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Prime Ideals in the C^*-algebra of a Nilpotent Group

By

Jean Ludwig, Bielefeld

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Abstract. We say that a locally compact group G has T_1 primitive ideal space if the group C^*-algebra, $C^*(G)$, has the property that every primitive ideal (i.e. kernel of an irreducible representation) is closed in the hull-kernel topology on the space of primitive ideals of $C^*(G)$, denoted by Prim G. This means of course that every primitive ideal in $C^*(G)$ is maximal. Long ago Dixmier proved that every connected nilpotent Lie group has T_1 primitive ideal space. More recently Poguntke showed that discrete nilpotent groups have T_1 primitive ideal space and a few month ago Carey and Moran proved the same property for second countable locally compact groups having a compactly generated open normal subgroup. In this note we combine the methods used in [3] with some ideas in [9] and show that for nilpotent locally compact groups G, having a compactly generated open normal subgroup, closed prime ideals in $C^*(G)$ are always maximal which implies of course that Prim G is T_1.

Introduction

We note first the following structure theorem for compactly generated nilpotent locally compact groups ([5]). Let G be a locally compact nilpotent compactly generated group. In G there exists a characteristic maximal compact subgroup K consisting of all elements whose powers form a relatively compact set. Furthermore the quotient group N/K is a Lie group, i.e., the connected component $(N/K)_0$ of (N/K) is open in N/K and $(N/K)_0$ is a connected nilpotent Lie group.

We need some

Definitions: Let H be a locally compact group, let W be a closed normal subgroup of H. Let I be a twosided closed H-invariant ideal in $C^*(W)$. We say that I is *H-maximal* if I is maximal among the set of all H-invariant closed twosided ideals in $C^*(W)$. A closed twosided H-invariant ideal I in $C^*(W)$ is said to be *H-prime* if for any two H-invariant twosided ideals I_1 and I_2 in $C^*(H)$ with $I_1 \cdot I_2 \subset I$ either $I_1 \subset I$ or $I_2 \subset I$. A closed H-invariant subset A in Prim W is said to be *H-irreducible* if for any two closed H-invariant subsets $A_1, A_2 \subset$ Prim W with $A \subset A_1 \cup A_2$ we have either $A \subset A_1$ or $A \subset A_2$.

Of course an ideal in $C^*(W)$ is H-prime if and only if its hull $h(I) = \{J \in \operatorname{Prim} W \mid J \supset I\}$ is H-irreducible. If $A \subset \operatorname{Prim} W$ is H-irreducible then necessarily any open H-invariant subset in A must be dense.

(1) If the group W is second countable, then every closed subset in Prim W is a Baire space (see [4]). Thus every H-irreducible subset A is the closure in Prim W of an H-orbit, i.e., if I is H-prime, then $I = \ker H \cdot \varrho := \bigcap_{h \in H} \ker h \cdot \varrho$ (for some $\varrho \in \hat{W}$).

This can be seen in the following way: Let $\{U_n\}_{n \in \mathbb{N}}$ be a basis of the topology of A. For every n, let $\tilde{U}_n := H \cdot U_n := \bigcup_{h \in H} h \cdot U_n$. Then $\{\bigcap \tilde{U}_n \mid n \in \mathbb{N}, U_n \neq \emptyset\} \neq \emptyset$, because otherwise as A is a Baire-space, the complement $A_{n_0} := (\tilde{U}_{n_0})^0$ of one of the $\tilde{U}_n's \neq \emptyset$ must contain an open set $\neq \emptyset$, whence for some $n_0' \in \mathbb{N}$, $\tilde{U}_{n_0} \cap \tilde{U}_{n_0'} = \emptyset$ i.e. $A_{n_0} \cup A_{n_0'} = A$ $(A_{n_0} \neq A,\ A_{n_0'} \neq A)$ whence, as A is H-irreducible, $A = A_{n_0}$ or $A = A_{n_0'}$ (a contradiction). If we choose now $\varrho \in \{\bigcap_n \tilde{U}_n \mid \tilde{U}_n \neq \emptyset\}$, then $H \cdot \varrho$ is dense in A. If an H-prime ideal I is of the form $I = \ker H \cdot \varrho$, ($\varrho \in \hat{W}$), we say that I is an H-orbit. It is clear that an H-orbit is always H-prime. Let us recall that a locally compact group G is $*$-regular (see [2]), if the space Prim G is homeomorphic with the space $\operatorname{Prim}^*(L^1(G))$ of the kernels of irreducible unitary representations of $L^1(G)$. Especially for any closed subset A of $\hat{G}$, the closure in $C^*(G)$ of $\ker A < L^1(G)$ is equal to $\ker A$ in $C^*(G)$. Any nilpotent group is $*$-regular. (see [2].)

Let now I denote a twosided closed ideal in $C^*(H)$. The *restriction* $r_W(I)$ of I to W is the twosided closed ideal in $C^*(W)$ defined by:

$$r_W(I) := \{\bigcap \ker(\varrho_{|W}) \mid \varrho \in \hat{H}; \varrho(I) = 0\} .$$

$r_W(I)$ is of course H-invariant. If J is a closed twosided H-invariant ideal in $C^*(W)$, define the *extension* $e_H(J)$ of J to be the (closed twosided) ideal in $C^*(H)$

$$e_H(J) = \{\bigcap_{\varrho} \ker \varrho \mid \varrho \in \hat{H}, \varrho|_W(J) = 0\} .$$

It is easy to see that for every twosided closed ideal I in $C^*(H)$

$$e_H(r_W(I)) \subset I . \tag{2}$$

If H is nilpotent, hence $*$-regular, we have for every H-invariant closed twosided ideal in $C^*(W)$:

$$r_W(e_H(J)) = J \tag{3}$$

see ([6]).

Let us observe that an H-prime ideal I in $C^*(W)$ is H-maximal if and only if its hull $h(I)$ in Prim (W) is H-minimal i.e. for any $\tau \in h(I)$, $H \cdot \tau$ is dense in $h(I)$.

Let G be as in the theorem. Let Max $(L^1(G))$ denote the set of all closed, maximal, twosided ideals in $L^1(G)$, let Prim $(L^1(G))$ denote the set of all the kernels of irreducible $*$-representations. It follows from the results of this paper, from the existence of minimal dense ideals in $L^1(G)$, from the fact that $L^1(G)$ is symmetric and that G has the Wiener property, that

$$\operatorname{Max}(L^1(G)) = \operatorname{Prim}^*(L^1(G)) = \operatorname{Prim}(L^1(G)) .$$

(See the considerations in the last part of [9] and [7].) The following (probably) wellknown lemma will be used several times.

1. Lemma: *Let H be a nilpotent locally compact group. Let $V \subset W$ be closed, normal subgroups of H. If the ideal I in $C^*(W)$ is H-prime then $J = r_V(I)$ is H-prime in $C^*(V)$.*

Proof: Let J_1, J_2 be two H-invariant twosided ideals in $C^*(V)$ such that $J_1 \cdot J_2 \subset J$. If $I_i = e_W(J_i)$, $i = 1, 2$, then by (3) $r_V(I_i) = J_i$; $i = 1, 2$ and $I_1 \cdot I_2 \subset e_W(J) \subset I$. Hence if I is H-prime $I_1 \subset I$ or $I_2 \subset I$ and so $J_1 = r_V(I_1) \subset r_V(I) = J$ or $J_2 = r_V(I_2) \subset J$. q.e.d.

2. Lemma: *Let G be a nilpotent locally compact group, K a compact normal subgroup of G. Assume that the connected component $(G/K)_0$ of (G/K) is a Lie group. Let G_0 be the pullback of $(G/K)_0$ to G. Then the hull $h(I)$ in* Prim (G_0) *of a G-prime ideal I in $C^*(G_0)$ is the closure of a G-orbit, i.e., I is a G-orbit.*

Proof: By 1. Lemma $r_K(I)$ is G-prime in $C^*(K)$ and so, as Prim K is discrete, $r_K(I) = \ker G \cdot \tau$ for some $\tau \in \hat{K}$. Let $M = M_\tau$ be the stabilizer of τ in G. Then, as G_0/K is connected, $G_0 \subset M$. Let $A = h(J) \subset$ $\subset$ Prim G_0, where: $J = r_{G_0}(I)$. A is G-invariant and A is the union of the subsets $A_{g \cdot \tau}$ $(g \in G)$ in Prim G_0, where $A_{g \cdot \tau} = \{I' \in \operatorname{Prim} G_0 \mid r_K(I') =$ $= \ker g \cdot \tau \lhd C^*(K)\}$. As K is compact, $A_{g \cdot \tau}$ is open and closed in Prim G_0 for every g. Hence if U is any open M-invariant subset in A_τ, $\tilde{U} = \bigcup_{g \in G} g \cdot U$ is open in A and G-invariant. Thus $\tilde{U}$ is dense in A, as A is G-irreducible. But then U is dense in A_τ, and so A_τ is M-irreducible.

Let now $K^0 = \ker \tau \lhd K$. K^0 is M-invariant and we may as well assume that $K^0 = (e)$. Then K is a compact Lie group. Furthermore, as G_0/K and K are Lie groups, G_0 is also Lie (see [8]). This implies that the connected component $(G_0)_0$ of G_0 is of finite index in G_0. Thus G_0 is separable and $A_\tau \subset \operatorname{Prim} G_0$ must be the closure of an M-orbit (by (1)). So finally A itself is the closure of a G-orbit. q.e.d.

3. Lemma: *Let H be a nilpotent locally compact group, W a closed normal connected subgroup, which is a Lie group. Then for any $\pi \in \hat{W}$, $\ker H \cdot \pi$ is H-maximal.*

Proof: The group H acts by the adjoint representation Ad on the Lie algebra w of the Lie group W, hence also by the coadjoint representation Ad* on the dual space w^* of w. In order to show that $\ker H \cdot \pi$ is H-maximal, it is enough to show that for the Kirillov orbit O of π in w^*, $H \cdot O = H \cdot \ell \; (\ell \in O)$ is H-maximal. By ABELS theorem (see [1]), it is enough to prove that for any $h \in H$, the eigenvalues of $\mathrm{Ad}^*(h)$ are of modulus one. But H is nilpotent, so there exists $m \in \mathbb{N}$, such
that for every $h \in H$, $x \in w$, $t \in \mathbb{R}$ $\underbrace{[h,[h,[h,\ldots,[h,\exp tx]\ldots\ldots]}_{m\text{-times}} = e$.
Differentiating this equation in t and evaluating for $t = 0$ we get $(\mathrm{Ad}(h) - 1)^m(x) = 0$ which means that $\mathrm{Ad}(h)$, hence also $\mathrm{Ad}^*(h)$, is unipotent.

4. Lemma: *Let G be a locally compact nilpotent group, let $N \subset W$ be closed normal subgroups. Assume that N is open in W and that W/N is central in G/N. Let $I \subset C^*(W)$ be a G-orbit and let J be a closed G-invariant twosided ideal in $C^*(W)$. If $r_N(I) \supset r_N(J)$, then there exists $\chi \in (W/N)\hat{}$ such that $I \supset \chi \cdot J$.*

Proof: As $r_N(I) \supset r_N(J)$, $e_W(r_N(I)) \supset e_W(r_N(J))$. W/N is a discrete abelian group, hence the dual group $X := (W/N)\hat{}$ is compact. Thus $X \cdot h(J)$ is closed in $\operatorname{Prim}(W)$ and it is easy to see that

$$h(e_W(r_N(J))) = X \cdot h(J)\,. \tag{4}$$

Now $I = \ker G \cdot \varrho$ for some $\varrho \in \hat{W}$, so $(\varrho \mid N)(r_N(J)) = 0$ and thus $\varrho \in h(e_W(r_N(J)) = 0$. Hence $\ker \varrho = \chi \cdot \ker \tau = \ker(\chi \otimes \tau)$ for some $\tau \in h(J)$ in $\hat{W}$ (by (4)). But then the fact that W/N is central in G/N implies that

$$g \cdot \ker \varrho = g \cdot (\chi \ker \tau) = \chi \cdot \ker(g\tau)$$

for every $g \in G$. Finally:

$$I = \ker(G \cdot \varrho) = \bigcap_{g \in G} g \cdot \ker \varrho = \chi \cdot (\bigcap_{g \in G} \ker g \cdot \tau) \supset \chi \cdot J .$$

q.e.d.

5. Lemma: *Let G be a locally compact nilpotent group, let $N \subset W$ be closed normal subgroups of G. Assume that N is open in W and that W/N is central in G/N. If every G-prime ideal in $C^*(N)$ is G-maximal, then every G-orbit in $C^*(W)$ is contained in a maximal G-orbit.*

Proof: We want to apply Zorn's lemma to the set $\tilde{J}$ of all G-orbits containing the G-orbit I ordered by inclusion. Let $\{I_\lambda\}_{\lambda \in \Lambda}$ be a chain in $\tilde{J}$. Let $J_\lambda := r_N(I_\lambda)$; $\lambda \in \Lambda$. As J_λ is G-prime for every λ (by 1. Lemma), and as $J_\lambda \supset J_{\lambda'}$, ιφ $\lambda > \lambda'$, by hypothesis we have:

$$J_\lambda = J_{\lambda_0} \text{ for every } \lambda \text{ for some } \lambda_0 .$$

So for every λ there exists $\chi_\lambda \in X = (W/N)\hat{}$ with

$$I_\lambda \subset \chi_\lambda \cdot I_{\lambda_0} \quad \text{(by 4. Lemma)} .$$

Passing to a subnet $\{\lambda_\mu\}_{\mu \in M}$ we may assume that $\{\chi_{\lambda_\mu}\}_{\mu \in M}$ converges in X to χ_0 (as X is compact).

So $\bigcup_{\lambda \in \Lambda} I_\lambda \subset \bigcup_{\mu \in M} I_{\lambda_\mu} \subset \chi_0 I_{\lambda_0}$ and $\{J_\lambda\}_\lambda$ has an upper bound. Applying Zorn's lemma we see that $\tilde{J}$ has a maximal element. q.e.d.

6. Lemma: *Let G be a locally compact nilpotent group, let $N \subset W$ be normal subgroups. Assume that N is open in W and that W/N is central in G/N. If every G-prime ideal in $C^*(N)$ is G-maximal, then so is every G-prime ideal in $C^*(W)$.*

Proof: We show first that every G-orbit $I = \ker G \cdot \varrho \, (\varrho \in \hat{W})$ is G-maximal. Let $J = \ker G \cdot \tau \, (\tau \in \hat{W})$ be a maximal G-orbit in $C^*(W)$ containing I (see 5. Lemma.) Then by hypothesis $r_N(J) \supset r_N(I)$ implies $r_N(J) = r_N(I)$ hence, by 4. Lemma, $\chi \cdot J \subset I$ for some $\chi \in (W/N)\hat{}$. But then $I = \chi \cdot J$ as J is maximal and so I is also maximal. Hence $I = J$. Let now I be G-prime in $C^*(W)$. Let $J_0' = \ker \varrho_0 \, (\varrho_0 \in \hat{W})$ be any element in $h(I)$. Then $C^*(W) \rhd J_0 = \ker G \cdot \varrho_0$ contains I and so $r_N(J_0) \supset r_N(I)$ implies again $r_N(I) = r_N(J_0)$. Hence for any $J' = \ker \varrho \; (\varrho \in \hat{W})$ in the hull $h(I)$ of I

$$r_N(J) = r_N(J_0) ,$$

where $J = \ker G \cdot \varrho \lhd C^*(W)$.

Thus $J = \chi_J \cdot J_0$ for some $\chi_J \in X := (W/N)\hat{}$ (by 4. Lemma and by the fact that J is G-maximal) and

$$I = \bigcap_{J' \in h(I)} J' = \bigcap_{\varrho \in \hat{W}, \varrho(I)=0} \ker(G \cdot \varrho) = \bigcap_{J} \chi_J \cdot J_0 .$$

Hence there exists a closed subset A of X so that

$$I = \ker A \cdot J_0 := \bigcap_{a \in A} a \cdot J_0 .$$

Let $X_0 = \{\chi \in X \mid \chi \cdot J_0 = J_0\}$. X_0 is a closed subgroup of X and we may assume that A is X_0-invariant, i.e., $A = X_0 \cdot A$. Let $\tilde{A} = A \bmod X_0$ in X/X_0. If $\tilde{A}$ contains more than one element there exist two closed X_0-invariant subsets A_1 and A_2 in A with $A_1 \not\subset A_2$, $A_2 \not\subset A_1$ and $A_1 \cup A_2 = A$. Let $I_i = \ker A_i J_0\,(i = 1, 2)$. Then $I_1 \cap I_2 = \ker A \cdot J_0 = I$ and $I_i\,(i = 1, 2)$ is G-invariant. Hence as I is G-prime $I \supset I_1$ or $I \supset I_2$. Assume that $I \supset I_1$, then $A_1 \cdot h(J_0) = A \cdot h(J_0)$. But this implies that $A \bmod X_0 = A_1 \bmod X_0$, a contradiction.

Thus we may assume that $A \setminus X_0$ contains only one X_0-coset or none which means that:

$$I = \chi \cdot J_0 \quad \text{for some} \quad \chi \in X .$$

But then I is G-maximal as J_0 is G-maximal and in fact I is a G-orbit. q.e.d.

Theorem: *Let G be a locally compact nilpotent group. Let N be a compactly generated normal open subgroup or more generally assume that G has a compact normal subgroup K so that the connected component of (G/K) is open in G/K and is a Lie group. Then every closed twosided prime ideal in $C^*(G)$ is maximal. Especially* Prim G *is* T_1.

Proof: Let $\tilde{N}$ denote the connected component of (G/K) and let N be the pullback of $\tilde{N}$ to G. Then N is an open normal subgroup of G. Let us show first that every G-prime ideal I in $C^*(N)$ is G-maximal. By (2. Lemma) I is a G-orbit, $I = \ker G \cdot \varrho$ for some $\varrho \in \hat{N}$. As N/K is connected, ϱ restricts to a multiple of some $\tau \in \hat{K}$. Let $M = M_\tau$ be the stabilizer of τ in G. M contains N. Let us show first that $I_0 = \ker M \cdot \varrho$ is M-maximal in $C^*(N)$. To do that we can again assume that the kernel K^0 of τ in K is trivial and hence that N is a Lie group. Let N_0 denote the connected component of N. N_0 is separable, hence every M-prime ideal in $C^*(N_0)$ is a G-orbit and thus by 3. Lemma is M-maximal.

Choose closed normal subgroups $H_0 = N \supset H_1 \supset \ldots \supset H_k = N_0$, so that $[M, H_i] \subset H_{i+1}$; $i = 0, 1, 2, \ldots, k-1$. Then H_{i+1} is open in H_i and H_i/H_{i+1} is central in M/H_{i+1} $i = 0, \ldots, k-1$. Applying 6. Lemma several times we see that every M-prime ideal in $C^*(N)$ is M-maximal. Especially I_0 is M-maximal.

We prove now that $I = \ker(G \cdot \varrho)$ is G-maximal. Let $\pi \in h(I) \subset \hat{N}$. i.e. $\pi(I) = 0$. We must show that $I = \ker G \cdot \pi$. There exists a net $\{x_\lambda\}_{\lambda \in \Lambda}$ in G so that $\ker \pi = \lim_\lambda x_\lambda \cdot \ker \varrho$ in $\operatorname{Prim} N$. But then $\{x_\lambda \cdot r_K(I)\}_\Lambda$ converges in $\operatorname{Prim} K$ to $r_K(\ker \pi) = \ker \tau'$ for some $\tau' \in \hat{K}$. As $\operatorname{Prim} K$ is discrete, for a subnet, also denoted for simplicity by Λ, $x_\lambda \cdot r_K(I) = x_{\lambda_0} \cdot r_K(I) = \ker \tau'$ for every λ, for some λ_0. Hence $x_{\lambda_0}^{-1} x_\lambda \cdot r_K(I) = r_K(x_{\lambda_0}^{-1} x_\lambda \cdot I) = \ker \tau' = r_K(I)$ for every λ and so $x_{\lambda_0}^{-1} x_\lambda \in M_\tau = M$ for every λ. Thus $x_{\lambda_0}^{-1} \cdot \ker \pi = \ker x_{\lambda_0}^{-1} \cdot \pi$ is contained in the closure of $M \cdot (\ker \varrho)$. As M-orbits are maximal, it follows that $\ker p \in h(M \cdot x_{\lambda_0}^{-1} \cdot \ker \pi)$ and hence that $\ker \varrho \in h(G \cdot \ker \pi)$, i.e., $I = \ker G \cdot \pi$.

We have now proved that every G-prime ideal in $C^*(N)$ is G-maximal. But N is open in G. Applying again several times 6. Lemma, as has been done before, it follows that every prime ideal in $C^*(G)$ is maximal. q.e.d.

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J. LUDWIG
Fakultät für Mathematik
Universität Bielefeld
D-4800 Bielefeld, Federal Republic of Germany