Delta y), \quad \sup_{0 \geqslant r(i-1)\Delta x} (U(c) - \beta V_{ij}$$
$$\left. + D_x^- V_{ij}(r(i-1)\Delta x - c) + D_y^+ V_{ij} b(j-1)\Delta y)\right\}.$$

Finally, inside the rectangular domain, a numerical approximation V of the solution $\bar{u}$ of the equation $L_0'(x, y, \bar{u}, \bar{u}_x, \bar{u}_y) = 0$ will satisfy, for all $1 < i < M$ and $1 < j < L$,

$$g(D_x^- V_{ij}, D_x^+ V_{ij}, D_y^+ V_{ij}) = 0.$$

Below and only to simplify the presentation, we restrict ourselves to the case of the H.A.R.A. utility functions. We note, however, that all our arguments can be easily extended to general utilities.

When U is a H.A.R.A. utility function, given by $U(c) = (1/\gamma)c^\gamma$ with $\gamma \in (0, 1)$, g takes the following form:

(i) if $(D_x^+ V_{ij})^{\frac{1}{\gamma-1}} < r(i-1)\Delta x$ and $(D_x^- V_{ij})^{\frac{1}{\gamma-1}} < r(i-1)\Delta x$ then

$$\begin{aligned} g(D_x^- V_{ij}, D_x^+ V_{ij}, D_y^+ V_{ij}) &= -\beta V_{ij} + \frac{1-\gamma}{\gamma}(D_x^+ V_{ij})^{\frac{\gamma}{\gamma-1}} \\ &\quad + r(i-1)\Delta x D_x^+ V_{ij} \\ &\quad + b(j-1)\Delta y D_y^+ V_{ij}, \end{aligned}$$

(ii) if $(D_x^+ V_{ij})^{\frac{1}{\gamma-1}} < r(i-1)\Delta x$ and $(D_x^- V_{ij})^{\frac{1}{\gamma-1}} > r(i-1)\Delta x$ then

$$\begin{aligned} g(D_x^- V_{ij}, D_x^+ V_{ij}, D_y^+ V_{ij}) &= -\beta V_{ij} + \frac{1-\gamma}{\gamma}(D_x^- V_{ij})^{\frac{\gamma}{\gamma-1}} \\ &\quad + r(i-1)\Delta x D_x^- V_{ij} \\ &\quad + b(j-1)\Delta y D_y^+ V_{ij}, \end{aligned}$$

(iii) if $(D_x^+ V_{ij})^{\frac{1}{\gamma-1}} > r(i-1)\Delta x$ and $(D_x^- V_{ij})^{\frac{1}{\gamma-1}} > r(i-1)\Delta x$ then

$$\begin{aligned} g(D_x^- V_{ij}, D_x^+ V_{ij}, D_y^+ V_{ij}) &= -\beta V_{ij} + \frac{1-\gamma}{\gamma}(D_x^+ V_{ij})^{\frac{\gamma}{\gamma-1}} \\ &\quad + r(i-1)\Delta x D_x^+ V_{ij} \\ &\quad + b(j-1)\Delta y D_y^+ V_{ij}, \end{aligned}$$

(iv) if $(D_x^+ V_{ij})^{\frac{1}{\gamma-1}} > r(i-1)\Delta x$ and $(D_x^- V_{ij})^{\frac{1}{\gamma-1}} < r(i-1)\Delta x$ then

$$\begin{aligned} g(D_x^- V_{ij}, D_x^+ V_{ij}, D_y^+ V_{ij}) &= -\beta V_{ij} + \frac{1-\gamma}{\gamma}(r(i-1)\Delta x)^{\frac{\gamma}{\gamma-1}} \\ &\quad + b(j-1)\Delta y D_y^+ V_{ij}. \end{aligned}$$

In addition to the above approximations one has to construct an approximation for the points located on the boundary of the discretized domain $\mathcal{D}$. We first consider the x and y-axis and we assume that the non-transaction region lies entirely in the first quadrant. Under this assumption, the x- and y-axis belong respectively to the *sell* and *buy* region and therefore, the value function solves $L_1(\bar{u}_x, \bar{u}_y) = 0$, on the x-axis and $L_2(\bar{u}_x, \bar{u}_y) = 0$, on the y-axis.

Let us remark that this is not always the case. Indeed, Shreve and Soner (preprint) recently examined the location of the free boundaries in the H.A.R.A. case and they showed that for a certain range of the market parameters the x and y-axis may be included in the non-transaction region. Although, in this case, the scheme we propose might work under appropriate modifications, we do not examine this case herein.

At the points located on the x-axis, V_{i1} satisfies

$$g_1(D_x^- V_{i1}, D_y^+ V_{i1}) = 0 \text{ for } 1 < i \leqslant M$$

where g_1 is given by (5.2). At the points located on the y-axis, the function V_{1j} satisfies

$$g_2(D_x^+ V_{1j}, D_y^- V_{1j}) = 0 \text{ for } 1 < j \leqslant L$$

where g_2 is given by (5.3).

One may notice that this method could not be applied to the real boundaries $x + (1 - \mu)y = 0$ $(y \geqslant 0)$ and $x + (1 + \lambda)y = 0$ $(y \leqslant 0)$. Indeed, the monotone approximation of L_2 requires the backward finite difference along the y-axis and the approximation of L_2 requires the backward finite difference on the x-axis and these values are not available at the boundaries of Ω due to the presence of the state constraint (1.4).

Next, at the point $i = 1$, $j = 1$, we impose the Dirichlet condition $V_{11} = 0$. Actually this value follows directly from the Variational Inequalities themselves evaluated at the origin.

Finally, we impose Neumann conditions at the points located on $x = (M-1)\Delta x$ and $y = (L - 1)\Delta y$. We have to assign given values to the normal derivatives but the results may vary strongly with the prescribed values, especially the location of the free boundaries.

First, it seems rather natural since the value function is unique only in the class of concave functions. Indeed, the following fact may happen: if we impose over-estimated conditions that allow the approximation to loose its concavity, the scheme may converge to another solution of the Variational Inequality instead of the value function.

Secondly, from the numerical experiments, it turns out that even if normal derivatives are set to *reasonable* values, the location of the free boundaries is still sensitive to the given values. Actually, it appears that the error is essentially concentrated near the boundary.

Such a phenomenon has been already noticed by Barles, Daher and Romano for the heat equation and the Black and Scholes formula (1991) and by Fitzpatrick and Fleming for an Investment-Consumption model in (1991). Finally, we impose some reasonable values on the boundary of a sufficiently large domain and compute the corresponding value function and the free boundaries. Then, we only take into account the results obtained inside the domain.

In the second half iteration, we solve the monodimensional heat equation using a Cranck Nickolson scheme. Such a scheme requires boundary conditions which are chosen as follows: on the x-axis, we impose Dirichlet conditions whose values are provided by the formula:

$$g_1(D_x^- V_{ij}, D_y^- V_{ij}) = 0.$$

At the points located on $y = (L-1)\Delta y$, we have already imposed Neumann conditions. Thus the second half-iteration consists of inverting a tridiagonal matrix.

Let us recall that we have to choose the time step in order that the scheme be monotone. At each step, we may add a sufficient condition for the monotonicity. Finally, at each step, we choose the greatest value among those which preserve the monotonicity of the scheme. Actually, it yields a time step which is not far from being constant but may evolve a little during the convergence.

Finally, we compute the approximation using the following algorithm:

Algorithm

1st step

- $V_{ij}^0 = (i-1)\Delta x + (j-1)\Delta y, 1 \leqslant i \leqslant M, 1 \leqslant j \leqslant L$
- C given

$(n+1)$st step

- V^n is given
- Construction of V_1:
$$V_{1,ij}^{n+1} = V_{ij}^n - \min(g_1, g_2) \times \Delta t, 1 < i < M, 1 < j < L.$$
- Construction of V on the boundaries $x = (M-1)\Delta x$ and $y = (L-1)\Delta y$:
$$V_{i1}^{n+1} = V_{i1}^n - g_1 \Delta t, \quad 1 < i \leqslant M,$$
$$V_{1j}^{n+1} = V_{1j}^n - g_2 \Delta t, \quad 1 < j \leqslant L,$$
$$V_{11}^{n+1} = 0.$$
- Construction of V_2.

First half-iteration

$$V_{2,ij}^{n+1/2} = V_{ij}^n + \Delta t g(D_x^- V_{ij}^n, D_x^+ V_{ij}^n, D_y^+ V_{ij}^n) \quad 1 < i < M, \ 1 < j < L.$$

Second half-iteration

$$\frac{V_{2,ij}^{n+1} - V_{2,ij}^{n+1/2}}{\Delta t} = \frac{1}{2}(\Delta y)^2 (j-1)^2 \left[\frac{1}{2}(V_{2,ij+1}^{n+1/2} + V_{2,ij-1}^{n+1/2} - 2V_{2,ij}^{n+1/2}) + \frac{1}{2}(V_{2,ij+1}^{n+1} + V_{2,ij-1}^{n+1} - 2V_{2,ij}^{n+1})\right],$$

given that $V_{2,iL}^{n+1} = V_{2,iL-1}^{n+1} + C$ for $1 < i < M, 1 < j < L$.

- Construction of V^{n+1} from V_1^{n+1} and V_2^{n+1}:
$$V_{ij}^{n+1} = \max(V_{1,ij}^{n+1}, V_{2,ij}^{n+1}) \quad 1 < i < M, \ 1 < j < L,$$
$$V_{iL}^{n+1} = V_{iL-1}^{n+1} + C \quad 1 < i \leqslant M,$$
$$V_{Mj}^{n+1} = V_{M-1j}^{n+1} + C \quad 1 < j \leqslant L.$$

- If $\sup_{ij} |V_{ij}^n - V_{ij}^{n+1}| < \epsilon$ then stop, where ϵ is a tolerance bound prescribed by the user.
- After the convergence is established, find the non-transaction region:

$$(NT) = \{((i-1)\Delta x, (j-1)\Delta y) \ \text{ where } \min\{L_0, L_1, L_2\} = L_0\}.$$

The algorithm has been implemented on a HP 735 workstation. In order to obtain satisfactory free boundaries, one has to let the algorithm converge for a few hours, the time depending highly on the choice of the utility function and the initial condition. To improve the efficiency of the algorithm, we tried second-order schemes for Hamilton-Jacobi equations based on ideas developed by Osher for problems of Conservation Laws (see Harten, Engquist, Osher and Chakravarthy (1986)). Moreover, we also plan to use adaptive mesh size techniques to make the algorithm more efficient. Finally, the precision near the origin has been really improved but the approximation of the free boundaries for large values is not yet satisfactory; actually, for some utility functions, the free boundaries almost desegregate. In the future, we plan to use ideas based on second-order filtered schemes whose convergence has been recently proved by Lions and Souganidis (forthcoming).

Numerical experiments

Next, we present numerical experiments corresponding to three different classes of utility functions

i) $U(c) = (1/\gamma)c^{\gamma}\ \gamma < 1$
ii) $U(c) = (1/\gamma 1)c^{\gamma 1} + (1/\gamma 2)c^{\gamma 2}$, with $0 < \gamma 1, \gamma 2 < 1$
iii) $U(c) = M - c^{-\alpha}\ \alpha > 0$.

Figures 1–3 show the computed non-transactional regions corresponding to the utility functions $U_1(c) = 2\sqrt{c}$, $U_2(c) = 3c^{1/3} + \frac{3}{2}c^{2/3}$ and $U_3(c) = 100 - (1/\sqrt{c})$.

In Figure 1 we also plot the location of the transaction regions obtained by the algorithm of Davis and Norman.

6. Conclusions

In this paper, we presented a class of numerical schemes for the value function and the optimal policies of an optimal investment and consumption model which was formulated as a singular stochastic control problem. Although, due to the presence of singular policies, the value function is not, in general, smooth, the convergence of the scheme is guaranteed by the uniqueness of viscosity solutions of the Hamilton-Jacobi-Bellman equation. The numerical scheme developed here can be applied to a number of control problems with singular policies as well as problems in which some state dynamics are governed by stochastic processes and some others by deterministic ones; the latter problems give rise to degenerate HJB equations which are, in general, very hard to solve.

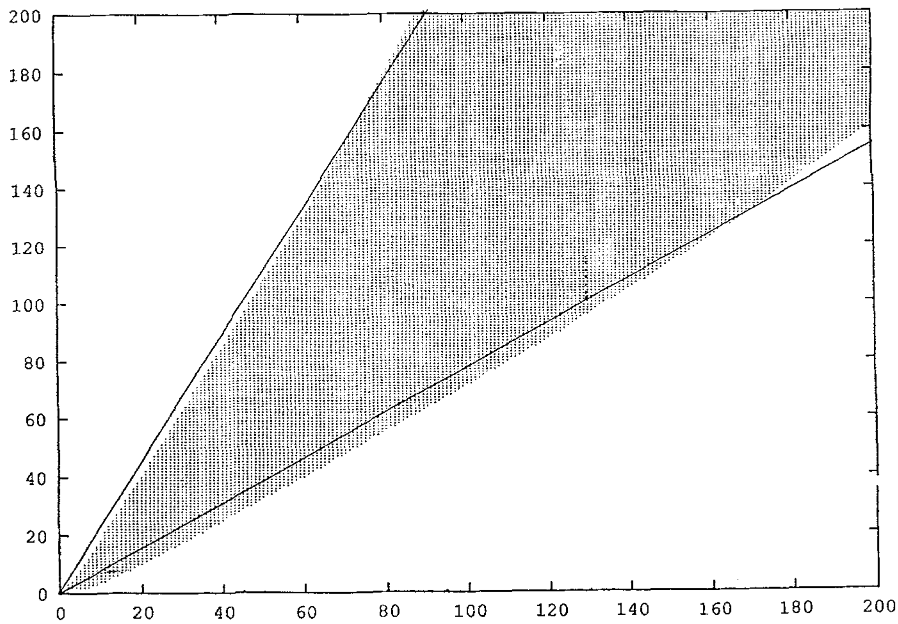

Fig. 1. $U(c) = 2\sqrt{c}$.

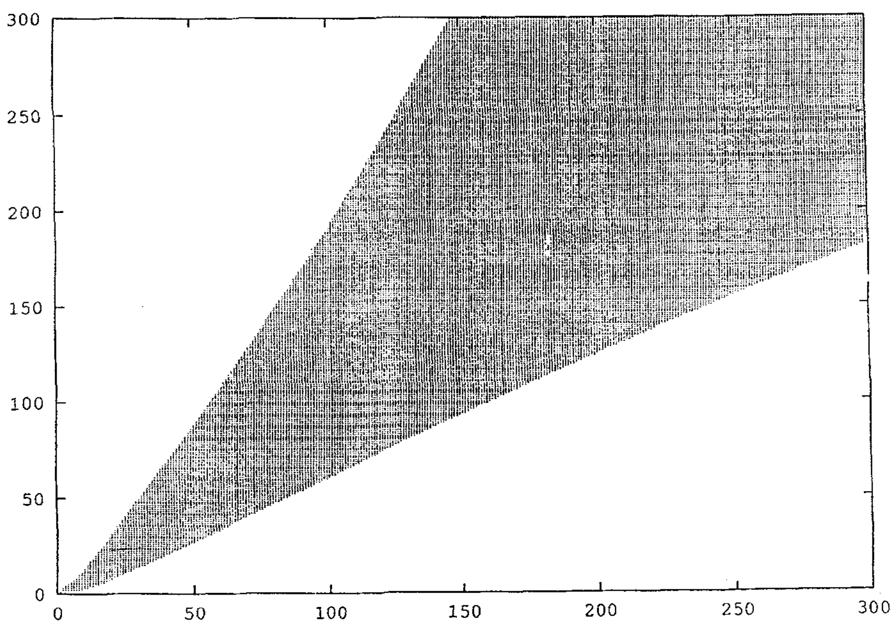

Fig. 2. $U(c) = 3c^{1/3} + \frac{3}{2}c^{2/3}$.

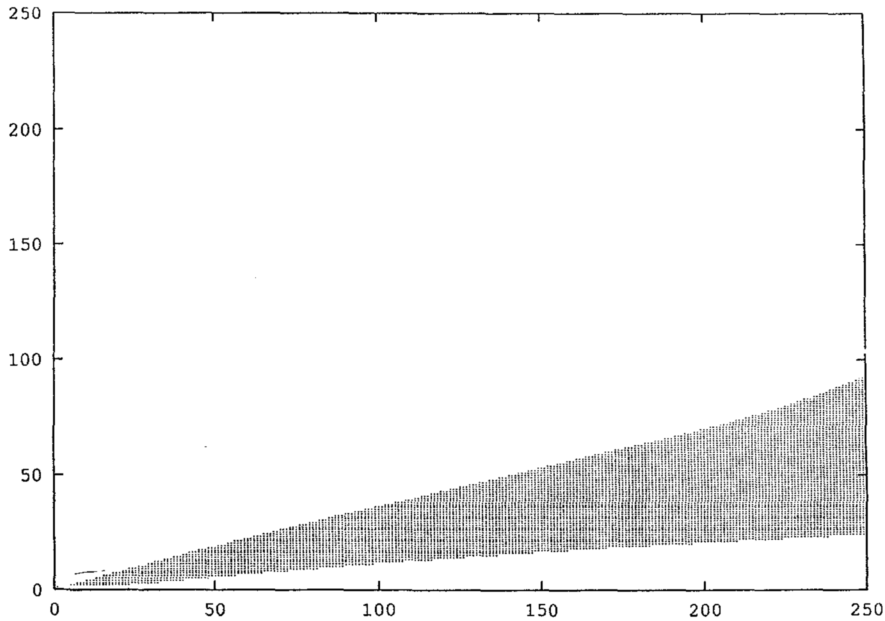

Fig. 3. $U(c) = 100 - 1/\sqrt{c}$.

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