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PERIODIC SOLUTIONS OF ASYMPTOTICALLY LINEAR HAMILTONIAN SYSTEMS

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We prove existence and multiplicity results for periodic solutions of time dependent and time independent Hamiltonian equations, which are assumed to be asymptotically linear. The periodic solutions are found as critical points of a variational problem in a real Hilbert space. By means of a saddle point reduction this problem is reduced to the problem of finding critical points of a function defined on a finite dimensional subspace. The critical points are then found using generalized Morse theory and minimax arguments.

1. Introduction

We shall study the existence of periodic solutions of Hamiltonian equations

(1) $$\dot{x} = Jh'(t,x) \ , \ (t,x) \in R \times R^{2n} \ ,$$

where

(2) $$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathcal{L}(R^{2n})$$

is the standard symplectic structure in R^{2n}, 1 denoting the identity matrix in R^n . The Hamiltonian function h belongs to $C^2(R \times R^{2n})$ and by $h'(t,.)$ we denote the gradient with respect to the x-variable.

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We shall assume that h depends periodically on the time t with period $T > 0$:

(3) $$h(t,x) = h(t+T,x) \ , \ (t,x) \in R \times R^{2n} \ .$$

Our aim is to find periodic solutions of the system (1) having period T. We consider asymptotically linear systems assuming that:

(4) $$Jh'(t,x) = Jb_\infty x + o(|x|) \ , \text{ as } |x| \to \infty$$

uniformly in t , for a symmetric and time independent matrix $b_\infty \in \mathcal{L}(R^{2n})$. In addition we shall require the Hessian of $h(t,.)$ to be uniformly bounded:

(5) $$-\beta \leq h''(t,x) \leq \beta \ , \ (t,x) \in R \times R^{2n} \ ,$$

for some constant $\beta > 0$. The following existence result has been proved in [1] (Theorem 12.4).

<u>*Theorem 1:*</u> *Let* $h(t,x)$ *be periodic in* t *with period* $T > 0$. *Then under the assumptions* (4) *and* (5) *the Hamiltonian system* (1) *possesses at least one* T-*periodic solution provided*

(6) $$\sigma(Jb_\infty) \cap i \frac{2\pi}{T} \mathbb{Z} = \emptyset \ .$$

Here and in the following $\sigma(Jb_\infty)$ *denotes the spectrum of the linear operator* Jb_∞ .

Condition (6) reminds of the nonresonance condition in Liapunovs theorem. It requires that the linear and time independent

Hamiltonian system $\dot{x} = Jb_\infty x$ does not possess a periodic solution of period T .

It should be said that in theorem 1 the constant solutions are admitted as periodic solutions. Namely for a time independent Hamiltonian function h , the system (1) possesses necessarily an equilibrium point, which is a constant periodic solution. In fact, if f is any C^2-function on R^m satisfying $f'(x) = b_\infty x + o(|x|)$, as $|x| \to \infty$, with a symmetric and nonsingular matrix $b_\infty \in \mathcal{L}(R^m)$, then there is an $x^* \in R^m$ such that $f'(x^*) = 0$. (see for instance [1], proposition 12.5).

For this reason we consider in the following systems which possess an equilibrium point, which we assume to be 0 , i.e. $Jh'(t,0) = 0$. The aim is to find T-periodic solutions, which are not trivial, i.e. $x(t) \neq 0$. More precisely we shall require, that

(7) $$Jh'(t,x) = Jb_0 x + o(|x|) \quad , \quad \text{as} \quad |x| \to 0$$

uniformly in t for a symmetric and time independent matrix $b_0 \in \mathcal{L}(R^{2n})$.

It turns out that such systems possess at least one nontrivial T-periodic solution, if the two linear Hamiltonian systems $\dot{x} = Jb_0 x$ and $\dot{x} = Jb_\infty x$ are different from each other. This difference will be measured by an integer

(8) $$i = i(b_0, b_\infty, \tfrac{2\pi}{T}) \in \mathbb{Z} \quad ,$$

which will be computed explicitly in terms of the period T and the imaginary eigenvalues of the linear systems $\dot{x} = Jb_0 x$ and $\dot{x} = Jb_\infty x$.

Our main result (theorem 2) guarantees the existence of a nontrivial T-periodic solution if $i(b_0, b_\infty, \frac{2\pi}{T}) > 0$ provided the nonresonance condition (6) for Jb_∞ holds true. For example, $i > 0$, if $b_0 < 0 < b_\infty$ or if $b_\infty < 0 < b_0$. We remark that no nonresonance condition for Jb_0 is required. We also point out, that only assumptions "at 0" and "at ∞" are required and none in the "interior". In this respect the existence statement is similar to the Poincaré-Birkhoff fixed point theorem; the condition $i > 0$ corresponds to the twist condition required in that theorem.

The above periodic solutions will be found as critical points of a variational problem in a real Hilbert space H. Due to the assumption (5) the problem can be reduced to the problem of finding nontrivial critical points of a function a which is defined on a finite dimensional subspace $Z \subset H$. As a consequence of the assumption (6) this function satisfies the so called Palais-Smale condition. Moreover due to the assumptions (4) and (6) the qualitative behavior of the function is known in a neighborhood of 0 and of "∞". The critical points are then found using the topological tools of the generalized Morse theory developped by C. Conley [5].

In the sections 4 and 5 we shall establish multiplicity results, however under additional assumptions on the function h "in the interior". We shall prove (theorem 4) for h even in the x-variable, i.e. $h(t,x) = h(t,-x)$ for all $(t,x) \in R \times R^n$, that the integer $i(b_0,b_\infty,\frac{2\pi}{T})$ is a lower bound of the number of nontrivial T-periodic aolutions of (1). Here standard minimax arguments based on the concept of genus as described by D.C. Clark [4] lead to the result. In section 5 we consider a time independent Hamiltonian function h. In this situation there is no "natural" period for the sought periodic solution given. Assuming h to be strictly convex we shall show (theorem 5) that for any positive number T satisfying $i(b_0,b_\infty,\frac{2\pi}{T})>0$ there are at least $i/2$ distinct, non constant, periodic solutions of the Hamiltonian system (1) having period T. Again the proof is based on minimax arguments. This time we use the fact, that the functional is invariant under a special S^1-action and apply the index-theory developed by E. Fadell and P. Rabinowitz [6].

The proofs of these results, already announced in a preliminary version in 1979, rest heavily on our previous work in [1]. Recently, V. Benci proved results related to our multiplicity results in section 5. His approach however is different from ours, but also based on minimax arguments.

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2. The index $i(b_0, b_\infty, \tau)$

In order to formulate our first existence statement, we shall introduce first an integer $i(b_0, b_\infty, \tau)$ for two symmetric matrices $b_0, b_\infty \in \mathcal{L}(R^{2n})$ and a positive number τ .

If $b \in \mathcal{L}(R^{2n})$ is symmetric, and $\mu \geq 0$ we consider the quadratic form on $R^{2n} \times R^{2n}$, defined as

$$(8) \qquad 2\mu\langle Jx_1, x_2\rangle - \langle bx_1, x_1\rangle - \langle bx_2, x_2\rangle ,$$

$(x_1, x_2) \in R^{2n} \times R^{2n}$. It is represented by the matrix $Q(\mu, b) \in \mathcal{L}(R^{4n})$:

$$(9) \qquad Q(\mu,b) = \mu\begin{pmatrix} 0 & J^T \\ J & 0 \end{pmatrix} - \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} .$$

Notice that $-J = J^T$. We denote in the following by $m^+(.)$, $m^0(.)$ and $m^-(.)$ the positive, the zero and the negative Morse index of a quadratic form or of a matrix representing this form. We observe that

$$(10) \qquad m^+(Q(\mu,b)) = m^-(Q(\mu,b)) = 2n$$

if $\mu > \max \{\alpha \in R \mid i\alpha \in \sigma(Jb)\}$.

In fact if $\mu > 0$ is sufficiently large then $m^+ = m^- = 2n$, which are the indices of the first matrix in (9). Moreover if μ decreases, these indices can change only at those values of μ , for which the matrix (9) is singular, that is $m^0(Q(\mu,b)) \neq 0$. This occurs precisely for those values of $\mu \in R$ for which $i\mu$ is a purely imaginary eigenvalue of Jb . Indeed assume $(x_1,x_2) \in R^{2n} \times R^{2n}$ is an eigenvector of (9) with eigenvalue 0 . Then, using $J^T = -J$, we have $bx_1 + \mu Jx_2 = 0$ and $bx_2 - \mu Jx_1 = 0$. Therefore $b(x_1+ix_2)$ $= \mu J(ix_1-x_2) = i\mu J(x_1+ix_2)$, hence $Jb(x_1-ix_2) = -i\mu(x_1+ix_2)$ and so $\pm i\mu \in \sigma(Jb)$, as claimed. From these remarks the assertion (10) is obvious.

Assume the two matrices $b_0,b_\infty \in \mathcal{L}(R^{2n})$ to be symmetric and let $\tau > 0$. Abbreviating $Q_\mu^0 = Q(\mu,b_0)$ and $Q_\mu^\infty = Q(\mu,b_\infty)$, we then define two integers $i^\pm = i^\pm(b_0,b_\infty,\tau)$ as follows:

$$(11) \qquad i^\pm = \frac{1}{2}\{m^\pm(Q_0^0) - m^\pm(Q_0^\infty)\} + \sum_{j=1}^{\infty}\{m^\pm(Q_{j\tau}^0) - m^\pm(Q_{j\tau}^\infty)\} .$$

Finally, we set:

$$(12) \qquad i(b_0,b_\infty,\tau) = \max\{i^+,i^-\} \in \mathbb{Z} .$$

In view of (10) the above sum is finite. As a sideremark we observe that $i^\pm(s^Tb_0s,t^Tb_\infty t,\tau) = i^\pm(b_0,b_\infty,\tau)$ for two symplectic matrices s and t , since $s^TJs = J$ for $s \in Sp(2n,R)$. Hence i is a symplectic invariant. Clearly $i^\pm(b_0,b_\infty,\tau) = 0$ if $b_0 = b_\infty$, or if both matrices Jb_0 and Jb_∞ have no imaginary eigenvalues (zero included). Also:

$$i^\pm(b_0,b_\infty,\tau) = \bar{m}^\mp(b_0) - \bar{m}^\mp(b_\infty)$$

if τ is sufficiently large.

We next compute the numbers $i^\pm$ explicitly in terms of the purely imaginary eigenvalues of Jb_0 and Jb_∞. If $b \in \mathcal{L}(R^{2n})$ is a real and symmetric matrix, then it is well known that with an eigenvalue $\lambda \in \sigma(Jb)$, also $-\lambda$, $\bar{\lambda}$ and $-\bar{\lambda}$ belong to $\sigma(Jb)$. That is, the eigenvalues of Jb occur in groups of four, in case they are complex, and in groups of two, in case they are real or purely imaginary. With P_λ we denote the projection onto the eigenspaces of such an eigenvalue group. Then $P_\lambda R^{2n}$ is a symplectic subspace on R^{2n}, which is invariant under Jb; in particular the dimension of $P_\lambda R^{2n}$ is even. Therefore R^{2n} decomposes into invariant symplectic subspaces corresponding to different eigenvalue groups. We now consider the symplectic subspaces, which belong to a pair of purely imaginary eigenvalues $\pm i\alpha$, $\alpha \neq 0 \in R$. It is important to observe that there is a preferred symplectically invariant choice of the signs of these eigenvalues which singles out half of them as "positively oriented". To explain this, we first assume $i\alpha$ to be a simple eigenvalue, so that $\dim P_{i\alpha} R^{2n} = 2$. There is a linear symplectic coordinate change in this subspace such that the corresponding Hamiltonian has the following normal form on R^2:

$$h(x,y) = \frac{1}{2}\,\alpha(x^2 + y^2) .$$

The number α is a symplectic invariant, and we call $i\alpha$ the "positively oriented" eigenvalue (of the pair $\pm i\alpha$). If the multiplicity of $i\alpha$ is $r > 1$, then $\dim P_{i\alpha} R^{2n} = 2r$. If we denote by $E_{i\alpha}$ the complex eigenspace belonging to the eigenvalue $i\alpha$, then

$$\frac{1}{2i} \langle v, J\bar{v} \rangle , \quad v \in E_{i\alpha}$$

defines a nondegenerate Hermitean form. If this form has an r_+-dimensional positive and an r_--dimensional negative subspace, where $r_+ + r_- = r$, then we set r_+ of the eigenvalues equal to $i\alpha$, $\alpha > 0$ and r_- of them equal to $-i\alpha$, so that

$$\underbrace{i\alpha, i\alpha, \ldots, i\alpha}_{r_+\text{-times}} \qquad \underbrace{-i\alpha, \ldots, -i\alpha}_{r_-\text{-times}}$$

are the positively oriented eigenvalues of the restriction of Jb onto $P_{i\alpha}R^{2n}$. If this restriction is symplectically diagonalizable, there is a symplectic change of coordinates, which puts the corresponding Hamiltonian into the normal form on R^{2r} :

$$h(x,y) = \frac{1}{2}\alpha \sum_{j=1}^{r_+} (x_j^2 + y_j^2) - \frac{1}{2}\alpha \sum_{j=1}^{r_-} (x_{r_++j}^2 + y_{r_++j}^2) .$$

We should say that a definite Hamiltonian is always symplectically diagonizable. For instance, if it is positively definite, its normalform is

$$h(x,y) = \frac{1}{2}\alpha \sum_{j=1}^{r} (x_j^2 + y_j^2) , \quad \alpha > 0$$

and $i\alpha, \ldots, i\alpha$ (r-times) are the positively oriented eigenvalues. We refer to J. Moser [7] for more details. After these remarks the indices of the quadratic form (9) are easily computed. We shall denote in the following by [M] the cardinality of a finite set M.

Lemma 1: *Assume the imaginary part of* Jb *is symplectically diagonalizable and denote by* $S = \{i\alpha_1, \ldots, i\alpha_s\}$ *the set of positively oriented imaginary eigenvalues. Then if* $\mu > 0$:

$$m^+(Q(\mu,b)) = 2n - 2\,[\,i\alpha \in S \mid \alpha \geq \mu] + 2\,[\,i\alpha \in S \mid \alpha < -\mu]$$
$$m^-(Q(\mu,b)) = 2n + 2\,[\,i\alpha \in S \mid \alpha > \mu] - 2\,[\,i\alpha \in S \mid \alpha \leq -\mu]$$
$$m^0(Q(\mu,b)) = 2\,[\,i\alpha \in S \mid \alpha = \mu] + 2\,[\,i\alpha \in S \mid \alpha = -\mu]$$

If, in addition, b *is a nonsingular matrix, then:*

$$m^{\pm}(b) = n \pm \{\,[\,i\alpha \in S \mid \alpha > 0\,] - [\,i\alpha \in S \mid \alpha < 0\,]\}\ .$$

We recall that in the special case, where the restrictions of b onto the imaginary subspaces are definite, then Jb is symplectically diagonalizable on these subspaces. The condition of being diagonalizable can be dropped if $i\mu \notin \sigma(Jb)$.

Proof: We have seen that $m^+ = m^- = 2n$ if $\mu > 0$ is sufficiently large. Moreover of μ decreases, these indices change precisely if $i\mu \in \sigma(Jb)$. Assume first that $i\alpha$, $\alpha \in R$ is a simple positively oriented eigenvalue of Jb . Then we put the Hamiltonian, restricted to the eigenspace of the pair $\pm i\alpha$, by means of a symplectic transformation into the form $\frac{1}{2}\,\alpha(x_1^2 + y_1^2)$. The restriction of the form $Q(\mu,b)$ onto the two copies of these subspaces becomes

$$\begin{aligned} &2\mu(x_1y_2 - x_2y_1) - \alpha(x_1^2 + y_1^2) - \alpha(x_2^2 + y_2^2) = \\ &= -\alpha\ \{(x_1 - \tfrac{\mu}{\alpha}\,y_2)^2 + (1 - (\tfrac{\mu}{\alpha})^2)\ y_2^2\} \\ &\quad -\alpha\ \{(x_2 + \tfrac{\mu}{\alpha}\,y_1)^2 + (1 - (\tfrac{\mu}{\alpha})^2)\ y_1^2\}\ . \end{aligned}$$

Consider the case $\alpha > 0$. Then we read off for $\mu > \alpha$:

$(m^+,m^0,m^-) = (2,0,2)$, for $\mu = \alpha$: $(m^+,m^0,m^-) = (0,2,2)$, and for $\mu < \alpha$: $(m^+,m^0,m^-) = (0,0,4)$. Similarly for the case $\alpha < 0$: if $-\mu < \alpha$, then $(m^+,m^0,m^-) = (2,0,2)$, if $-\mu = \alpha$, then $(m^+,m^0,m^-) = (2,2,0)$, and if $-\mu > \alpha$, then $(m^+,m^0,m^-) = (4,0,0)$. Therefore, in case $\alpha > 0$, the index m^+ changes by -2 and m^- by $+2$ if μ crosses α from above, and, in case $\alpha < 0$, the index m^+ changes by $+2$ and m^- by -2 if μ crosses $-\alpha = |\alpha|$ from above. If $\pm i\alpha$ is not a simple eigenvalue pair then, by our assumption, the restriction of $Q(\mu,b)$ onto the eigenspace is a sum of quadratic forms of the above type and the Lemma follows.

As a consequence we find the following explicit expressions for the integers $i^{\pm}(b_0,b_\infty,\tau)$.

<u>Lemma 2</u>: *Let* $b_0,b_\infty \in (R^{2n})$ *be symmetric and* $\tau > 0$. *Assume the imaginary parts of* Jb_0 *and* Jb_∞ *are symplectically diagonalizable, and denote by* $S^0 = \{i\alpha_1^0,\dots,i\alpha_{s_0}^0\}$ *and* $S^\infty = \{i\alpha_1^\infty,\dots,i\alpha_{s_\infty}^\infty\}$ *the sets of positively oriented imaginary eigenvalues of* Jb_0 *and* Jb_∞ . *Then:*

$$\begin{aligned} i_+ &= \frac{1}{2}\{m^+(Q_0^0) - m^+(Q_0^\infty)\} \\ &\quad + 2\sum_{j=1}^{\infty} \{ [i\alpha^0 \in S^0 \mid \alpha^0 < -j\tau] - [i\alpha^0 \in S^0 | \alpha^0 \geq j\tau\} \\ &\quad - 2\sum_{j=1}^{\infty} \{ [i\alpha^\infty \in S^\infty \mid \alpha^\infty < -j\tau] - [i\alpha^\infty \in S^\infty | \alpha^\infty \geq j\tau]\} \\ i_- &= \frac{1}{2}\{m^-(Q_0^0) - m^-(Q_0^\infty)\} \\ &\quad + 2\sum_{j=1}^{\infty} ([i\alpha^0 \in S^0 | \alpha^0 > j\tau] - [i\alpha^0 \in S^0 | \alpha^0 \leq -j\tau]) \end{aligned}$$

$$- 2 \sum_{j=1}^{\infty} ([i\alpha^{\infty} \in S^{\infty} | \alpha^{\infty} > j\tau] - [i\alpha^{\infty} \in S^{\infty} | \alpha^{\infty} \leq -j\tau]).$$

If b_0, b_∞ *are non singular then*

$$\frac{1}{2} \{m^+(Q_0^0) - m^+(Q_0^\infty)\} = - \frac{1}{2} \{m^-(Q_0^0) - m^-(Q_0^\infty)\}$$

$$= [S^0 | \alpha^0 < 0] - [S^0 | \alpha^0 \geq 0] - [S^\infty | \alpha^\infty < 0] + [S^\infty | \alpha^\infty \geq 0]$$

The assumption that, Jb_0 *and* Jb_∞ *are symplectically diagonalizable on the imaginary subspace can be dropped provided* $\sigma(Jb_0) \cap i\tau\mathbb{Z} = \emptyset$ *and* $\sigma(Jb_\infty) \cap i\tau\mathbb{Z} = \emptyset$.

3. The time dependent case

a) The statements:

After these preliminaries we formulate the main results and some consequences:

Theorem 2: Assume $h(t,x) \in C^2$ *to be periodic in* t *with period* $T > 0$ *satisfying* (4), *and assume*

$$(13) \quad \begin{aligned} Jh'(t,x) &= Jb_0 x + o(|x|) \quad , \quad |x| \to 0 \\ Jh'(t,x) &= Jb_\infty x + o(|x|) \quad , \quad |x| \to \infty \end{aligned}$$

uniformly in t *for two symmetric time independent matrices* $b_0, b_\infty \in \mathcal{L}(R^{2n})$. *Then if*

$$i(b_0, b_\infty, \frac{2\pi}{T}) > 0 ,$$

there is at least one nontrivial T-*periodic solution of the*

Hamiltonian system $\dot{x} = Jb'(t,x)$, *provided* $\sigma(Jb_\infty) \cap i\frac{2\pi}{T}\mathbb{Z} = \emptyset$. This statement generalizes theorem (12.11) in [1], where the additional assumption $\sigma(Jb_0) \cap i\frac{2\pi}{T}\mathbb{Z} = \emptyset$ is required. The interest of this statement lies in the fact, that the index i is explicitely computable in terms of the normal modes of the two linear systems $\dot{x} = Jb_0 x$ and $\dot{x} = Jb_\infty x$. This leads to various existence statements. We mention two simple special cases.

<u>*Corollary 1*</u>: *Assume* h *as in the theorem, and assume* $b_0 < o < b_\infty$ *or* $b_\infty < 0 < b_0$, *then the Hamiltonian system has at least one non-trivial* T-*periodic solution provided* $\sigma(Jb_\infty) \cap i\frac{2\pi}{T}\mathbb{Z} = \emptyset$.

In fact in this case Jb_0 and Jb_∞ are symplectically diagonalizable and if $b_0 < 0 < b_\infty$ then $\alpha^0 < 0$ for $i\alpha^0 \in S^0$ and $\alpha^\infty > 0$ for $i\alpha^\infty \in S^\infty$, and therefore by Lemma 2,

$$i^+ = m^-(b_0) + 2\sum_{j=1}^{\infty}\{[S^0|\alpha^0 < -j\tau] + [S^\infty|\alpha^\infty \geq j\tau]\} \tag{14}$$

which is greater than 2n for every $\tau > 0$. Similarly for the case $b_\infty < 0 < b_0$.

More generally, if $b_0 \leq 0 < b_\infty$, then we conclude, in view of (14), a T-periodic solution if either $m^-(b_0) > 0$, or if there is an integer $j \geq 1$ such that $j\tau < \alpha_k^\infty$ for some $i\alpha_k^\infty \in S^\infty$, where $\tau = \frac{2\pi}{T}$.

<u>*Corollary 2:*</u> *Assume* h *as in the theorem. If* Jb_0 *is hyperbolic*

(i.e. has no imaginary eigenvalues), and if the restriction of b_∞ *onto the imaginary eigenspace of* Jb_∞ *is definite (e.g. if* Jb_∞ *possesses only one pair of simple imaginary eigenvalues), then the Hamiltonian system has at least one non-trivial* T*-periodic solution provided* $\sigma(Jb_\infty) \cap i\,\frac{2\pi}{T}\,\mathbb{Z} = \emptyset$.

Indeed, in this case $\mathcal{J}^0 = \emptyset$ and we find by Lemma 2, in the positive definite case,

$$i^+ = [\,S^\infty | \alpha^\infty > 0] + 2 \sum_{j=1}^{\infty} [\,S^\infty | \alpha^\infty \geq j\tau] \ ,$$

which is greater than zero. In the negative definite case we have

$$i^- = [\,S^\infty | \alpha^\infty < 0] + 2 \sum_{j=1}^{\infty} [\,S^\infty | \alpha^\infty \leq j\tau] \ .$$

b) Proof of theorem 2

We first reformulate the problem as an abstract variational problem in a Hilbert space in order to apply the abstract results of [1]. We let

$$T > 0 \quad \text{and} \quad \tau = \frac{2\pi}{T} \ ,$$

and consider the Hilbert space $H = L_2(0,T;R^{2n})$. In H we define the linear operator $A : \mathrm{dom}(A) \subset H \to H$ by $\mathrm{dom}(A) = \{u \in H^1(0,T;R^{2n}) | u(0) = u(T)\}$ and

(15) $$Au = -J\dot{u} \ , \quad u \in \mathrm{dom}(A) \ .$$

We also define the continuous potential operator $F : H \to H$ by

(16) $$F(u)(t) = h'(t,u(t)) \ ,$$

whose potential $\phi(u)$ is given by

(17) $$\phi(u) = \int_0^T h(t,u(t))dt\,.$$

Clearly F is the gradient of ϕ , i.e. $\phi'(u) = F(u)$. Writing the equation (1) in the form $-J\dot{x} = h'(t,x)$, we see that every solution $u \in \mathrm{dom}(A)$ of the equation

(18) $$Au = F(u)$$

defines (by T-periodic continuation) a classical T-periodic solution of (1). Conversely,every T-periodic solution of (1) defines (by restriction) a solution u of the equation (18). The equation (18) is the Euler equation of the variational problem extr $\{f(u)$, $u \in \mathrm{dom}(A)\}$, where

(19) $$f(u) = \frac{1}{2}\langle Au,u\rangle - \phi(u)\ ,$$

or in classical notation

$$\mathrm{extr} \int_0^T \{\frac{1}{2}\langle\dot{x},Jx\rangle - h(t,x(t))\}\,dt\ ,$$

$x(0) = x(T)$. The following properties of the operator A are readily verified:

Lemma 3: The operator A *is selfadjoint and has a pure point spectrum* $\sigma(A) = \tau\mathbb{Z}$. *Every eigenvalue* $\lambda \in \sigma(A)$ *has multiplicity* $2n$ *and the eigenspace* $E(\lambda) = \ker(\lambda - A)$ *is spanned by the orthogonal basis:*

$$t \to e^{t\lambda J}e_k = (\cos\lambda t)e_k + (\sin\lambda t)Je_k,\ k = 1,2,\dots,2n\ ,$$

where $\{e_k \mid 1 \le k \le 2n\}$ *is the standard basis of* R^{2n} . *In particular,* $\ker(A) = R^{2n}$ *that is, it consists of the constant functions.*

Since $h'(t,0) = 0$, the potential operator F satisfies $F(0) = 0$, and it follows from the assumption (5) that

$$(20) \qquad -\beta|u-v|^2 \le \langle F(u)-F(v),u-v\rangle \le \beta|u-v|^2$$

for every $u,v \in H$. Introducing the bounded symmetric operators $B_0,B_\infty \in \mathcal{L}(H)$ by

$$B_0\, u(t) = b_0\, u(t) \;,\; B_\infty\, u(t) = b_\infty\, u(t) \;,$$

we derive from our assumption (13), that

$$F'(0) = B_0 \quad \text{and} \quad F'(\infty) = B_\infty \;,$$

where the last equation means $\lim\limits_{|u| \to \infty} |u|^{-1}\, |F(u)-B_\infty u| = 0.$

We finally observe that the condition $\sigma(Jb_\infty) \cap i\tau\mathbb{Z} = \emptyset$ of theorem 2 is equivalent to the statement $0 \notin \sigma(A-B_\infty)$.

The estimate (20) for the nonlinearity F allows to reduce the problem of finding a nontrivial solution of the equation (18) to the problem of finding nontrivial critical points of a function defined on the following finite dimensional subspace $Z = PH \subset H$, where P is the projection

$$P = \int_{-\beta}^{\beta} dE_\lambda$$

onto the eigenspace of A belonging to the eigenvalues contained in $(-\beta,\beta)$; here E_λ is the spectral resolution of the selfadjoint operator A . We can assume that $\beta \notin \sigma(A)$.

Lemma 4: There are a function $a \in C^2(Z,R)$ *and an injective* C^1*-map* $u : Z \to H$ *satisfying* $u(0) = 0$ *and* $\operatorname{Im} u \subset \operatorname{dom}(A)$ *with the following properties:*

(i) $a(0) = 0$, $a'(0) = 0$ *and* $z \in Z$ *is a critical point of* a, *i.e.* $a'(z) = 0$, *if and only if* $u(z)$ *is a solution of the equation* $Au = F(u)$. a *is of the form:*

$$a(z) = \frac{1}{2} <Au(z),u(z)> - \phi(u(z)) \quad .$$

(ii) *If* $0 \notin \sigma(A-B_\infty)$, *then* a *satisfies the Palais-Smale condition.*

(iii) *The operators* B_0 *and* B_∞ *commute with the projection* P , *and there is a constant* $\delta > 0$ *such that*

$$\frac{1}{2} <(A-B_\infty)z,z>-\delta \le a(z) \le \frac{1}{2} <(A-B_\infty)z,z> + \delta$$

for every $z \in Z$. *Moreover*

$$a''(0) = (A-B_0)|Z.$$

Proof: The proof follows from [1], Proposition 2.1, Proposition 4.5, and Lemma 7.2, observing that we have the freedom to make β in the estimate (4) large.

It remains to find nontrivial critical points of the function a,which by the Lemma behaves like $< (A-B_0)z,z >$ in a neighborhood of 0 and like $< (A-B_\infty)z,z >$ in a neighborhood of ∞. A critical point is guaranteed by the following crucial lemma, which can be proved using the generalized Morse theory developed by C. Conley [5].

Lemma 5: *Assume* $0 \notin \sigma(A-B_\infty)$, *then the function* a *has a nontrivial critical point if*

(21) $$m^+((A-B_\infty)|Z) \notin [\, m^+(a''(0)) \, , \, m^+(a''(0)) + m^0(a''(0))] \, .$$

Proof: See Proposition 9.3 in [1] .

Observe that in the special case, where, in addition, $0 \notin \sigma(A-B_0)$ the above condition is simply

$$m^+((A-B_\infty)|Z) \neq m^+((A-B_0)|Z) \, .$$

Since $0 \notin \sigma(A-B_\infty)$ we have $m^-((A-B_\infty)|Z) = \dim Z - m^+((A-B_\infty)|Z)$, and therefore condition (21) es equivalent to

$$m^+((A-B_0)|Z) - m^+((A-B_\infty)|Z) > 0$$

or

$$m^-((A-B_0)|Z) - m^-((A-B_\infty)|Z) > 0 \, .$$

We shall show that the left hand sides of these two inequalities agree with $i^+(b_0,b_\infty,\tau)$ and $i^-(b_0,b_\infty,\tau)$, for $\tau = \frac{2\pi}{T}$. Indeed, if $E(\lambda)$ denotes the eigenspace of A for the eigenvalue $\lambda \in \sigma(A) = \tau\mathbb{Z}$,

$\tau = \frac{2\pi}{T}$, we find by Lemma 3:

$$E(\lambda)+E(-\lambda)=\{ \tfrac{2}{T} (\cos \lambda t)x_1+ \tfrac{2}{T}(\sin \lambda t)x_2,\ 0 \le t \le T \mid x_1,x_2 \in R^{2n}\} .$$

Let now $\lambda = j\tau \in \sigma(A)$, $j \ge 1$. The restriction of the operator $A-B_0$ (resp. $A - B_\infty$) onto the subspace $E(\lambda) + E(-\lambda) \subset H$ defines a quadratic form, which agrees with the quadratic form $Q(\mu,b)$ defined by (8), for $\mu = j\tau$ and $b = b_0$ (resp. $b = b_\infty$). Therefore, if we choose β so large that $\sigma(B_0)$ and $\sigma(B_\infty)$ are contained in $(-\beta,\beta)$ it follows from (11) that

(22)
$$\begin{aligned} m^+((A-B_0)|Z) - m^+((A-B_\infty)|Z) &= i^+(b_0,b_\infty,\tau) \\ m^-((A-B_0)|Z) - m^-((A-B_\infty)|Z) &= i^-(b_0,b_\infty,\tau) , \end{aligned}$$

where $\tau = \frac{2\pi}{T}$. Theorem 2 now follows by Lemma 5 and Lemma 4.

c) Remarks

Remark 1: There is a curious relation of the above existence theorem to the Poincaré-Birkhoff fixed point theorem. We consider a Hamiltonian system in 2 dimensions $h(t,x) = h(t+T,x)$ for $(t,x) \in R \times R^2$, $h \in C^2(R \times R^2)$, which satisfies

(23)
$$\begin{aligned} Jh'(t,x) &= \alpha^0 Jx + o(|x|) , \quad \text{as} \quad |x| \to 0 \\ Jh'(t,x) &= \alpha^\infty Jx + o(|x|) , \quad \text{as} \quad |x| \to \infty , \end{aligned}$$

for two real numbers $\alpha^0 \ne \alpha^\infty$, say $\alpha^0 < \alpha^\infty$. The two linear systems $\dot{x} = \alpha^0 Jx$ and $\dot{x} = \alpha^\infty Jx$ represent two harmonic oscillators with frequencies α^0 resp. α^∞ . One verifies easily that in this special case the conditions $i(b_0,b_\infty,\tau) > 0$ and $\sigma(Jb_\infty) \cap i\tau\mathbb{Z} = \emptyset$, $\tau = \frac{2\pi}{T}$, required in theorem 2, are equivalent to the following conditions: there is an integer $j \in \mathbb{Z}$, such that

$$\alpha^0 < j\tau < \alpha^\infty \quad , \tag{24}$$

and $\alpha^\infty \neq k\tau$, for every $k \in \mathbb{Z}$. We shall assume in the following only condition (24) which is always satisfied if $\alpha^0 < 0 < \alpha^\infty$. Introducing symplectic polar coordinates

$$x_1 + ix_2 = \sqrt{2R}\, e^{i\theta} \quad ,$$

the system $\dot{x} = Jh'(t,x)$ becomes

$$\begin{aligned} \dot{\theta} &= -\hat{H}_R(t,\theta,R) \\ \dot{R} &= +\hat{H}_\theta(t,\theta,R) \end{aligned} \tag{25}$$

with $\hat{H}(t,\theta,R) = H(t,x)$ periodic in θ of period 2π. The flow of (25) gives rise to measure preserving homeomorphisms of the annulus $0 \le \theta \le 2\pi$ $0 < R < \infty$. In the covering space $-\infty < \theta < \infty$, $R > 0$ we define the measure preserving map

$$\phi : (\theta_0, R_0) \to (\theta_1, R_1) = (\theta(T) - 2\pi j, R(T))$$

where $\theta(t)$ and $R(t)$ are solutions of the equations (25) having the initial conditions $\theta(0) = \theta_0$ and $R(0) = R_0$. Since $x = 0$ is an equilibrium point of $\dot{x} = Jh'(t,x)$, the "inner" boundary, $R_0 = 0$, remains invariant under ϕ, and on $R_0 = 0$

$$\theta_1 - \theta_0 = T\alpha^0 - 2\pi j = 2\pi \left(\frac{\alpha^0}{\tau} - j \right) < 0 \tag{26}$$

by our assumptions (23) and (24). If $R_0 \neq 0$ then the corresponding circle of radius R_0 is not necessarily invariant under ϕ, but if R_0 is sufficiently large we find in view of (23) and (24)

$$(27)\qquad \begin{aligned} \theta_1 - \theta_0 &= T\alpha^{\infty} - 2\pi j + 0\left(\frac{1}{R_0}\right) \\ &= 2\pi\left(\frac{\alpha^{\infty}}{\tau} - j\right) + 0\left(\frac{1}{R_0}\right) > 0 . \end{aligned}$$

Therefore on account of (26) and (27) the twist condition required in the Poincaré-Birkhoff fixed point theorem for a measure preserving homeomorphism of an annulus is satisfied, [3], and we conclude two fixed points for ϕ in $0 < R \leq R_0$ which by construction give rise to nontrivial T-periodic solutions. In order to prove this statement we did not require condition (5). Incidently it also follows by means of this fixed point theorem that ϕ has infinitely many periodic points, which correspond to infinitely many distinct periodic solutions having periods nT, $n \in \mathbb{N}$.

Summarizing we have seen that the condition $i > 0$ in theorem 2 corresponds to the twist condition required in the Poincaré-Birkhoff fixed point theorem. It has to be said that a genuine generalization of this fixed point theorem to symplectic mappings of higher dimensions has not been found. Of course, if additional assumptions are imposed in the interior such results do exist, see [8].

Remark 2: An existence statement similar to theorem 2 cannot be expected for general asymptotically linear equations, as the following example shows:

$$\dot{x} = -y\theta + (1-\theta)x$$

$$\dot{y} = x\theta + (1-\theta)y ,$$

where $\theta = \theta(r^2)$, $r^2 = x^2 + y^2$, and where θ is a function with compact support $\theta(0) = 1$, $\dot{\theta}(0) = 0$ and $\theta(r^2) < 1$ for $r^2 \neq 0$. For every solution of this equation:

$$\frac{d}{dt} (x^2+y^2) = \frac{1}{2} (1-\theta)(x^2+y^2) ,$$

which is greater than 0 , if $x^2+y^2 \neq 0$. Therefore $x = y = 0$ is the only periodic solution.

Remark 3: The existence statements so far require the linear Hamiltonian systems $Jb_0 x$ and $Jb_\infty x$ to be independent of t . Although the general time dependent case is not worked out yet, we give a special result. Assume that $b_0(t)$ depends periodically on t with period T and assume that $b_\infty = \alpha_\infty 1$ with a constant $\alpha_\infty \in R$. We denote by $\sigma(B_0)$ the spectrum of the bounded operator $B_0 u(t) = B_0(t)\, u(t)$ in H . It can be shown that if there is an integer $j \in \mathbb{Z}$ such that either

$$\sigma(B_0) < j\tau < \alpha_\infty$$

or

$$\alpha_\infty < j\tau < \sigma(B_0) ,$$

where $\tau = \frac{2\pi}{T}$, then there is a nontrivial T-periodic solution of $\dot{x} = Jh'(t,x)$, provided $\alpha_\infty \notin \tau\mathbb{Z}$. In fact this statement is an immediate application of Corollary 9.5 of [1].

4. The Hamiltonian is even

In the following we shall restrict the class of Hamiltonian systems under considerations even further, in order to prove multiplicity results. We do not only make assumptions on the systems "at 0" and "at ∞" but also in the "interior". We first assume the function h to be even in the x-variable and prove the following:

Theorem 3: Let h *be as in theorem 2 and assume, in addition, that*

(28) $$h(t,x) = h(t,-x)$$

for all $(t,x) \in R \times R^{2n}$ *. Then if* $i = i(b_0, b_\infty, \frac{2\pi}{T}) > 0$, *there are at least* i *nontrivial pairs* $(x(t), -x(t))$ *of periodic solutions having period* T *, provided* $\sigma(Jb_\infty) \cap i\frac{2\pi}{T}\mathbb{Z} = \emptyset$.

Proof: It follows from (28) that the potential operator F in (16) is odd, and we conclude from [1] Proposition 3.2, that the function a of Lemma 4 is an even function:

(29) $$a(z) = a(-z) .$$

Following D.C. Clark [4] closely we shall apply minimax arguments based on the genus in order to find critical points of a . We recall, that the genus g(.) is defined for closed symmetric subsets C of $Z \setminus \{0\}$, and $g(C)$ is the least integer k such that there exists an odd continuous map from C into $R^k \setminus \{0\}$; we set $g(\emptyset) = 0$. This genus has the following properties (see for example [4]):

(g1) $C_1 \subset C_2$ implies $g(C_1) \leq g(C_2)$

(g2) If $C_1 \cap C_2 = \emptyset$, then $g(C_1 \cup C_2) = \max \{g(C_1), g(C_2)\}$

(g3) If there exists an odd homeomorphism of C onto the k-sphere, then $g(C) = k + 1$.

(g4) If Z_m is an m-dimensional subspace of Z and $C \cap Z_m = \emptyset$ then $g(C) \leq m$.

For the function a in Lemma 4 which now satisfies also (29), we define:

$$c_k(a) := \inf_{g(C) \geq k} \sup a(C) . \tag{30}$$

Clearly, $c_1(a) \leq c_2(a) \leq \ldots$, and if

$$-\infty < c := c_n(a) = c_{n+1}(a) = \ldots = c_{n+k}(a) < 0 \tag{31}$$

then $g(K_c) \geq k + 1$, where $K_c := \{z \in Z \mid a(z) = c \text{ and } a'(z) = 0\}$. This assertion is theorem 8 in [4]. The following Lemma is also implicitly contained in [4].

<u>*Lemma 6:*</u> *Let* $\psi \in C^2(Z,R)$ *be an even function satisfying the Palais-Smale condition, and* $\psi(0) = 0$. *Assume that*

(i) *There is an r-dimensional subspace* Z^- *of* Z *and a positive number* ρ *such that* $\psi(z) < 0$ *for all* $z \in Z^-$ *satisfying* $|z| = \rho$.

(ii) *there is an* s*-dimensional subspace* Z^+ *of* Z*, such that* ψ *is bounded below on* Z^+

(iii) $r + s - \dim Z > 0$.

Then $-\infty < c_k(\psi) < 0$ *for all* k *satisfying* $\dim Z - s < k \le r$ *, and* ψ *has at least* $s + r - \dim Z$ *nonzero pairs* $(z,-z)$ *of critical points.*

Proof: Property (g3) and condition (i) imply that there is a $C \subset Z \setminus \{0\}$ with $g(C) = r$ such that $\sup \psi(C) < 0$ and hence $c_r(\psi) < 0$. If $\tau \in R$ we introduce the notation $\psi_\tau = \psi^{-1}((-\infty,\tau])$. Due to condition (ii) there is $\tau \in R$ with $\psi_\tau \cap Z^+ = \emptyset$, hence by (g4), $g(\psi_\tau) \le \dim Z - s$. Therefore if j satisfies $\dim Z - s < j \le \dim Z$, and if $g(C) \ge j$ (such a set exists by (g3)), then $\sup \psi(C) \ge \tau$ and hence $c_j(\psi) \ge \tau > -\infty$. The Lemma now follows from (iii) on account of (31).

To conclude the proof of Theorem 3 we denote by E_α^-, E_α^0, E_α^+ , $\alpha = 0,\infty$ the subspaces on which the symmetric operator $(A-B_\alpha)|Z$, $\alpha = 0,\infty$, is negative definite, zero, and positive definite respectively. By assumption $\sigma(Jb_\infty) \cap i\tau\mathbb{Z} = \emptyset$, hence $0 \notin \sigma(A-B_\infty)$ and so $E_\infty^- + E_\infty^+ = Z$. Setting $r = \dim E_0^-$ and $s = \dim Z - \dim E_\infty^-$ we find by (11)

$$\begin{aligned} r + s - \dim Z &= \dim E_0^- - \dim E_\infty^- \\ &= m^-((A-B_0)|Z) - m^-((A-B_\infty)|Z) \\ &= i^+(b_0,b_\infty,\tau) \ . \end{aligned}$$

Assuming $i^- > 0$, the assumptions of Lemma 6 are satisfied for the function $\psi = a$ with the subspaces $Z^- = E_0^-$ and $Z^+ = E_\infty^+$. This follows from Lemma 4 (iii). We conclude that a has i^- pairs of nontrivial critical points, which by Lemma 4(i) correspond to pairs $(x(t),-x(t))$ of periodic solutions. Similarly, if $i^+ > 0$ we consider $\psi = -a$ and take $Z^- = E_0^+$, $Z^+ = E_\infty^-$. Setting $r = \dim E_0^+$ and $s = \dim E_\infty^-$ we find this time

$$\begin{aligned} r + s - \dim Z &= \dim E_0^+ - \dim E_\infty^+ \\ &= m^+((A-B_0)|Z) - m^+((A-B_\infty)|Z) \\ &= i^+(b_0,b_\infty,\tau) . \end{aligned}$$

Consequently, by Lemma 6, there are i^+ pairs of nontrivial periodic solutions and Theorem 3 is proved.

From now on we shall assume the Hamiltonian function h to be time independent. In this case there is no "natural" period for the sought periodic solutions given. The function h is an integral and we could ask for periodic solutions on a given integral surface. But instead of prescribing this integral we look for periodic solutions having a prescribed period T. Such periods are described by the next theorem.

Theorem 4: *Assume* h *satisfies the assumptions of theorem 2, and assume, in addition,* h *even and independent of* t.

Let T *be any positive number satisfying* $i(b_0,b_\infty,\frac{2\pi}{T}) > 0$.

Then $\frac{i}{2}$ *is a lower bound for the number of distinct, nontrivial pairs* $(x(t),-x(t))$ *of T-periodic solutions of the equation* $\dot{x} = Jh'(x)$, *provided* $\sigma(Jb_\infty) \cap i\frac{2\pi}{T}\mathbb{Z} = \emptyset$.

Remark: Theorem 4 does not claim that the periodic solutions found are not constant. Additional singular points of the Hamiltonian vector field could be among these periodic solutions. It is good to know that there is always an additional singular point if $m^-(b_\infty) \notin [m^-(b_0), m^-(b_0) + m^0(b_0)]$ (see [1] Proposition 12.5).

It turns out in the proof of theorem 4, that under further artifical restrictions on h , the periodic solutions are nonconstant. In fact, we shall show that if h satisfies the assumptions of theorem 4 and if, in addition,

$$h(x) \geq 0 \quad (\text{resp. } h(x) \leq 0) \quad , x \in R^{2n} \quad ,$$

then if $i^+ = i^+(b_0, b_\infty, \frac{2\pi}{T}) > 0$ (resp. $i^- > 0$), there are at least $i^+/2$ (resp. $i^-/2$) nonconstant pairs of periodic solutions having period T .

Proof: Since the Hamiltonian vector field is independent of the time, with every solution $x(t)$ also $x(t+s)$ is a solution for every fixed $s \in R$. As a consequence, the equation (18) is invariant under a unitary representation of $S^1 = \{z \in C \mid |z| = 1\}$ in H . In fact, denote by $E_T \subset H$ the dense subspace of H consisting of all $u \in H$ such that u is the restriction of a T-periodic function $\hat{u} \in C(R, R^{2n})$, and define for $u \in E_T$

(32) $$U_\sigma u(t) = \hat{u}(t+s) \quad \text{for all} \quad t \in R$$

where

$$\sigma = e^{i\tau s} \in S^1 \quad , s \in R.$$

Since U_σ defines a continuous linear operator from E_T to H , it has a continuous extension which we denote again by U_σ . Clearly $\sigma \to U_\sigma : S^1 \to \mathcal{L}(H)$ is a strongly continuous unitary representation of the circle group S^1 . Moreover $U_\sigma A \subset AU_\sigma$ and $F \circ U_\sigma = U_\sigma \circ F$ for $\sigma \in S^1$. Consequently, if $u \in \text{dom}(A)$ is a solution of (18) i.e. of $Au = F(u)$, then every element of the orbit $\mathcal{O}(u)$ is also a solution, where

$$\mathcal{O}(u) = \{U_\sigma u \mid \sigma \in S^1\} \quad .$$

We now claim that distinct orbits $\mathcal{O}(u)$ consisting of solutions of (18) correspond to geometrically distinct T-periodic solutions of the equation $\dot{x} = Jh'(x)$. Here we call two nonconstant solutions of an ordinary differential equation $\dot{x} = f(x)$ geometrically not distinct, if one is a reparametrization of the other, that is,if there is a C^1-diffeomorphism $\phi : \mathbb{R} \to \mathbb{R}$, so that $x_1(t) = x_2(\phi(t))$. Suppose that ϕ is such a reparametrization. Then we claim $\phi(t) = t+s$. In fact, since $\dot{x}_1 = f(x_1)$ and $\dot{x}_2 = f(x_2)$ we conclude that $f(x_1(t))(\dot{\phi}(t)-1) = 0$ for all t . By assumption the solution x_1 is not a constant, hence $f(x_1(t)) \neq 0$ for every $t \in R$, and therefore $\dot{\phi}(t) = 1$ for $t \in R$ which proves the claim.

Since F and A commute with the unitary representation, the function a of Lemma 4 is invariant, i.e.

(33) $$a(U_\sigma z) = a(z)\ , \qquad \sigma \in S^1$$

for all $z \in Z$. This follows from [1] Proposition 3.2. Therefore if z^* is a critical point of a, then its orbit $\mathcal{O}(z^*) = \{U_\sigma z^* \mid \sigma \in S^1\}$ consists of critical points of a. But by Lemma 4 and the previous considerations, different critical orbits give rise to different periodic orbits.

Assume now $i = i(b_0, b_\infty, \frac{2\pi}{T}) > 0$, then it has been shown in the proof of Theorem 3, that for some integer ℓ the critical levels of ψ satisfy

$$-\infty < c_{\ell+1}(\psi) \leq \ldots \leq c_{\ell+i}(\psi) < 0\ ,$$

where ψ is either equal to a or equal to -a . Let m be equal to $\frac{i}{2}$ if i is even and equal to $[\frac{i}{2}] + 1$ if i is odd. Then if m of the critical levels are distinct, there are at least m distinct pairs $\mathcal{O}(z) \cup \mathcal{O}(-z)$ of critical orbits of a, giving rise to m distinct pairs $(x(t), -x(t))$ of T-periodic solutions and Theorem 4 is proved in this case. Otherwise there is an integer j such that

$$c = c_{j+1}(\psi) = c_{j+2}(\psi) = c_{j+3}(\psi) < 0$$

and $g(K_c(\psi)) \geq 3$ on account of (31). But for every $z \in Z,\ z \neq 0$, $g(\mathcal{O}(z) \cup \mathcal{O}(-z)) \leq 2$ (see [2] Lemma 6.1). Therefore by property (g2) of the genus we conclude that there are infinitely many distinct orbits in $K_c(\psi)$. This finishes the proof of Theorem 4.

It remains to prove the statement in the remark following Theorem 4. We need a lemma. Recall the notation of section 3b and Lemma 4.

Lemma 7: *Assume* h *to be independent of* t *. Then* $u(z^*) \in \mathrm{Ker}(A)$ *if and only if* $z^* \in \mathrm{Ker}\ a$ *, and in this case* $u(z^*) = z^*$ *. Moreover, if* $z^* \in \mathrm{Ker}\ A$ *, then* $z^* \in R^{2n}$ *(by Lemma 3) and*

$$a(z^*) = -Th(z^*).$$

Proof: Assume $u(z^*) \in \mathrm{Ker}(A)$. Since $\mathrm{Ker}(A) \subset Z$ it follows from the definition of the map u (see [1] Section 3) that $u(z^*) = z^*$. Conversely, if $z^* \in \mathrm{Ker}(A)$, then by Lemma 3, $z^* \in R^{2n}$ is a sonstant vector, hence $F(z^*) \in \mathrm{Ker}(A)$, where F is defined by (16) (note h is independent of t). It then follows from Proposition 2.1 in [1] that $u(z^*) = z^*$ (since $R(\ker a) \doteq S(\mathrm{Ker}\ A) = \{0\}$, in the notation of that proposition). By the representation $a(z) = \frac{1}{2}\langle Au(z),u(z)\rangle-\phi(u(z))$ of Lemma 4 i, we conclude for $z^* \in \mathrm{Ker}\ A$, in view of the definition (17) for ϕ :

$$a(z^*) = -\phi(z^*) = -\int_0^T h(z^*)\, dt = -Th(z^*) ,$$

and the Lemma is proved.

In order to prove the statement in the remark following Theorem 4 we shall assume $i^-(b_0,b_\infty,\tau) > 0$ and $h(x) \le 0$ for every

$x \in R^{2n}$. Assume that one of the solutions found by Theorem 4 is a constant solution. By Lemma 7 it is represented by $u(z^*)=z^* \in Ker(A)$, z^* a critical point of a . Since in view of Lemma 6, the critical values of a are all negative, we have $a(z^*) < 0$. On the other hand, since $h(z^*) \leq 0$, we find by Lemma 7, $a(z^*) = -Th(z^*) \geq 0$, which is a contradiction. Hence the solutions found are nonconstant. The other case is proved similarly.

5. Convex Hamiltonians

For a strictly convex Hamiltonian function h , the system $\dot{x} = Jh'(x)$ has many periodic solutions. For instance, every "energy surface" $\{x \in R^{2n} \mid h(x) = \text{const}\}$ carries at least one nonconstant periodic solution. This has been proved by P. Rabinowitz [9] and A. Weinstein [10]. In contrast to these solutions having prescribed "energy" we look for nonconstant periodic solutions having prescribed periods and prove the following multiplicity result.

Theorem 5: Suppose $h \in C^2(R^{2n})$ *is strictly convex with bounded second derivative. Assume, in addition, that*

$$Jh'(x) = Jb_0 x + o(|x|) , \quad |x| \to 0$$

$$Jh'(x) = Jb_\infty x + o(|x|) , \quad |x| \to \infty$$

for two symmetric positive definite matrices $b_0, b_\infty \in \mathcal{L}(R^{2n})$.

Then if T *is a positive number satisfying* $i=i(b_0,b_\infty,\frac{2\pi}{T}) > 0$, *there are at least* $i/2$ *nonconstant -periodic solutions of the* *system* $\dot{x} = Jh'(x)$, *provided* $\sigma(Jb_\infty) \cap i\,\frac{2\pi}{T}\,\mathbb{Z} = \emptyset$.

<u>Remarks:</u> We first observe that since $b_0, b_\infty > 0$, the index i is an even integer. In fact Jb_0 and Jb_∞ are symplectically diagonalizable, and $S^\infty = \{i\alpha_k^\infty \mid \alpha_k^\infty > 0 \ , \ k = 1,2,\ldots,n\}$, $S^0 = \{i\alpha_k^0 \mid \alpha_k^0 > 0 \ , \ k = 1,2, \ldots,n\}$. Hence by Lemma 2:

$$i_+(b_0,b_\infty,\tau) = 2 \sum_{j=1}^{\infty} \left([S^\infty \mid \alpha^\infty \geq j\tau] - [S^0 \mid \alpha^0 \geq j\tau] \right) \tag{33}$$

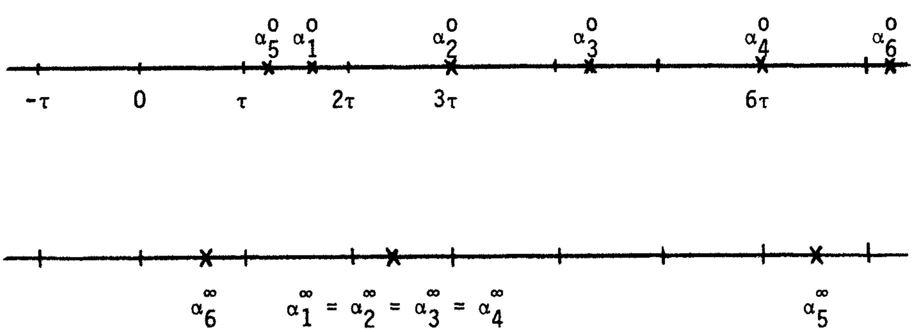
$$i_-(b_0,b_\infty,\tau) = 2 \sum_{j=1}^{\infty} \left([S^0 \mid \alpha^0 > j\tau] - [S^\infty \mid \alpha^\infty > j\tau] \right) . \tag{34}$$

This can be rewritten as

$$i_+ = 2 \sum_{k=1}^{n} \left(\sum_{\alpha_k^0 < \tau j \leq \alpha_k^\infty} 1 \ - \sum_{\alpha_k^\infty < j\tau \leq \alpha_k^0} 1 \right)$$

$$i_- = 2 \sum_{k=1}^{n} \left(\sum_{\alpha_k^\infty \leq \tau j < \alpha_k^0} 1 \ - \sum_{\alpha_k^0 \leq j\tau < \alpha_k^\infty} 1 \right)$$

As an illustration we consider an example in R^{2n} , $n = 6$. We give a possible distribution of the normal modes of the two linear systems at 0 and at ∞ by the following diagram:



By the above formulars for i_+ and i_- one finds $\frac{1}{2}i_+ = 1+5-1-2-4-7$ $= -8 < 0$ and $\frac{1}{2}i_- = 2+3+7-1-5 = 6$. Hence $\frac{1}{2}i = \frac{1}{2}i_- = 6$.

It should be said that "in general" $\max\{i^+, i^-\} = i \to \infty$ as $\tau \to 0$. For instance, let

$$\ell_+ := \sum_{\substack{1 \le k \le n \\ \alpha_k^\infty > \alpha_k^0}} (\alpha_k^\infty - \alpha_k^0) \ , \ \ell_- := \sum_{\substack{1 \le k \le n \\ \alpha_k^0 > \alpha_k^\infty}} (\alpha_k^0 - \alpha_k^\infty) \ ,$$

then clearly $i_+ \to \infty$ and $i_- \to -\infty$ as $\tau \to 0$ provided $\ell_+ > \ell_-$.

The convexity assumption on h in Theorem 5 can be replaced by other conditions. For example the following statement will follow from the proof of Theorem 5: Assume h satisfies the requirements of Theorem 5 with the convexity assumption replaced by the assumption $h(x) \ge 0$ for every $x \in R^{2n}$. If then $i_+(b_0, b_\infty, \frac{2\pi}{T}) > 0$, there are at least $\frac{1}{2} i_+$ nonconstant T-periodic solutions provided $\sigma(Jb_\infty) \cap i\frac{2\pi}{T}\mathbb{Z} = \emptyset$. Here the index $i_+(b_0, b_\infty, \frac{2\pi}{T})$ is given by (33). For example the integer i^+ is positive if the two systems at 0 and at ∞ are separated in the sense that there is an integer $j \ge 1$ such that $\alpha_k^0 < j\tau < \alpha_\ell^\infty$ for every $k, \ell = 1,2,\dots,n$; in fact we then find $i_+ \ge 2n$.

<u>Proof of Theorem 5:</u> We have to establish critical points of the function $a \in C^2(Z)$ of Lemma 4. Since h does not depend on t , the function a is invariant under the unitary representation (32) of S^1, i.e. $a(U_\sigma z) = a(z)$, $\sigma \in S^1$ and $z \in Z$. Analogous to the proof of

Theorem 4 we shall apply minimax and maximin arguments in order to find critical points of a . This time however the function a is not even and we replace the genus, which is suited dealing with functionals which are invariant under a $\mathbb{Z}_2$-action, by the Fadell-Rabinowitz index for a special S^1-action. This cohomological index is introduced in connection with actions of compact Lie groups in [6]. In order to realize the set up of [6] we shall complexify the even dimensional subspace $Z \subset H$ as follows: recalling Lemma 3 we take the eigenvectors $u_{\lambda,k}(t) = e^{t\lambda J} e_k$ of the eigenvalue $\lambda \in \tau\mathbb{Z}$ of A . Let $\sigma(A) \cap (-\beta,\beta) = \{-\tau m, -\tau(m-1),\dots,-\tau,0,\tau,\dots,\tau m\}$. The linear maps $R^2 \to C$, $x_1 u_{\lambda k} + x_2 u_{\lambda,k+n} \to x_1 + ix_2$ define an isomorphism of the eigenspace E_λ onto C^n , hence an isomorphism of Z onto $C^n \times \dots \times C^n$ (2m+1-times). Consequently we may assume from now on that

$$Z = C^n \times \dots \times C^n \text{ (2m+1-times)} . \tag{35}$$

The S^1-action U_σ , given by (32), is by this isomorphism carried over into the following group action of S^1 on Z :

$$\sigma \cdot \zeta = (\sigma^{-m} \zeta_{-m}, \sigma^{-(m-1)}\zeta_{-(m-1)},\dots,\zeta_0, \dots, \sigma^m \zeta_m) \quad , \tag{36}$$

where $\sigma \in S^1 = \{\lambda \in C \mid |\lambda| = 1\}$ and where $\zeta = (\zeta_{-m},\dots,\zeta_m) \in Z$. The fixed point set of this action is the subspace $N_0 = \mathrm{Ker}(A)$:

$$N_0 = \{(0,\dots,0,\zeta_0,0,\dots,0) \mid \zeta_0 \in C^n\} \quad . \tag{37}$$

With $\mathcal{F}$ we denote in the following the family of subsets $X \subset Z \setminus \{0\}$, which are invariant under the S^1-action (36), i.e. $S^1 \cdot X = X$. Then the following crucial result is taken from [6].

Lemma 8: *Let* Z , $N_0 \subset Z$, *and the* S^1*-action be as introduced above. Then there is a map* $\gamma : \mathcal{F} \to \mathbb{N} \cup \{\infty\}$ *satisfying:*

(γ0) *If* $X = \emptyset$ *then* $\gamma(X) = 0$, *if* $X \neq \emptyset$ *then* $\gamma(X) \geq 1$.

(γ1) *If* $X, Y \in \mathcal{F}$ *then* $\gamma(X \cup Y) \leq \gamma(X) + \gamma(Y)$.

(γ2) *If* $X, Y \in \mathcal{F}$ *and if* $\phi : X \to Y$ *is a continuous and equivariant map from* X *into* Y , *then* $\gamma(X) \leq \gamma(Y)$. *Equality holds in particular if* ϕ *is a homeomorphism.*

(γ3) *If* $X \in \mathcal{F}$ *is compact, then there is a compact neighborhood* $N \in \mathcal{F}$, *such that* $\gamma(X) = \gamma(N)$.

(γ4) *If* $X \cap N_0 = \emptyset$ *then* $\gamma(X) = \infty$.
If $X \cap N_0 = \emptyset$ *and* X *compact, then* $\gamma(X) < \infty$.

(γ5) *If* $E \subset Z$ *is an invariant linear subspace with* $\dim_{\mathbb{C}} E = k$, *and satisfying* $E \cap N_0 = \{0\}$, *then* $\gamma(E \cap S_R) = k$ *for every* $R > 0$.

(γ6) *If* $1 < \gamma(X) < \infty$, *then the orbit space* X/S^1 *is an infinite set.*

For each $n \in \mathbb{N}$ we define the family of subsets $\Gamma_n = \{X \in \ | n \leq \gamma(X) < \infty\}$ and define for a function ψ on Z the minimax levels $c_n(\psi)$ by

$$c_n(\psi) = \inf_{X \in \Gamma_n} \sup \psi(X) ,$$

provided, of course, that $\Gamma_n \neq \emptyset$. It is obvious that $c_1(\psi) \le c_2(\psi) \le \ldots$ The basic properties of these minimax levels are contained in the following lemma.

Lemma 9: *Assume* $\psi \in C^2(Z,R)$ *is invariant under the special* S^1*-action* (35) *and satisfies the Palais-Smale condition. Suppose that, for some integers* n *and* k :

$$-\infty < c = c_n(\psi) = c_{n+1}(\psi) = \ldots = c_{n+k}(\psi) < 0 .$$

Then, if $\gamma(K_c) < \infty$, *we have* $\gamma(K_c) \ge k + 1$, *where* $K_c = K_c(\psi) = \{\zeta \in Z \mid \psi(\zeta) = c$ *and* $\psi'(\zeta) = 0\}$ *is the critical set of* ψ *on the level* c .

In view of Lemma 7, the proof follows almost literally the proof of ([4], Theorem 8), see also ([2], Lemma 4.5). The next statement is the analogon of Lemma 6 for the special S^1-action instead of the Z_2-action.

Lemma 10: *Assume* $\psi \in C^2(Z,R)$ *with* $\psi(0) = 0$ *is invariant under the* S^1*-action* (32) *and satisfying the Palais-Smale condition. Suppose that*

(i) *There is an invariant subspace* $Z^- \subset Z$, *with* $Z^- \cap N_0 = \{0\}$ *and* $\dim_C Z^- = r$, *such that* $\psi(z) < 0$ *for all* $z \in Z^-$ *satisfying* $|\zeta| = \rho$, *some fixed* $\rho > 0$.

(ii) *There is an invariant subspace* $Z^+ \subset Z$ *with* $N_o \subset Z^+$ *and* $\dim_C Z^+ = s$, *such that* ψ *is bounded below on* Z^+.

(iii) $r + s > \dim_C Z$.

Then $-\infty < c_k(\psi) < 0$ *for all* $k \in \mathbb{N}$ *satisfying* $\dim_C Z - s < k \le r$, *and* ψ *has at least* $s + r - \dim_C Z$ *distinct nonzero critical orbits provided* $K_{c_k}(\psi) \cap N_o = \emptyset$.

Proof: For $a \in R$ we denote by ψ_a the set $\psi^{-1}(-\infty,a] \subset Z$, and by S we denote the unit sphere in Z . By assumption (i) there is a positive $\sigma > 0$ such that $\psi(\zeta) \le -\sigma < 0$ for $\zeta \in \rho S \cap Z^- =: B_\rho$, hence $B_\rho \subset \psi_{-\sigma}$ and by $(\gamma 2)$ of Lemma 8, $\gamma(B_\rho) \le \gamma(\psi_{-\sigma})$. Since $N_o \cap Z^- = \{0\}$, we have $\gamma(B_\rho) = r$ by $(\gamma 5)$ and therefore $c_r(\psi) \le -\sigma < 0$ and so $c_k(\psi) < 0$ for $k \le r$. In view of assumption (ii) we can pick $\tau < 0$ with $\psi_\tau \cap Z^+ = \emptyset$. If $\pi : Z \to (Z^+)^\perp$ denotes the orthogonal projection, then $\pi : \psi_\tau \to (Z^+)^\perp \setminus \{0\}$ is continuous and equivariant, hence by $(\gamma 2)$ and $(\gamma 5)$ we find $\gamma(\psi_\tau) \le \dim_C (Z^+)^\perp = \dim_C Z - s$, since $N_o \cap (Z^+)^\perp = \{0\}$. Let $j > \dim_C Z - s$ and assume there is $X \in \mathcal{F}$ such that $j \le \gamma(X) < \infty$, then $\sup \psi(X) \ge \tau$. If, in addition, $j \le r$, then there is indeed such a set in view of the assumptions (i), (iii) and on account of $(\gamma 5)$. Consequently, $c_j(\psi) \ge \tau > -\infty$ for $\dim_C Z - s <$ $< j \le r$ and the statement follows from Lemma 9, in case the levels $c_k(\psi)$ are different and $K_{c_k} \cap N_o = \emptyset$. If, however, two or more of these levels coincide, say they are equal to c , then by $(\gamma 6)$ the orbit space K_c/S^1 is an infinite set. This finishes the proof of Lemma 10.

In order to prove Theorem 5 we first assume $i^+(b_0,b_\infty,\frac{2\pi}{T}) > 0$ and apply Lemma 10 to the function $\psi = -a$. Let E_α^-, E_α^0, E_α^+, $\alpha = 0,\infty$, be the subspaces on which $(B_\alpha - A)|_Z$ is negative, zero, and positive, respectively. These subspaces are clearly invariant under the S^1-action since $A-B_\alpha$ commutes with the action. Set $Z^- = E_0^-$ and $Z^+ = E_\infty^+$, then, by Lemma 4 iii, $N_0 \cap Z^- = \{0\}$ and $N_0 \subset Z^+$ since by assumption, $b_0,b_\infty > 0$. Moreover, with $r = \dim_C Z^-$ and $s = \dim_C Z^+$,

$$r + s - \dim_C Z = \dim_C E_0^- + \dim_C E_\infty^+ - \dim_C Z$$

$$= \dim_C E_0^- - \dim_C E_\infty^-$$

$$= \frac{1}{2}\, i^+(b_0,b_\infty,\frac{2\pi}{T}) > 0 \ .$$

Consequently the assumptions of Lemma 10 are met in view of Lemma 4 (iii), and we conclude that $-\infty < c_k(\psi) < 0$ for all $k \in \mathbb{N}$ satisfying $\dim_C Z - s < k \le r$. We shall show that $K_{c_k}(\psi) \cap N_0 = \emptyset$. Assume $z^* \in N_0 \cap K_{c_k}(\psi)$, then in particular $z^* \in \mathrm{Ker}(A)$, hence by Lemma 7, $u(z^*) = z^* \in \mathrm{Ker}(A)$ and $u(z^*)$ is a constant periodic solution, hence a singular point of the vector field $Jh'(x)$. But since h is strictly convex, the only singular point of $Jh'(x)$ is $x = 0$, hence $u(z^*) = z^* = 0$ and $a(z^*) = 0$ in contradiction to $c_k(\psi) < 0$. Therefore $K_{c_k}(\psi) \cap N_0 = \emptyset$ as claimed, and we conclude, that there are at least $\frac{1}{2}\, i^+$ nonzero critical orbits of a, hence at least $\frac{1}{2}\, i^+$ nonconstant T-periodic solutions of $\dot{x} = Jh'(x)$.

If, on the other hand, $i^-(b_0,b_\infty,\frac{2\pi}{T}) > 0$, one proceeds similarly. This time, however, one considers the "maximin" levels:

$$\hat{c}_k(\psi) = \sup_{X \in \Gamma_n} \inf \psi(X) .$$

The statement then follows from the dual version of Lemma 10 applied to $\psi = -a$. This finishes the proof of Theorem 5.

Remark: In Theorem 5 the technical difficulty caused by the fixed point set $N_o = \mathrm{Ker}(A)$ of the S^1-action can be avoided under the additional assumption $0 < \alpha \leq h''(x) \leq \beta < \infty$. In fact in this case we can define the subspace $Z = PH \subset H$ differently, namely by

$$P = \int_{-\beta}^{\varepsilon} dE_\lambda + \int_{\varepsilon}^{\beta} dE_\lambda \quad ,$$

for some $0 < \varepsilon < \min\{\tau,\alpha\}$. It then follows that $Z \cap N_o = \emptyset$ and the above S^1-action on Z is now fixed point free, which simplifies the proof considerably. The abstract results in [1] still hold true with the projections P_- and P_+ in there given by

$$P_- = \int_{-\infty}^{-\beta} dE_\lambda + P_o \quad , \quad P_+ = \int_{\beta}^{\infty} dE_\lambda \quad ,$$

P_o being the projection onto $\mathrm{Ker}(A)$.

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