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# On Spline Systems

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In this note an effort is made to give an abstract approach to some features of the theory of spline interpolation within the framework of functional analysis. For this purpose we introduce the concept of spline systems allowing us to formulate certain minimal properties and their analogues which we obtain by duality technique. Finally, as an illustration we describe the linkage of these results to classical spline interpolation theory.

## 0. Basic Notation

All vector spaces we shall consider in this paper will be defined over the field $\mathbf{R}$ of real numbers.

Let $E$, $F$ be two topological vector spaces. We shall denote by $L(E, F)$ the vector space of all $\mathbf{R}$-linear mappings from $E$ into $F$ continuous with respect to the initially given topologies and by $L(E)$ the algebra $L(E, E)$ over $\mathbf{R}$ consisting of all continuous endomorphisms of $E$.

As usual the kernel and the image of any mapping $f \in L(E, F)$ will be denoted by Ker $f$ and Im $f$ respectively.

A mapping $f \in L(E, F)$ is called an *epimorphism* iff it is an open surjective mapping from $E$ onto $F$, i. e. iff $f$ is a surjective strict morphism. Recall that if $f$ has a right inverse in $L(F, E)$ we are sure that $f$ is an epimorphism.

Furthermore any idempotent mapping $f_1 \in L(E)$ is called a *projector* in $E$. Then $f_2 = id_E - f_1$ is the *supplementary projector* of $f_1$.

Let $H$ be a prehilbert space. The scalar product of $H$ is written $(x, y) \to (x|y)$ and the canonical norm induced by it will be denoted by $\|\cdot\|_H$.

Finally, if $M$ and $N$ are orthogonal subsets of $H$, i. e. if $(x|y) = 0$ for all pairs $(x, y) \in M \times N$, we shall employ the notation $M \perp N$.

## 1. Spline Systems

In the following definition we are going to introduce the key notion for our developments.

Let $E$ be a topological vector space and $H$ be a (separated) prehilbert space. A quadruple

$$(E, p_1, u, H)$$

is called a *spline system*, provided the following four conditions are satisfied by the linear mappings $p_1$ and $u$:

(i) $p_1 \in L(E)$ is a projector (having $p_2$ as the supplementary projector in $E$),

(ii) $u \in L\,(E, H)$ is an epimorphism,

(iii) $\mathrm{Ker}\, u \subseteqq \mathrm{Im}\, p_1$,

(iv) $\mathrm{Im}\,(u \circ p_1) \perp \mathrm{Im}\,(u \circ p_2)$.

In section 3 an explicit example of a spline system will be discussed in some detail.

Let $\pi : E \to E/\mathrm{Ker}\, u$ be the canonical epimorphism and

$$\bar{u} : E/\mathrm{Ker}\, u \to H$$

be the injection associated with $u$. Then we have the canonical factorization

$$u = \bar{u} \circ \pi \tag{1}$$

and by condition (ii) the mapping $\bar{u}$ represents a (toplinear) isomorphism of the quotient topological vector space $E/\mathrm{Ker}\ u$ onto the prehilbert space $H$.

Condition (iii) is equivalent to $\mathrm{Ker}\ u \subseteqq \mathrm{Ker}\, p_2$. Hence there exists exactly one continuous linear mapping $\bar{p}_2 : E/\mathrm{Ker}\, u \to E$ such that

$$p_2 = \bar{p}_2 \circ \pi. \tag{2}$$

We may now state

**Theorem 1.** *Let* $(E, p_1, u, H)$ *be any spline system in the sense defined above. Then the mapping*

$$P_2 = u \circ \bar{p}_2 \circ \bar{u}^{-1}$$

*is an orthogonal projector in the prehilbert space* $H$.

*Proof*. Evidently $P_2 \in L(H)$. To prove that $P_2$ is an idempotent endomorphism of $H$, first of all we remark that the relation $p_2 \circ p_2 = p_2$ implies by (2) the identity

$$p_2 \circ \bar{p}_2 = \bar{p}_2. \tag{3}$$

Then by (1), (2), (3) we obtain

$$\begin{aligned} P_2 \circ P_2 &= u \circ \bar{p}_2 \circ \bar{u}^{-1} \circ u \circ \bar{p}_2 \circ \bar{u}^{-1} \\ &= u \circ \bar{p}_2 \circ \bar{u}^{-1} \circ \bar{u} \circ \pi \circ \bar{p}_2 \circ \bar{u}^{-1} \\ &= u \circ p_2 \circ \bar{p}_2 \circ \bar{u}^{-1} \\ &= P_2. \end{aligned}$$

Consequently $P_2$ is a projector in $H$. It remains to verify that $P_2$ is symmetric relative to the bilinear form $(x, y) \to (x|y)$ on $H \times H$. To this end we observe that we can derive by means of (1) and (2)

$$P_2 \circ u = u \circ \bar{p}_2 \circ \bar{u}^{-1} \circ \bar{u} \circ \pi = u \circ p_2.$$

Suppose that $x, y$ are arbitrary elements of $H$. Condition (ii) for spline systems enables us to choose elements $x_0 \in E$, $y_0 \in E$ such that $x = u(x_0)$ and $y = u(y_0)$. Then by condition (iv)

$$\begin{aligned} (P_2(x)|y) &= (u \circ p_2(x_0) \mid u(y_0)) \\ &= (u \circ p_2(x_0) \mid u \circ p_2(y_0)) \\ &= (u(x_0) \mid P_2 \circ u(y_0)) \\ &= (x \mid P_2(y)). \end{aligned}$$

This completes the proof.

If $P_1 \in L(H)$ denotes the supplementary orthogonal projector of $P_2$, we can infer from the preceding proof the following relations which will be used repeatedly.

*Lemma 1. For $i = 1, 2$ we have the relation*

$$P_i \circ u = u \circ p_i.$$

Combining Theorem 1 with the projection theorem of Hilbert space we obtain by virtue of the preceding lemma the following minimal property:

**Theorem 2.** *Let $(E, p_1, u, H)$ be a spline system and suppose that $H$ is a Hilbert space. For any $x_0 \in E$ there is one and only one point $x \in \operatorname{Im} P_i$ such that*

$$\| u(x_0) - x \|_H = \inf_{z_0 \in E} \| u(x_0) - u \circ p_i(z_0) \|_H$$

*where $i = 1, 2$. We have $x = u \circ p_i(x_0)$.*

## 2. Dual Aspects

Given a spline system $(E, p_1, u, H)$, in the sequel we shall suppose that $E$ denotes a locally convex topological vector space and $H$ represents a Hilbert space. Let us form the topological duals $E'$ and $H'$ associated with $E$ and $H$ respectively and suppose that both vector spaces are equipped with the strong dual topologies. In addition $H'$ carries the Hilbert space structure transported by the canonical isometric isomorphism $j : H \to H'$. It will not be useful for our purposes to identify the spaces $H$ and $H'$.

As is well known, the transposed linear mappings

$$q_i = {}^t p_i \in L(E'), \ (i = 1, 2),$$
$$v = {}^t u \in L(H', E'),$$
$$Q_i = {}^t P_i \in L(H'), \ (i = 1, 2),$$

exist. Clearly $v$ is an injective linear mapping. Hence its induced inverse $w = v^{-1}$ is well defined on $\operatorname{Im} v$ and represents a surjective linear mapping

$$w : \operatorname{Im} v \to H'.$$

Lemma 1 yields

*Lemma 2. For* $i = 1, 2$ *the relation*

$$Q_i = w \circ q_i \circ v$$

*is valid.*

It follows from Theorem 1 that $Q_i = j \circ P_i \circ j^{-1}$ $(i = 1, 2)$, are orthogonal projectors in the Hilbert space $H'$ each supplementary to the other. Thus by the projection theorem and Lemma 2 we are led to

**Theorem 3.** *Let* $(E, p_1, u, H)$ *be a spline system, $E$ being a locally convex topological vector space and $H$ being a Hilbert space. For any* $x_0' \in \operatorname{Im} v$ *there is exactly one point* $x' \in \operatorname{Im} Q_i$ *such that*

$$\| w(x_0') - x' \|_{H'} = \inf_{z_0' \in \operatorname{Im} v} \| w(x_0') - w \circ q_i(z_0') \|_{H'}$$

*where* $i = 1, 2$. *We have* $x' = w \circ q_i(x_0')$.

Let us proceed to a special case. From (1) we deduce the identity $\operatorname{Im} v = \operatorname{Im} {}^t\pi$. Hence we obtain by (2) the inclusion

$$\operatorname{Im} q_2 \subseteqq \operatorname{Im} v. \tag{4}$$

If $x_0'$ denotes any point of $E'$, it is possible to apply the previous theorem to $q_2(x_0') \in \operatorname{Im} v$ in place of $x_0'$. Select any point $y_0' \in E'$ such that

$y_0' - x_0' \in \operatorname{Im} v$ is verified. By Lemma 2 we get $q_1(y_0' - x_0') \in \operatorname{Im} v$. Hence by (4)

$$z_0' = q_1(y_0') - x_0' \in \operatorname{Im} v.$$

Setting $i = 1$, Theorem 3 yields a

*Corollary. Let $(E, p_1, u, H)$ be a spline system having the same properties as in Theorem 3. For any $x_0' \in E'$ we have*

$$\| w \circ q_2(x_0') \|_{H'} = \inf_{y_0' \in x_0' + \operatorname{Im} v} \| w(x_0' - q_1(y_0')) \|_{H'}.$$

## 3. An Example of a Spline System

It is the purpose of this and of the last section to give an illustration of the preceding developments.

Let $I = [a, b]$ be any non-trivial compact interval in the real line $\mathbf{R}$ and $m \geq 1$ any fixed integer. We denote by $C_{\mathbf{R}}^{m-1}(I)$ the vector space of all real-valued functions $f$ which are defined on $I$ and have continuous derivatives $D^i f$ of order $0 \leq i \leq m-1$ in $I$, addition and multiplication by scalars being defined in the usual way. Clearly, at the end points of $I$ the corresponding left resp. right derivatives take the place of $D^i f$.

We shall provide $C_{\mathbf{R}}^{m-1}(I)$ with the topology of uniform convergence of functions and of their derivatives of order $\leq m - 1$. If we denote by $\|.\|_\infty$ the ČEBYŠEV norm on $I$, this topology is induced on $C_{\mathbf{R}}^{m-1}(I)$ by the norm

$$f \to \| f \|_{C^{m-1}} = \sup_{0 \leq i \leq m-1} \| D^i f \|_\infty,$$

and clearly it turns $C_{\mathbf{R}}^{m-1}(I)$ into a Banach space.

Following I. J. SCHOENBERG [4] we consider the (algebraic) vector subspace $K_{\mathbf{R}}^{2,m}(I)$ of $C_{\mathbf{R}}^{m-1}(I)$ which consists of all functions $f \in C_{\mathbf{R}}^{m-1}(I)$ such that $D^{m-1} f$ is an *absolutely continuous* function on $I$ and the function $D^m f$ (more precisely, the equivalence class defined by $D^m f$ which lies automatically in $L_{\mathbf{R}}^1(I)$) is an element of the Hilbert space $L_{\mathbf{R}}^2(I)$. The space $K_{\mathbf{R}}^{2,m}(I)$ is equipped with the natural norm

$$f \to \| f \|_{K^{2,m}} = \sup (\| f \|_{C^{m-1}}, \| D^m f \|_{L^2}).$$

Evidently the canonical injection $K_{\mathbf{R}}^{2,m}(I) \to C_{\mathbf{R}}^{m-1}(I)$ is continuous, i. e. the topology of $K_{\mathbf{R}}^{2,m}(I)$ defined by $\|.\|_{K^{2,m}}$ is finer than the relative topology induced by $C_{\mathbf{R}}^{m-1}(I)$.

**Theorem 4.** *$K_{\mathbf{R}}^{2,m}(I)$ is a Banach space.*

*Proof.* Let $(f_n)_{n\geq 1}$ be any Cauchy sequence in the space $K_{\mathbf{R}}^{2,m}(I)$. Evidently $(f_n)_{n\geq 1}$ is a Cauchy sequence in $C_{\mathbf{R}}^{m-1}(I)$ and $(D^m f_n)_{n\geq 1}$ is a Cauchy sequence in $L_{\mathbf{R}}^2(I)$. By completeness, these sequences have limits $f \in C_{\mathbf{R}}^{m-1}(I)$ resp. $F \in L_{\mathbf{R}}^2(I)$ in the topologies associated with these vector spaces. Since $F \in L_{\mathbf{R}}^1(I)$, it suffices to prove that relative to the topology of pointwise convergence for all $t \in I$

$$\lim_{n\to\infty} (D^{m-1}f_n(t) - D^{m-1}f_n(a)) = \int_a^t F(s)\,ds.$$

But we have with a constant $c > 0$

$$| D^{m-1} f_n(t) - D^{m-1} f_n(a) - \int_a^t F(s)\,ds \,| \leq$$

$$\leq \| D^m f_n - F \|_{L^1} \leq c \,.\, \| D^m f_n - F \|_{L^2}$$

for all $n \geq 1$ and all points $t \in I$. This proves the theorem.

Let us denote by $P_{m-1}(I)$ the vector space of the restrictions $\mathrm{rest}_I P$ to the interval $I$ of all polynomial functions $P$ of degree $\leq m-1$ with real coefficients. We shall consider $P_{m-1}(I)$ as a (closed) vector subspace of the Banach space $K_{\mathbf{R}}^{2,m}(I)$.

**Theorem 5.** *The linear mapping $D^m$ belongs to the space $L(K_{\mathbf{R}}^{2,m}(I), L_{\mathbf{R}}^2(I))$ and is an epimorphism with* Ker $D^m = P_{m-1}(I)$.

*Proof.* It follows immediately from the definition of the topology carried by the vector space $K_{\mathbf{R}}^{2,m}(I)$ that the linear mapping

$$D^m : K_{\mathbf{R}}^{2,m}(I) \to L_{\mathbf{R}}^2(I)$$

is continuous. Given an integer $n$ with $1 \leq n \leq m$, let us consider the linear mapping

$$J^n : L_{\mathbf{R}}^2(I) \ni F \to \left(t \to \frac{1}{(n-1)!} \int_a^t F(s)\,.\,(t-s)^{n-1}\,ds\right).$$

Then $D^i \circ J^n = J^{n-i}$ for $0 \leq i \leq n$ with the usual conventions concerning the identity mapping. In particular we obtain

$$D^{m-1} \circ J^m(F) = J^1(F) : t \to \int_a^t F(s)\,ds,$$

$$D^m \circ J^m(F) = F \in L_{\mathbf{R}}^2(I)$$

and

$$\| J^m(F) \|_{K^{2,m}} \leq c_m \,.\, \| F \|_{L^2}$$

with an appropriate constant $c_m > 0$. It follows that $J^m \in L(L^2_{\mathbf{R}}(I), K^{2,m}_{\mathbf{R}}(I))$ is a right inverse of the mapping $D^m$. Hence $D^m$ is an epimorphism. The last assertion comes from the fact that $f \in \operatorname{Ker} D^m$ if and only if $f \in C^{m-1}_{\mathbf{R}}(I)$ and $D^{m-1}f$ is a constant function on the interval $I$.

*Remark.* In view of the fact that $K^{2,m}_{\mathbf{R}}(I)$ is a Banach space (Theorem 4), it suffices to verify that $D^m$ is a continuous linear mapping from $K^{2,m}_{\mathbf{R}}(I)$ *onto* the Hilbert space $L^2_{\mathbf{R}}(I)$. By the open mapping theorem $D^m$ is then an epimorphism. However, the procedure above makes use only of arguments which are completely elementary in character.

Let us briefly recall what a (natural polynomial) *spline function* is (see e. g. AHLBERG-NILSON [1], T. N. E. GREVILLE [2], I. J. SCHOENBERG [4]). Let $n$ denote an integer, $n \geq m \geq 1$, and let $(t_i)_{1 \leq i \leq n}$ be $n$ numbers within the interval $I$ ordered in the following way:

$$a < t_1 < t_2 < \ldots < t_{n-1} < t_n < b. \tag{5}$$

By a spline function of degree $2m-1$ having the $n$ points $(t_i)_{1 \leq i \leq n}$ as nodes we mean a function $S \in C^{2m-2}_{\mathbf{R}}(\mathbf{R})$ with the property

$$\operatorname{rest}_{]-\infty, t_1[} S \in P_{m-1}(]-\infty, t_1[),$$

$$\operatorname{rest}_{]t_i, t_{i+1}[} S \in P_{2m-1}(]t_i, t_{i+1}[), \; 1 \leq i \leq n-1,$$

$$\operatorname{rest}_{]t_n, +\infty[} S \in P_{m-1}(]t_n, +\infty[).$$

We may then state the following interpolation theorem.

**Theorem 6.** *Given $n$ numbers $(t_i)_{1 \leq i \leq n}$ within the interval $I \subset \mathbf{R}$, ordered as in* (5) *and an n-tuple $(f_1, \ldots, f_n)$ of real numbers then there exists one and only one (natural polynomial) spline function $S$ of degree $2m-1$ with nodes $(t_i)_{1 \leq i \leq n}$ such that*

$$S(t_i) = f_i$$

*for $1 \leq i \leq n$.*

We shall not prove here the interpolation theorem but we refer to T. N. E. GREVILLE [2] and I. J. SCHOENBERG [5].

Let us denote by $S_i$ $(1 \leq i \leq n)$ the restrictions to $I$ of the fundamental functions of spline interpolation relative to the nodes $(t_i)_{1 \leq i \leq n}$. In the sequel we regard the functions $S_i$ as elements of the Banach space $K^{2,m}_{\mathbf{R}}(I)$. Moreover let $\varepsilon_{t_i} \in K^{2,m}_{\mathbf{R}}(I)'$, $(1 \leq i \leq n)$, be the Dirac measure (unit point mass) placed at the node $t_i$ and let $\langle . , . \rangle$ denote the bilinear form associated with the topological duality $(K^{2,m}_{\mathbf{R}}(I), K^{2,m}_{\mathbf{R}}(I)')$. To any function $f \in K^{2,m}_{\mathbf{R}}(I)$ we shall assign the linear combination

$$SP_n(f): I \ni t \to \sum_{i=1}^{n} < f, \varepsilon_{t_i} > . S_i(t).$$

Clearly $f \to SP_n(f)$ defines a linear mapping of the vector space $K_{\mathrm{R}}^{2,m}(I)$ into itself. It is easy to see that $\mathrm{Im}\, SP_n$ is the subspace of $K_{\mathrm{R}}^{2,m}(I)$ generated by $\{S_i \mid 1 \leq i \leq n\}$. For convenience we set

$$\mathrm{Im}\, SP_n = \Sigma_{m,n}(I).$$

**Theorem 7.** *Let the nodes* $(t_i)_{1 \leq i \leq n}$ *be chosen within the interval* $I$ *as in* (5) *and* $1 \leq m \leq n$. *Then*

$$(K_{\mathrm{R}}^{2,m}(I), SP_n, D^m, L_{\mathrm{R}}^2(I))$$

*forms a spline system.*

*Proof.* To verify property (i) of spline systems we observe that for any function $f \in K_{\mathrm{R}}^{2,m}(I)$

$$\| SP_n(f) \|_{K^{2,m}} \leq c_{m,n} \cdot \| f \|_{K^{2,m}}$$

where $c_{m,n} = \sum_{i=1}^{n} \| S_i \|_{K^{2,m}}$. Hence $SP_n \in L(K_{\mathrm{R}}^{2,m}(I))$. The fact that the linear mapping $SP_n$ is idempotent follows at once by Theorem 6. Next we refer to Theorem 5 which is applied to establish condition (ii). In addition Theorem 5 yields $\mathrm{Ker}\, D^m = P_{m-1}(I)$. By virtue of Theorem 6

$$SP_n(P_{m-1}(I)) = P_{m-1}(I).$$

Hence property (iii) follows. Finally, property (iv) of spline systems reads in the present case

$$\int_a^b D^m SP_n(f)(t) . (D^m f(t) - D^m SP_n(f)(t))\, dt = 0$$

for all functions $f \in K_{\mathrm{R}}^{2,m}(I)$. But this is equivalent to the so-called integral relation, well-known from spline interpolation theory. See AHLBERG-NILSON [1] and T. N. E. GREVILLE [2].

## 4. Classical Minimal Properties

Besides the fact that the preceding theorem yields a concrete example of a spline system, it enables us to apply the results we have derived in sections 1 and 2 for general spline systems to the case $(K_{\mathrm{R}}^{2,m}(I), SP_n, D^m, L_{\mathrm{R}}^2(I))$. Recall that $\Sigma_{m,n}(I)$ denotes the closed vector subspace of $K_{\mathrm{R}}^{2,m}(I)$ of spline functions (restricted to $I$) having degree $2m-1$ and the points $(t_i)_{1 \leq i \leq n}$ in the ordering (5) as nodes.

Let us begin with Theorem 2. We obtain in the case $i=1$ the following

**Theorem 8** (First minimal property). *Let $f \in K_{\mathbf{R}}^{2,m}(I)$ be given. For any spline function $s \in \Sigma_{m,n}(I)$ we have in the Hilbert space $L_{\mathbf{R}}^2(I)$*

$$\| D^m f - D^m s \|_{L^2} \geq \| D^m f - D^m \circ SP_n(f) \|_{L^2}$$

*with equality iff*

$$s \equiv SP_n(f) \bmod P_{m-1}(I).$$

We proceed to apply the case $i = 2$ of Theorem 2 to our special spline system and obtain for any $f \in K_{\mathbf{R}}^{2,m}(I)$

$$\| D^m f - g \|_{L^2} = \inf_{h \in K_{\mathbf{R}}^{2,m}(I)} \| D^m f - D^m h + D^m \circ SP_n(h) \|_{L^2},$$

where $g = D^m f - D^m \circ SP_n(f)$. Consequently

$$\| D^m \circ SP_n(f) \|_{L^2} \leq \| D^m f \|_{L^2},$$

with equality iff $f \equiv SP_n(f) \bmod P_{m-1}(I)$. Hence follows

**Theorem 9** (Second minimal property). *For any function $f \in K_{\mathbf{R}}^{2,m}(I)$ we have*

$$\| D^m f \|_{L^2} \geq \| D^m \circ SP_n(f) \|_{L^2}.$$

*Equality holds iff*

$$f = SP_n(f).$$

Our last application is concerned with Schoenberg's approximation theorem. It will be obtained as a simple and direct consequence of the Corollary of Theorem 3.

Let $(b_i)_{1 \leq i \leq n}$ denote real numbers. Then the real Radon measure on $I$,

$$T = \sum_{i=1}^{n} b_i \, \varepsilon_{t_i}$$

whose support is contained in the set $\{t_i \mid 1 \leq i \leq n\}$ of given nodes is an element of the dual space $K_{\mathbf{R}}^{2,m}(I)'$. For any function $f \in K_{\mathbf{R}}^{2,m}(I)$ we have the identities

$$\langle f, T \rangle = \sum_{i=1}^{n} b_i f(t_i) = \sum_{i=1}^{n} b_i SP_n(f)\,(t_i) =$$
$$= \langle SP_n(f), T \rangle = \langle f, {}^t SP_n(T) \rangle.$$

Hence

$$T = {}^t SP_n(T).$$

In addition let a continuous linear form $L \in C_{\mathbf{R}}^{m-1}(I)'$ be given. Clearly $T - L$ can be considered as an element of the space $K_{\mathbf{R}}^{2,m}(I)'$.

Suppose that $\mathrm{rest}_{P_{m-1}(I)}(T-L) = 0$. It is possible to apply Peano's theorem (see A. SARD [3, Chap. 1]). It yields for any function $f \in K_{\mathbf{R}}^{2,m}(I)$ the identities

$$\langle f, T-L\rangle = (D^m f \mid K) = \langle D^m f, j_{L^2}(K)\rangle = \langle f, {}^tD^m \circ j_{L^2}(K)\rangle \qquad (6)$$

where $K \in L_{\mathbf{R}}^2(I)$ denotes the Peano kernel associated with the linear form $T-L$ and $j_{L^2}$ denotes the canonical isometric isomorphism from $L_{\mathbf{R}}^2(I)$ onto its topological dual $L_{\mathbf{R}}^2(I)'$. It follows that

$$T \in L + \mathrm{Im}\, {}^tD^m.$$

Hence we get by virtue of the aforementioned Corollary

**Theorem 10** (I. J. SCHOENBERG [4]). *Let the continuous linear form* $L \in C_{\mathbf{R}}^{m-1}(I)'$ *be given. If the discrete Radon measure*

$$T = \sum_{i=1}^{n} b_i\, \varepsilon_{t_i} \in K_{\mathbf{R}}^{2,m}(I)'$$

*on* $I$ *has the property*

$$\mathrm{rest}_{P_{m-1}(I)}\,(T-L) = 0$$

*then the inequality*

$$\| ({}^tD^m)^{-1} (L-T) \|_{L^{2'}} \geq \| ({}^tD^m)^{-1} (L-{}^tSP_n(L)) \|_{L^{2'}}$$

*holds and*

$${}^tSP_n(L) = \sum_{i=1}^{n} \langle S_i, L\rangle . \varepsilon_{t_i}.$$

In view of (6), the preceding theorem states that, loosely speaking, when the continuous linear form $L$ is approximated by $T$, the remainder $L-T$ turns out to have a Peano kernel with minimal $L^2$ norm if the coefficients $(b_i)_{1\leq i\leq n}$ of $T$ are equal to the values $\langle S_i, L\rangle$ of $L$ at the fundamental functions $(S_i)_{1\leq i\leq n}$ of spline interpolation with nodes $(t_i)_{1\leq i\leq n}$. In this sense the discrete Radon measure

$${}^tSP_n(L)$$

represents a best approximation of $L$ on $I$.

It is the topic of forthcoming papers to expose more detailed investigations of the topological duality $(K_{\mathbf{R}}^{2,m}(I), K_{\mathbf{R}}^{2,m}(I)')$, to point out especially the connections to elementary distribution theory and to give applications of our notion of spline system to the theory of $L$-splines.

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