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On the Structure of Compact Torsion Groups

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Abstract. It is shown that every profinite torsion group has a finite series of closed characteristic subgroups in which each factor either is a pro-p-group for some prime p or is isomorphic to a Cartesian product of isomorphic finite simple groups.

§ 1. Introduction

It has been known for some time that all compact Hausdorff torsion groups are profinite. A proof of this result may be found in [3], p. 69. Examples of infinite profinite torsion groups are provided by, for instance, Cartesian products of finite groups of bounded exponent, with the product topology. In [2], HERFORT has shown that if G is a profinite torsion group then the set $\pi(G)$ of primes p for which G has non-trivial Sylow p-subgroups is finite. Here we use this result, the results of HALL and HIGMAN [1] and the classification of the finite simple groups to establish a stronger statement:

Theorem 1. *Let G be a compact Hausdorff torsion group. Then G has a finite series*

$$1 = G_0 \leqslant G_1 \leqslant \ldots \leqslant G_n = G \qquad (*)$$

of closed characteristic subgroups, in which each factor G_i/G_{i-1} *either is* (i) *a pro-p-group for some prime p or is* (ii) *isomorphic (as a topological group) to a Cartesian product of isomorphic finite simple groups.*

Here, of course, by a *characteristic subgroup* we mean a subgroup invariant under all *continuous* automorphisms, and we shall always understand Cartesian products of finite groups to be equipped with the product topology.

Cartesian products of groups isomorphic to a fixed finite simple group S may be regarded as relatively well understood: they are locally finite and satisfy the laws of S, and in particular they have finite exponent. Theorem 1 therefore demonstrates rather clearly that a number of difficult open problems concerning profinite groups are essentially problems about pro-p-groups. Thus, for example, since the class of locally finite groups and the class of groups of finite exponent are both closed with respect to extensions, Theorem 1 implies the

Corollary. (i) *If all torsion pro-p-groups are locally finite, then so are all compact Hausdorff torsion groups.*

(ii) *If all torsion pro-p-groups have finite exponent, then so do all compact Hausdorff torsion groups.*

The assertion (ii) also follows from the theorem of HERFORT [2] mentioned above. We do not care to comment on the hypotheses in (i) and (ii), except to remark that all compact Hausdorff torsion groups of finite exponent are locally finite if and only if the restricted Burnside problem has a positive solution.

In [5], MCMULLEN studied the question of the existence of infinite compact Hausdorff groups having no infinite abelian subgroups. For brevity we write $\mathfrak{U}$ for the class of groups with these properties. Clearly $\mathfrak{U}$-groups are torsion-groups. MCMULLEN showed that the existence of a group G in $\mathfrak{U}$ would entail the existence of pro-p-groups in $\mathfrak{U}$. This, too, follows from Theorem 1, since it is very easy to check that the first infinite factor G_i/G_{i-1} in the series (*) for G given by Theorem 1 lies in $\mathfrak{U}$ and is a pro-p-group for some p. Indeed, using the profinite version of the Schur—Zassenhaus theorem and proceeding by induction on j, one can show that each infinite factor group G/G_j lies in $\mathfrak{U}$, so that each infinite factor in (*) lies in $\mathfrak{U}$ and is a pro-p-group for some p.

Structural information of the type given in Theorem 1 can be obtained under weaker conditions than periodicity. We shall deduce Theorem 1 from the following two results:

Theorem 2. *Let p be a prime and G a profinite group whose Sylow p-subgroups are torsion groups. Then G has a finite series of closed characteristic subgroups in which each factor either is pro-(p-soluble) or is isomorphic to a Cartesian product of non-abelian finite simple groups of order divisible by p.*

Theorem 3. *Let p be an odd prime and G a pro-(p-soluble) group whose Sylow p-subgroups are torsion-groups. Then G has a finite series of closed characteristic subgroups in which each factor is either a pro-p-group or a pro-p'-group.*

The deduction of Theorem 1 from these results is so straightforward that we can give it immediately. Let G be a compact Hausdorff torsion group. Then G is profinite, and by Theorem 1 of HERFORT [2], $\pi(G)$ is finite. Arguing by induction on the number of primes in $\pi(G)$ we obtain from Theorems 2 and 3 a finite series of closed characteristic subgroups of G in which each factor either is a pro-p-group for some prime p or is isomorphic to a Cartesian product of finite simple groups. Clearly it suffices to show that each factor C of the latter type accords with Theorem 1. A Cartesian product of a family $(S_\lambda; \lambda \in \Lambda)$ of finite groups can only be a torsion group if the finite groups S_λ have bounded exponent. However the classification of the finite simple groups has as a consequence that there are only finitely many isomorphism classes of finite simple groups of any given exponent. It therefore follows that C can be written in the form

$$C = C_1 \times \ldots \times C_r,$$

where each group C_i is a Cartesian product of isomorphic simple groups and is characteristic in C. The series

$$1 \leqslant C_1 \leqslant C_1 \times C_2 \leqslant \ldots \leqslant C_1 \times \ldots \times C_r = C$$

consists of closed characteristic subgroups of C and has factors isomorphic to the groups C_i, and thus Theorem 1 follows.

The role played in Theorem 1 by the classification of the finite simple groups deserves some comment. The consequence of the classification which we have just used — that there are only finitely many isomorphism classes of finite simple groups of any given exponent — cannot be avoided because it is in fact implied by Theorem 1. This is because any Cartesian product of finite groups of bounded exponent is a profinite torsion group. It seems possible that Theorem 1 could be proved using only this consequence of the classification; certainly it can be proved using only the slightly stronger statement that for each finite set π of primes there are only finitely many isomorphism classes of finite simple π-groups. On the other hand, it seems unlikely that one could prove a satisfactory weaker version of Theorem 1 without some appeal to the classification theorem. In our proof of Theorem 2 we have made use of another well known consequence of the classification theorem, the validity of the Schreier conjecture: that $\mathrm{Aut}\, S/\mathrm{Inn}\, S$ is soluble for every finite simple group S.

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§ 2. Reductions

We shall prove three results of an elementary nature and use them to relate Theorems 2 and 3 to results concerning finite groups. If X is a subset of a group G, we say that X has finite exponent if there is an integer $e > 0$ such that $x^e = 1$ for all $x \in X$; the least such integer e is called the exponent of X.

Lemma 1. (i) *If G is a profinite torsion group, then there is an open normal subgroup H of G one of whose cosets tH in G has finite exponent.*

(ii) *If G is a profinite group and for some prime p a Sylow p-subgroup P of G is a torsion group, then there is an open normal subgroup K of G such that some coset $t(K \cap P)$ of $K \cap P$ in P has finite exponent.*

Proof. (i) Write $X_e = \{g \in G; g^e = 1\}$ for each integer $e > 0$. Then each set X_e is closed and $G = \bigcup (X_e;\, e > 0)$. It follows from Baire's category theorem (see for example [4], p. 200) that some set X_e has non-empty interior and therefore contains a coset tH of the required type.

(ii) Applying (i) to the group P with the subgroup topology, we find an open subgroup H of P and an element $t \in P$ such that tH has finite exponent. Since H is open, there is an open subgroup K of G such that $K \cap P \leqslant H$; and clearly K can be chosen to be normal in G. The result follows.

By a *class* of groups we understand a class, in the usual sense, which contains all trivial groups and which is closed under isomorphic images; if $\mathfrak{X}$ is a class of finite groups, the statement that G is a pro-$\mathfrak{X}$-group means that G is a projective limit of $\mathfrak{X}$-groups.

Lemma 2. *Let $\mathfrak{X}_1, \ldots, \mathfrak{X}_n$ be classes of finite groups closed with respect to normal subgroups and subdirect products and let $\mathfrak{X}$ be the class of groups H having a series*

$$1 = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_n = H$$

with $H_i/H_{i-1} \in \mathfrak{X}_i$ for each i. Then every pro-$\mathfrak{X}$-group G has a series

$$1 = G_0 \leqslant G_1 \leqslant \ldots \leqslant G_n = G$$

of closed characteristic subgroups such that G_i/G_{i-1} is a pro-$\mathfrak{X}_i$-group for each i.

The *proof* of this by induction on n is straightforward and may be omitted.

Lemma 3. *Let $\mathfrak{Y}$ be a class of groups consisting of non-abelian finite simple groups, and let $\mathfrak{X}$ be the class of finite direct products of $\mathfrak{Y}$-groups. Then a profinite group G is a pro-$\mathfrak{X}$-group if and only if it is isomorphic (as a topological group) to a Cartesian product of $\mathfrak{Y}$-groups.*

Proof. If G is a Cartesian product of $\mathfrak{Y}$-groups, then G is the projective limit of the groups G/K with K running through all intersections of finitely many kernels of projection maps; each such G/K is clearly in $\mathfrak{X}$ and so G is a pro-$\mathfrak{X}$-group.

Conversely, suppose that G is a pro-$\mathfrak{X}$-group, and let $\mathscr{M}$ be the family of open maximal normal subgroups of G. If K is an open normal subgroup with $G/K \in \mathfrak{X}$ then K must be an intersection of subgroups in $\mathscr{M}$; thus since G is a pro-$\mathfrak{X}$-group we have

$$\bigcap (M; M \in \mathscr{M}) = 1,$$

so that the map

$$\varphi: G \to \mathrm{Cr}\,(G/M;\; M \in \mathscr{M})$$

is an injection. Since each M is open, φ is an injection of compact groups; thus it will suffice to show that φ is surjective. If $M \in \mathscr{M}$ and if K_1, K_2 are normal subgroups of G such that $G = MK_1 = MK_2$, then we have $G = M(K_1 \cap K_2)$: for otherwise, $K_1 \cap K_2 \leqslant M$ and the derived group G' of G satisfies

$$G' = [MK_1, MK_2] \leqslant M[K_1, K_2] \leqslant M(K_1 \cap K_2) = M,$$

contradicting the fact that G/M is non-abelian. Using this and an easy induction we see that, if $M_1, \ldots, M_n$ are distinct elements of $\mathscr{M}$, then $G = M_i(\bigcap_{j \neq i} M_j)$ for $i = 1, \ldots, n$. It follows that the image of the map

$$G \to G/M_1 \times \ldots \times G/M_n$$

contains the canonical image in the direct product of each group G/M_i, so that this map is surjective. In other words, given $M_i g_i \in G/M_i$ for $i = 1, \ldots, n$ there is an element $g \in G$ such that $M_i g_i = M_i g$ for each i. Since this holds for all finite subfamilies $(M_1, \ldots, M_n)$ of $\mathcal{M}$, we conclude that $G\varphi$ is dense in $\mathrm{Cr}(G/M; M \in \mathcal{M})$. However, being compact, it is also closed; and therefore φ is surjective, as required.

In § 3 we shall prove the following two results:

Theorem 2*. *Let p be a prime and K a normal subgroup of a finite group H. If a Sylow p-subgroup Q of K has a coset tQ in H of exponent dividing* p^a, *then K has a series*

$$1 = K_0 \leqslant K_1 \leqslant \ldots \leqslant K_{2a+1} = K \qquad (*)$$

such that K_i/K_{i-1} *is p-soluble for i odd and is a direct product of non-abelian simple groups of order divisible by p for i even.*

Theorem 3*. *Let p be an odd prime and K a normal p-soluble subgroup of a finite group H. If a Sylow p-subgroup Q of K has a coset tQ in H of exponent dividing* p^a, *then K has p-length at most* $2a$.

If in Theorem 3* we take $tQ = Q$, the assertion becomes the familiar relationship between the p-length of K and the exponent of its Sylow subgroup furnished by Theorem A of HALL and HIGMAN [1]; and examples in [1] show that the bound in Theorem 3* is best possible. Examples to show that the bound implicit in Theorem 2* is best possible are provided by wreath powers of suitably chosen groups $\mathrm{GL}(2, q)$.

Using the three lemmas above, one can deduce Theorems 2 and 3 easily from Theorems 2* and 3*. We indicate only the proof that Theorem 2* implies Theorem 2; the proof that Theorem 3* implies Theorem 3 is similar and may be omitted.

Let G be as in Theorem 2, and write $\mathfrak{S}_p$ for the class of non-abelian finite simple groups of order divisible by p. By Lemma 1 there is an open normal subgroup K of G whose Sylow p-subgroup has a coset in G of finite p-power exponent, p^a say. Each finite continuous image H of G satisfies the hypotheses of Theorem 2*, and so the image of K in each such H has a series of the form (*). Therefore each H has a series

$$1 = H_0 \leqslant H_1 \leqslant \ldots \leqslant H_n = H,$$

with $n = 2a + 1 + |G/K|$, such that H_i/H_{i-1} is p-soluble for i odd and is a direct product of $\mathfrak{S}_p$-groups for i even. On applying Lemma 2, we find in G a series of closed characteristic subgroups

$$1 = G_0 \leqslant G_1 \leqslant \ldots \leqslant G_n = G$$

such that G_i/G_{i-1} is pro-(p-soluble) for i odd and is a projective limit of direct products of $\mathfrak{S}_p$-groups for i even; the factors of the latter type are isomorphic to Cartesian products of $\mathfrak{S}_p$-groups by Lemma 3.

§ 3. Finite Groups

It now remains to prove Theorems 2* and 3*. The bulk of the proof of Theorem 2* is contained in the following lemma. It is the proof of assertion (ii) that we appeal to the validity of the Schreier conjecture. Again we write $\mathfrak{S}_p$ for the class of all non-abelian finite simple groups of order divisible by the prime p.

Lemma 4. *Let p be a prime and K a normal subgroup of a finite group H, and suppose that a coset tQ in H of a Sylow p-subgroup Q of K has exponent dividing p^a, where $a \geqslant 1$. Write*

$$A = \langle u^{p^{a-1}}; u \in tQ \rangle$$

and write N for the normal closure of A in H.

(i) *Let L/M be a perfect chief factor of H such that $L \leqslant K$ and such that p divides $|L/M|$. (Thus L/M is a direct product of $\mathfrak{S}_p$-groups.) Then N normalizes the simple direct factors of L/M.*

(ii) *There are normal subgroups K_1, K_2 of $K \cap N$ with $K_1 \leqslant K_2$ such that K_1 is p-soluble, K_2/K_1 is a direct product of $\mathfrak{S}_p$-groups and $(K \cap N)/K_2$ is soluble.*

Proof. (i) Passing to a homomorphic image of H if necessary, we may clearly assume that $M = 1$. Thus L is now a direct product $S_1 \times \ldots \times S_d$ of $\mathfrak{S}_p$-groups which are permuted by conjugation by H, and for each i, $Q \cap S_i$ is a Sylow p-subgroup of S_i so is non-trivial. We choose an element $u \in tQ$ and one of the groups S_i. If $u^{p^{a-1}}$ does not normalize S_i, then $\langle u \rangle$ permutes the conjugates of S_i under $\langle u \rangle$ faithfully, so that the groups $S_i^{u^j}$ for $0 \leqslant j < p^a$ are distinct and generate their direct product. On the other hand, if $h \in (Q \cap S_i) \setminus 1$, then we have $uh \in tQ$, and so $1 = u^{p^a} = (uh)^{p^a}$. Therefore

$$h^{u^{p^a-1}} \ldots h^u h = 1\,,$$

and since

$$h^{u^j} \in S_i^{u^j} \quad \text{for} \quad 0 \leqslant j < p^a$$

this is evidently a contradiction. It follows that the intersection of the normalizers of all subgroups S_i contains $u^{p^{a-1}}$ for each $u \in tQ$, and thus, since this intersection is normal in H, it also contains N.

(ii) Let $\mathscr{C}$ be a maximal series of H-invariant subgroups of $K \cap N$ and let $\mathscr{C}_1$ be a maximal series of N-invariant subgroups of $K \cap N$ containing $\mathscr{C}$. By (i) N normalizes the simple direct factors of the perfect factors in $\mathscr{C}$ of order divisible by p, and so each factor in $\mathscr{C}_1$ is either p-soluble or an $\mathfrak{S}_p$-group. Let $C_1, \ldots, C_r$ be the centralizers in $K \cap N$ of the factors in $\mathscr{C}_1$ which are $\mathfrak{S}_p$-groups, and set $K_1 = C_1 \cap \ldots \cap C_r$. Intersecting $\mathscr{C}_1$ with the subgroup K_1, we see that K_1 is p-soluble. Let K_2/K_1 be the smallest normal subgroup of $(K \cap N)/K_1$ such that $(K \cap N)/K_2$ is soluble. Each group $(K \cap N)/C_i$ is isomorphic to a group of automorphisms of an $\mathfrak{S}_p$-group containing the inner automorphisms, and so, since Schreier's conjecture holds, $K_2 C_i/C_i \in \mathfrak{S}_p$ for each i. Thus each group $K_2/(C_i \cap K_2)$ lies in $\mathfrak{S}_p$, and $K_2/K_1 = K_2/\bigcap (C_i \cap K_2)$ is a direct product of $\mathfrak{S}_p$-groups, as required.

*Proof of Theorem 2**. If $a = 0$ then $Q = 1$ so that K is a p'-group and the result holds. We may therefore assume $a > 0$ and argue by induction. Define the subgroup N as in Lemma 4. The image in H/N of Q is a Sylow p-subgroup of KN/N, and the image of the coset tQ has exponent dividing p^{a-1}. Induction therefore yields series of length $2a - 1$ in KN/N and in the isomorphic image $K/(K \cap N)$ of KN/N; let the series in $K/(K \cap N)$ be

$$(K \cap N)/(K \cap N) \leqslant K_3/(K \cap N) \leqslant \ldots \leqslant K_{2a}/(K \cap N) \leqslant$$
$$\leqslant K_{2a+1}/(K \cap N) = K/(K \cap N).$$

If K_1 and K_2 are the subgroups of $K \cap N$ given by Lemma 4 (ii), then the series

$$1 \leqslant K_1 \leqslant \ldots \leqslant K_{2a+1} = K$$

for K clearly has the desired properties.

For the proof of Theorem 3* we need the following result.

Lemma 5. *Let p be an odd prime and K a p-soluble normal subgroup of a finite group H, and suppose that a coset tQ in H of a Sylow p-subgroup Q of K has exponent dividing p^a, where $a \geqslant 1$. Write $A = \langle u^{p^{a-1}}; u \in tQ\rangle$, and write N, N_1 respectively for the normal closures in H of A and the derived group A' of A. Then*

(i) *N_1 acts trivially on each chief p-factor L/M of H satisfying $L \leqslant K$;*

(ii) *N acts trivially on each chief p-factor L/M of H satisfying* $L \leqslant K$ *and* $N_1 \cap K \leqslant M$; *and*

(iii) $N \cap K$ *has p-length at most two.*

Proof. Let L/M be chief p-factor satisfying $L \leqslant K$, and write $C = C_H(L/M)$, so that H/C acts faithfully by conjugation on L/M. Since it also acts irreducibly it has no non-trivial normal p-subgroups, and so Theorems 2.1.1 and 2.1.2 of Hall and Higman [1] may be applied. Let $u \in tQ$. If $h \in Q$ then $uh \in tQ$, and so $u^{p^a} = (uh)^{p^a} = 1$, and

$$h^{u^{(pq-1)}} \dots h^{u^2} h^u h = 1 ,$$

where $q = p^{a-1}$. Because Q is a Sylow p-subgroup of K we have $L/M \leqslant Q\,M/M$ and therefore $L = M(Q \cap L)$, so that every element of L is congruent to an element of $Q \cap L$ modulo M. Thus, in additive notation, the automorphism of L/M induced by u satisfies the polynomial equation

$$x^{pq-1} + \dots + x^2 + x + 1 = 0 .$$

Thus either $u^q = u^{p^{a-1}}$ acts trivially on L/M, or the automorphism induced by u is exceptional in the sense of [1], p. 10. Since this applies for each $u \in tQ$, it follows from Theorem 2.1.2 of [1] that A induces an elementary abelian group of automorphisms in L/M. We conclude that both A' and its normal closure N_1 act trivially on L/M, and (i) holds.

Now suppose that $N_1 \cap K \leqslant M$. Let $u \in tQ$ and $h \in Q \cap L$, and write

$$y = h^{q-1} \dots h .$$

Then $[(uh)^q, u^q] \in A' \leqslant N_1$. On the other hand

$$[(uh)^q, u^q] = [u^q y, u^q] = [y, u^q]$$

and this lies in K since $y \in L \leqslant K$. Thus

$$[(uh)^q, u^q] \in N_1 \cap K \leqslant M .$$

Transferring to additive notation and recalling that $L = M(Q \cap L)$, we find that the automorphism induced in L/M by each element $u \in tQ$ satisfies

$$(x^{q-1} + \dots + x + 1)(x^q - 1) = 0 ,$$

and so satisfies $(x-1)^{2q-1} = 0$. If on the other hand u induces an automorphism of order p^a, then by the Corollary to Theorem 2.1.1 of

[1] its minimal polynomial has the form

$$(x-1)^k = 0,$$

where $k \geqslant (p-1)p^{a-1}$. Since

$$(p-1)p^{a-1} = (p-1)q > 2q-1,$$

this is impossible. Therefore u^q acts trivially on L/M. Since this holds for each $u \in tQ$, we conclude that both A and N act trivially on L/M, and (ii) holds.

Finally, since by (i) N_1 acts trivially by conjugation on each chief p-factor L/M of G with $L \leqslant N_1 \cap K$, the group $N_1 \cap K$ is p-nilpotent. Similarly (ii) implies that $(N \cap K)/(N_1 \cap K)$ is p-nilpotent. It therefore follows that $N \cap K$ has p-length at most two, and the proof of Lemma 5 is complete.

*Proof of Theorem 3**. If $a = 0$ then K is a p'-group and the result holds. If $a > 0$ then we define N as in Lemma 5. Induction applied to the group H/N shows that KN/N has p-length at most $2(a-1)$. Therefore $K/(K \cap N)$ has p-length at most $2(a-1)$. By Lemma 5, $K \cap N$ has p-length at most two, and Theorem 3* follows.

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