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EXISTENCE OF THE DISPLACEMENTS FIELD FOR AN ELASTO-PLASTIC BODY SUBJECT TO HENCKY'S LAW AND VON MISES YIELD CONDITION

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We give "necessary" and sufficient conditions on body and traction forces for the existence of the displacements field for an elasto-plastic body subject to Hencky's law and Von Mises yield condition.

Let Ω be a bounded domain in $\mathbb{R}^3$ and let $u:\Omega \longrightarrow \mathbb{R}^3$ represent the displacements field of a plastic body occupying the domain Ω in unstrained position, then the deformation energy of the body, assuming the Von Mises yield condition and Hencky's law hold (see [3], [15]), is

$$\int_\Omega \phi(\varepsilon^D(u)) + \frac{K_0}{2}\int_\Omega (\operatorname{div} u(x))^2 dx$$

where

$$\phi(\varepsilon^D(u)) \begin{cases} \frac{1}{2}|\varepsilon^D(u)|^2 & \text{if} \quad |\varepsilon^D(u)| \leq 1 \\ |\varepsilon^D(u)| - \frac{1}{2} & \text{if} \quad |\varepsilon^D(u)| \geq 1 \end{cases}$$

and

$$\varepsilon^D(u) = \varepsilon(u) - \frac{1}{3}\operatorname{trace}(\varepsilon(u))\, I$$

is the deviator of the deformation tensor $\varepsilon(u)$ whose

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components are

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\right)$$

We shall be concerned in this paper with the problem of the existence for the field of displacements u of a plastic body subject to body forces f in Ω, to a traction F on some part Γ_N of the boundary (Neumann conditions) and with a prescribed value g for the displacement (Dirichlet conditions) on some other part Γ_D of the boundary. We are led then to the problem

$$(P_1) \quad \begin{cases} \text{minimize the functional} \\ \mathcal{F}(u) = \int_\Omega \phi(\varepsilon^D(u)) + \frac{K_0}{2}\int_\Omega (\operatorname{div} u)^2 + \int_\Omega f u + \int_{\Gamma_N} F u \\ u = g \quad \text{on } \Gamma_D \end{cases}$$

The analogy between problem (P_1) and the problem of finding graphs of prescribed mean curvature

$$\begin{cases} \int_\Omega F(\nabla v) + \int_\Omega f v + \int_{\Gamma_N} \alpha v \longrightarrow \inf \quad , \quad F(p) = \sqrt{1+p^2} \\ \\ v = \psi \quad \text{on } \Gamma_D \quad , \quad v \in BV(\Omega) \end{cases}$$

or, more generally, $|\nabla v| \leq F(\nabla v) \leq a|\nabla v| + b$ considered for example in [9],[7], is manifest. Therefore one is led to use the direct method of calculus of variations, looking for a solution to problem (P_1) in a suitable space $P(\Omega)$ where the functional $\mathcal{F}(u)$ is coercive and lower semicontinuous, and where the minimizing sequences are relatively compact. Following the analogy, one could try to work in the space of the functions u whose first derivatives are measures, and more precisely in the space

$$\tilde{P}(\Omega) = BV(\Omega, \mathbb{R}^3) \cap \{u \mid \operatorname{div} u \in L^2(\Omega)\}$$

Unfortunately, no Korn's inequality is available on $H^{1,1}$ see [12], therefore the functional in (P_1) is not coercive on $\widetilde{P}(\Omega)$.

In fact, as suggested in [17], [18], we shall look for a minimum point for problem (P_1) in the space

$$P(\Omega) = \left\{ u \in L^1(\Omega, \mathbb{R}^3) \,\middle|\, \operatorname{div} u \in L^2(\Omega),\ \varepsilon_{ij}(u) \text{ is a bounded measure } \forall\, i,j = 1,2,3 \right\}$$

Our methods will be very close to those used in [7], [9] [8] .

We refer to [18] for an approach to problem (P_1) by duality methods and limit analysis.

The paper is divided into three sections.

In section 1 we collect some properties of the space $BD(\Omega)$ of functions of bounded deformation; this space has been introduced in [12], [17], [20]. Our exposition will parallel closely the theory of BV functions [2], [10] so it will be somewhat different from the quoted ones. A comprehensive reference is [11], so we shall not prove the results proved there.

In section 2 we shall give a semicontinuous extension of the functional $\int_\Omega \phi(\varepsilon^D(u))$ to the space $BD(\Omega)$ following [7] and we shall relax the Dirichlet boundary condition following [8], [9], [7]. We prove then that the original functional and the relaxed one have the same infimum and we give a "necessary" and sufficient condition (theorems 2.4 , 2.5) on the forces f , F for the existence of a generalized solution to our problem.

We note that, as it is mathematically clear and physically reasonable, the functional in (P_1) is not bounded from below unless we put some "smallness" conditions on f and F .

Our condition for the existence differs from those given for the mean curvature equation in [6], [9], [4], and the reason why those conditions are not workable here is the lack of a coarea formula and of a sharp trace estimate for BD functions.

Finally, in section 3 , we shall give a few more readable sufficient conditions on f , F for the existence of the displacements field, and we shall discuss a few questions and extensions.

1. Functions of bounded deformation

Let Ω be an open set in $\mathbb{R}^n$. For a vector valued function $u \in L^1_{loc}(\Omega,\mathbb{R}^n)$ we denote by $\varepsilon(u)$ the deformation tensor associated to u . Recall that $\varepsilon(u)$ is the symmetric tensor of order two whose components are the distributions

$$\varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i}\right) \quad ; \quad i,j = 1,\dots,n$$

For a function $\varphi \in C_0^\infty(\Omega,\mathbb{R}^{n^2})$, $\varphi = \{\varphi_{ij}\}_{i,j=1,\dots,n}$ we have

$$\langle \varepsilon(u),\varphi\rangle = -\frac{1}{2}\int_\Omega \left(u^i \frac{\partial \varphi_{ij}}{\partial x_j} - u^j \frac{\partial \varphi_{ij}}{\partial x_i}\right) dx$$

For every open set $A \subset \Omega$ and for every function u in $L^1_{loc}(\Omega,\mathbb{R}^n)$ we set

$$\int_A |\varepsilon(u)| = \sup\left\{\langle \varepsilon(u),\varphi\rangle \; ; \; \varphi \in C_0^\infty(\Omega,\mathbb{R}^{n^2}),\ \mathrm{spt}\,\varphi \subset A \ , \ \sum_{i,j}\varphi_{ij}^2 \leq 1\right\}$$

It is well known that $\varepsilon(u)$ is a vector valued Radon measure in Ω if and only if $\int_A |\varepsilon(u)| < +\infty$ for all open sets $A \subset\subset \Omega$,moreover,in that case, the number $\int_A |\varepsilon(u)|$ equals the total variation in A of the measure $\varepsilon(u)$ so we can define a set function

$$B \longmapsto \int_B |\varepsilon(u)| \qquad B \subset \Omega$$

which is a positive (outer) measure in Ω .

DEFINITION - $BD(\Omega)$ denotes the linear space of the functions $u \in L^1(\Omega, \mathbb{R}^n)$ whose deformation tensor is a (Radon) measure of bounded variation in Ω, i.e.

$$BD(\Omega) = \left\{ u \in L^1(\Omega, \mathbb{R}^n) \;\middle|\; \|u\|_{BD(\Omega)} < +\infty \right\}$$

where

$$\|u\|_{BD(\Omega)} = \int_\Omega |u|\,dx + \int_\Omega |\varepsilon(u)|$$

It is easily seen that $BD(\Omega)$ is a Banach space with the norm $\|\cdot\|_{BD(\Omega)}$ and that the space $C^\infty(\Omega, \mathbb{R}^n)$ is not dense in $BD(\Omega)$. Moreover : $u \in BD(\Omega)$ if and only if

$$\varepsilon_{\alpha\alpha}(u) = \alpha \cdot \nabla(\alpha \cdot u) = \alpha^i \alpha^j \frac{\partial u^i}{\partial x_j}$$

is a bounded Radon measure for all $\alpha \in \mathbb{R}^n$.

The space $BD(\Omega)$ has been introduced in [12] and studied in [20], [17], [14], and [11], where a comprehensive account of the theory can be found.

Obviously, the space $BV(\Omega, \mathbb{R}^n)$, i.e. the space of $\mathbb{R}^n$ valued functions whose first derivatives are measures of bounded variation in Ω, is contained in $BD(\Omega)$; as we already mentioned this inclusion is strict since no Korn's inequality is available in $H^{1,1}$, see [12], [11].

We have

THEOREM 1.1 - (lower semicontinuity of the deformation) Let u, u_h be functions in $L^1_{loc}(\Omega, \mathbb{R}^n)$ with $u_h \longrightarrow u$ weakly, i.e. for each $\psi \in C_0^\infty(\Omega)$

$$\lim_{h \to \infty} \int_\Omega u_h^i \psi = \int_\Omega u^i \psi$$

then

$$\int_\Omega |\varepsilon(u)| \leq \liminf_{h \to \infty} \int_\Omega |\varepsilon(u_h)|$$

Proof: for every function $\varphi \in C_0^\infty(\Omega, \mathbb{R}^n)$ with $|\varphi| \leq 1$ we have

$$\langle \varepsilon(u), \varphi \rangle = \lim_{h\to\infty} \langle \varepsilon(u_h), \varphi \rangle \leq \liminf_{h\to\infty} \int_\Omega |\varepsilon(u_h)|$$

and taking the supremum for all φ the theorem follows.

q.e.d.

Obviously, theorem 1.1 also holds for the deviator $\varepsilon^D(u)$ of the deformation tensor, we recall that $\varepsilon^D(u)$ is defined as

$$\varepsilon^D(u) = \varepsilon(u) - \frac{1}{n}\,\mathrm{trace}(\varepsilon(u))\,I$$

We shall now list a few simple facts whose simple proof we omit.

i) $\displaystyle\int_{A_1} |\varepsilon(u)| + \int_{A_2} |\varepsilon(u)| = \int_{A_1\cup A_2} |\varepsilon(u)|$ for A_1, A_2 disjoint Borel sets

$\displaystyle\int_{A_1} |\varepsilon(u)| \leq \int_{A_2} |\varepsilon(u)|$ for $A_1 \subset A_2$

$\displaystyle\lim_{h\to\infty} \int_{A_h} |\varepsilon(u)| = \int_{\cup A_h} |\varepsilon(u)|$ for $A_h \subset A_{h+1}$, $h \in \mathbb{N}$

ii) let $A \subset\subset \Omega$ (i.e. A is open, $\bar{A}$ is compact, $\bar{A} \subset \Omega$) then

$$\int_A |\varepsilon(u * \psi)| \leq \int_{\mathbb{R}^n} |\psi|\,dx \int_\Omega |\varepsilon(u)|$$

for $\psi \in C_0^\infty(\mathbb{R}^n)$ with $\mathrm{diam}(\mathrm{spt}\,\psi) < \mathrm{dist}(A, \partial\Omega)$, moreover, for every sequence of mollifiers $\{\psi_h\}$ there exists $\bar{h}$ such that

$$\int_A |\varepsilon(u * \psi_h)| \leq \int_\Omega |\varepsilon(u)| \qquad \text{for} \quad h \geq \bar{h}$$

iii) let $\{\psi_h\}$ be a sequence of mollifiers, then

$$\int_{\mathbb{R}^n} |\varepsilon(u * \psi_h)| \longrightarrow \int_{\mathbb{R}^n} |\varepsilon(u)| \qquad \forall u \in BD(\mathbb{R}^n)$$

and for $A \subset\subset \Omega$

$$\lim_{h\to\infty} \int_A |\varepsilon(u * \psi_h)| \le \int_{\bar{A}} |\varepsilon(u)| \qquad \forall u \in BD(\Omega)$$

In particular, using the semicontinuity theorem 1.1 , if

$$\int_{\partial A} |\varepsilon(u)| = 0$$

then

$$\lim_{h\to\infty} \int_A |\varepsilon(u * \psi_h)| = \int_A |\varepsilon(u)|$$

<u>PROPOSITION 1.2</u> - <u>Let</u> u <u>be a</u> BD($\mathbb{R}^n$) <u>function with compact support, then we have</u>:

a) (<u>Poincaré inequality</u>)

$$\int_{\mathbb{R}^n} |u|\,dx \le c_1(n)\, \mathrm{diam}(\mathrm{spt}\,u) \int_{\mathbb{R}^n} |\varepsilon(u)|$$

b) (<u>Sobolev-Poincaré inequality</u>)

$$\left(\int_{\mathbb{R}^n} |u|^{\frac{n}{n-1}} \right)^{\frac{n-1}{n}} \le c_2(n) \int_{\mathbb{R}^n} |\varepsilon(u)|$$

<u>Proof</u>: due to iii) it is sufficient to show a) and b) for smooth functions with compact support in $\mathbb{R}^n$. Then a) is almost obvious, for b) see for example [19].

q.e.d.

As already stated, the space $C^\infty(\Omega,\mathbb{R}^n) \cap BD(\Omega)$ is not dense in $BD(\Omega)$, anyway, by iii) , for every function $u \in BD(\Omega)$ there exists a sequence $\{u_h\} \subset C^\infty(\mathbb{R}^n,\mathbb{R}^n)$ such that

$$u_h \longrightarrow u \qquad \text{in} \qquad L^1(\mathbb{R}^n,\mathbb{R}^n)$$

$$\int_{\mathbb{R}^n} |\varepsilon(u_h)| \longrightarrow \int_{\mathbb{R}^n} |\varepsilon(u)|$$

More generally the following is true.

<u>THEOREM 1.3</u> - <u>Let</u> Ω <u>be an open set in</u> $\mathbb{R}^n$ <u>and let</u>

$u \in BD(\Omega)$, <u>then there exists a sequence</u> $\{u_h\} \subset C^\infty(\Omega, \mathbb{R}^n) \cap BD(\Omega)$ <u>such that</u>

$$u_h \longrightarrow u \qquad \text{in} \qquad L^1(\Omega, \mathbb{R}^n)$$

$$\int_\Omega |\varepsilon(u_h)| \longrightarrow \int_\Omega |\varepsilon(u)|$$

<u>Proof</u>: the idea of the proof is as in [13] and [1] . We take a sequence of open sets $\Omega_1, \Omega_2, \ldots$, with regular boundary, such that

$$\Omega_K \subset\subset \Omega_{K+1} \quad , \quad \bigcup_{K=1}^\infty \Omega_K = \Omega$$

and we set

$$V_0 = \Omega_2 \quad , \quad V_K = \Omega_{3K+2} \setminus \bar{\Omega}_{3K-1}$$

then we take a sequence of functions φ_K with

$$\varphi_0 \in C_0^\infty(\Omega_4) \quad , \quad \varphi_0 = 1 \quad \text{in } \Omega_3$$

$$\varphi_K \in C_0^\infty(\Omega_{3K+4} \setminus \bar{\Omega}_{3K}) \quad , \quad \varphi_K = 1 \quad \text{in } \Omega_{3K+3} \setminus \bar{\Omega}_{3K+1}$$

$$\sum_{K=1}^\infty \varphi_K = 1 \qquad \text{in } \Omega$$

and a sequence of functions $\psi_{\tau_K} \in C_0^\infty(\mathbb{R}^n)$ such that

$$\psi_{\tau_K} \geq 0 \quad , \quad \operatorname{spt} \psi_{\tau_K} \subset \{x \in \mathbb{R}^n \mid |x| < \tau_K\} \quad , \quad \int_{\mathbb{R}^n} \psi_{\tau_K} = 1$$

Proceeding as in [1] it is now easy to see that one can find the numbers τ_K so that the function

$$u_h = \sum_{K=1}^\infty \psi_{\tau_K} * (u \varphi_K)$$

verifies

$$\int_\Omega |u_h - u| < \frac{1}{h}$$

$$\int_\Omega |\varepsilon(u_h)| < \int_\Omega |\varepsilon(u)| + \frac{1}{h}$$

and this, together with the lower semicontinuity of the deformation, proves the theorem.

q.e.d.

Remarks. 1. If u_h is as in theorem 1.3 then one also has

i) $u_h|_{\partial\Omega} = u|_{\partial\Omega}$ for all h

(see the existence of the trace in theorem 1.4, provided Ω has a Lipschitz boundary)

ii) $\int_A |\varepsilon(u_h)| \longrightarrow \int_A |\varepsilon(u)|$ for all open sets $A \subset \Omega$ such that $\int_{\partial A} |\varepsilon(u)| = 0$

iii) $\int_\Omega |\varepsilon_{ij}(u_h)| \longrightarrow \int_\Omega |\varepsilon_{ij}(u)|$ for all $i,j = 1,\dots,n$

iv) $\int_\Omega |\varepsilon^D(u_h)| \longrightarrow \int_\Omega |\varepsilon^D(u)|$

2. In case $u \in BD(\Omega)$ and $\operatorname{div} u \in L^2(\Omega)$ one can find the approximating functions u_h such that

$$\int_\Omega (\operatorname{div}(u-u_h))^2 dx < \frac{1}{h}$$

also holds.

Let Ω be a domain with Lipschitz boundary, then the trace of u on $\partial\Omega$ is well defined for each $u \in BD(\Omega)$ as an $L^1(\partial\Omega, \mathbb{R}^n)$ function. In fact the following theorem has been proved by Strang and Temam [17] .

THEOREM 1.4 - There exists a linear operator

$$\gamma : BD(\Omega) \longrightarrow L^1(\partial\Omega, \mathbb{R}^n)$$

such that

$$\gamma(u) = u|_{\partial\Omega}$$

for all $u \in BD(\Omega) \cap C^0(\bar{\Omega}, \mathbb{R}^n)$. The following trace estimate holds

$$(1.1) \qquad \int_{\partial\Omega} |\gamma(u)| \, d\mathcal{H}^{n-1} \leq c(n,\Omega) \|u\|_{BD(\Omega)}$$

moreover, for all i, j and for every $\varphi \in C^1(\bar{\Omega}, \mathbb{R}^n)$ the following Green's formula holds

$$(1.2) \qquad \int_\Omega \left(u^j \frac{\partial \varphi}{\partial x_i} + u^i \frac{\partial \varphi}{\partial x_j}\right) dx + 2\int_\Omega \varphi \, \varepsilon_{ij}(u) = \int_\Omega \varphi \left(\gamma^i(u)\nu_j + \gamma^j(u)\nu_i\right) d\mathcal{H}^{n-1}$$

where $\nu = (\nu_1, \ldots, \nu_n)$ is the unit outward normal vector to $\partial\Omega$ and $\gamma^i(u)$ is the i^{th} component of $\gamma(u)$.

Actually one can prove the estimate

$$(1.1)' \qquad \int_{\partial\Omega} |\gamma(u)| \, d\mathcal{H}^{n-1} \leq \alpha_1(n,L) \int_\Omega |\varepsilon(u)| + \alpha_2(\Omega) \int_\Omega |u| \, dx$$

where $\alpha_1(n,L)$ depends only on the dimension n of the ambient space and on the Lipschitz constant L of the boundary of Ω.

By the same method used in [1] (see theorem 6) for BV functions, one can prove the continuity of the trace operator in the following sense: if

$$\begin{cases} u_h \longrightarrow u & \text{in} \quad L^1(\Omega, \mathbb{R}^n) \\ \int_\Omega |\varepsilon(u_h)| \longrightarrow \int_\Omega |\varepsilon(u)| & \end{cases}$$

then

$$\gamma(u_h) \longrightarrow \gamma(u) \qquad \text{in} \quad L^1(\partial\Omega, \mathbb{R}^n)$$

From now on we shall simply denote $\gamma(u)$ as $u|_{\partial\Omega}$ or u.

We shall need in the following an explicit formula for the deformation $\int_\Gamma |\varepsilon(u)|$ on an $(n-1)$-dimensional surface Γ where u can be discontinuous. We shall obtain such a formula in the next theorem (where we confine ourselves to the case Γ is the boundary of an open set).

Let $u \in BD(\mathbb{R}^n)$ and let Ω be an open set with

Lipschitz boundary. Set

$$u^- = \text{trace of } u|_{\Omega} \text{ on } \partial\Omega$$

$$u^+ = \text{trace of } u|_{\mathbb{R}^n\setminus\Omega} \text{ on } \partial\Omega$$

then we have

<u>THEOREM 1.5</u> - <u>Let</u> $\nu(x)$ <u>be the outward unit normal vector to</u> $\partial\Omega$ <u>at</u> x <u>and set</u>

$$\tau_{ij}(p) = \frac{1}{2}(p_i\nu_j + p_j\nu_i) \quad , \text{ for } p \in \mathbb{R}^n$$

$$\tau = \{\tau_{ij}\}_{ij=1,\dots,n}$$

<u>then we have for all</u> $u \in BD(\mathbb{R}^n)$

i) $$\int_{\partial\Omega} \varepsilon_{ij}(u) = -\int_{\partial\Omega} \tau_{ij}(u^+ - u^-)\, d\mathcal{H}^{n-1}$$

ii) $$\int_{\partial\Omega} |\varepsilon(u)| = \int_{\partial\Omega} |\tau(u^+ - u^-)|\, d\mathcal{H}^{n-1}$$

iii) $$\int_{\mathbb{R}^n} |\varepsilon(u)| = \int_{\Omega} |\varepsilon(u)| + \int_{\partial\Omega} |\tau(u^+ - u^-)| + \int_{\mathbb{R}^n\setminus\Omega} |\varepsilon(u)|$$

<u>Proof</u>: i) Write formula (1.2) for $u|_{\Omega}$ and $u|_{\mathbb{R}^n\setminus\Omega}$ with $\varphi \equiv 1$ and sum.

ii) Using Green's formula (1.2) in Ω and in $\mathbb{R}^n\setminus\Omega$ we get for $\varphi_{ij} \in C_0^\infty(\mathbb{R}^n)$

$$\sum_{ij=1}^{n} \int_{\mathbb{R}^n} \varepsilon_{ij}(u)\varphi_{ij} = \sum_{ij=1}^{n} \Big\{ \int_{\Omega} \varepsilon_{ij}(u)\varphi_{ij} + \int_{\mathbb{R}^n\setminus\Omega} \varepsilon_{ij}(u)\varphi_{ij} - \\ - \int_{\partial\Omega} \tau_{ij}(u^+ - u^-)\varphi_{ij}\, d\mathcal{H}^{n-1} \Big\}$$

taking the supremum of both members for $\sum_{ij=1}^{n} \varphi_{ij}^2 \leq 1$ we obtain

$$\int_{\partial\Omega} |\varepsilon(u)| \leq \int_{\partial\Omega} |\tau(u^+ - u^-)|\, d\mathcal{H}^{n-1}$$

Let now $\varphi_{ij}^h \in C_0^\infty(\mathbb{R}^n)$ be such that

$$\sum_{ij=1}^{n} (\varphi_{ij}^h)^2 \leq 1 \quad , \quad \operatorname{spt} \varphi_{ij}^h \subset U_h$$

where we have set $U_h = \left\{ y \in \mathbb{R}^n \mid \operatorname{dist}(y, \partial\Omega) < \frac{1}{h} \right\}$ and suppose moreover that

$$\varphi_{ij}^h \longrightarrow \frac{\tau_{ij}(u)}{|\tau(u)|} \qquad \text{in} \qquad L^1(\partial\Omega)$$

For all h we have

$$\int_{\partial\Omega} \tau_{ij}(u^+ - u^-)\varphi_{ij}^h \, d\mathcal{H}^{n-1} \leq \int_{U_h} |\varepsilon(u)| + \int_{\Omega\cap U_h} |\varepsilon(u)| + \int_{(\mathbb{R}^n\setminus\Omega)\cap U_h} |\varepsilon(u)|$$

and going to the limit for $h \to \infty$ we get

$$\int_{\partial\Omega} |\tau(u^+ - u^-)| \, d\mathcal{H}^{n-1} \leq \int_{\partial\Omega} |\varepsilon(u)|$$

which concludes the proof of ii) .
iii) is obvious.

q.e.d.

We shall also need the analogous of theorem 1.5 for the deviator $\varepsilon^D(u)$ of $\varepsilon(u)$.

Set

$$\tau^D(p) = \tau(p) - \frac{1}{n}\operatorname{trace}(\tau(p)) I = \tau(p) - \frac{p\cdot\nu}{n} I$$

It is immediate that

$$\int_{\Omega} \varepsilon^D(u) = \int_{\partial\Omega} \tau^D(u) \, d\mathcal{H}^{n-1}$$

and that

$$\int_{\Omega} \varepsilon_{ij}^D(u)\varphi + \int_{\Omega} \left\{ \frac{1}{2}\left(u^i \frac{\partial\varphi}{\partial x_j} + \frac{\partial\varphi}{\partial x_i}\right) + \frac{u\cdot\nabla\varphi}{n}\delta_{ij} \right\} dx =$$

$$\int_{\partial\Omega} \varphi \, \tau_{ij}^D(u) \, d\mathcal{H}^{n-1}$$

moreover we have

THEOREM 1.6 - In the hypotheses of theorem 1.5 we have also

$$\int_{\partial\Omega} |\varepsilon^D(u)| = \int_{\partial\Omega} |\tau^D(u^+ - u^-)| \, d\mathcal{H}^{n-1}$$

Proof: the same as for theorem 1.5 .

q.e.d.

Let us remark here that one has, for regular functions,

$$|\varepsilon(u)|^2 = |\varepsilon^D(u)|^2 + \left|\frac{\operatorname{div} u}{n} I\right|^2$$

because $\varepsilon^D(u)$ and $\frac{1}{n}(\operatorname{div} u) I$ are orthogonal with respect to the inner product

$$a \cdot b = \sum_{i,j=1}^{n} a_{ij} b_{ij}$$

so we get

$$\int_{\Omega} |\varepsilon(u)| = \int_{\Omega} \left\{ |\varepsilon^D(u)|^2 + \left|\frac{\operatorname{div} u}{n} I\right|^2 \right\}^{1/2}$$

which holds, by approximation, for all $u \in BD(\Omega)$.

We also have

$$|\tau(p)| = \left\{ |\tau^D(p)|^2 + \left|\frac{p \cdot \nu}{n} I\right|^2 \right\}^{1/2}$$

PROPOSITION 1.7 - i) Let $\alpha \in \mathbb{R}^n$, $|\alpha| = 1$, $f \in BV(\Omega)$ and denote by $\nabla_{\alpha^\perp} f$ the projection of ∇f on the orthogonal space to α , then we have

$$\int_{\Omega} |\varepsilon(\alpha f)| = \int_{\Omega} \left\{ |\nabla_\alpha f|^2 + \frac{1}{2} |\nabla_{\alpha^\perp} f|^2 \right\}^{1/2}$$

where the right member denotes the total variation in Ω of the $\mathbb{R} \times \mathbb{R}^{n-1}$ valued measure $(\nabla_\alpha f , \nabla_{\alpha^\perp} f)$.

ii) Let $u \in L^1(\partial\Omega, \mathbb{R}^n)$ and set $u_\nu = u \cdot \nu$, $u_\tau = u - u_\nu \nu$, then we have

$$\int_{\partial\Omega} |\tau(u)| = \int_{\partial\Omega} \left\{ u_\nu^2 + \frac{1}{2} |u_\tau|^2 \right\}^{1/2}$$

Proof: i) Take a smooth function f and an orthonormal basis $\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$ with $e_1=\alpha$. We then have

$$|\varepsilon(\alpha f)|^2 = \sum_{ij=1}^{n} \frac{1}{4}\left(\alpha_j\langle\nabla f,e_i\rangle + \alpha_i\langle\nabla f,e_j\rangle\right)^2$$

where $\alpha_1=1$ and $\alpha_2 = \alpha_3 = \alpha_4 = \dots = \alpha_n = 0$, hence

$$|\varepsilon(\alpha f)| = \left(\langle\nabla f,e_1\rangle^2 + \frac{1}{2}\sum_{j=2}^{n}\langle\nabla f,e_j\rangle^2\right)^{1/2} = \left(|\nabla_\alpha f|^2 + \frac{1}{2}|\nabla_{\alpha^\perp} f|^2\right)^{1/2}$$

Integrating over Ω we get i) for smooth functions and, by approximation, we get the result for all $f \in BV(\Omega)$.
ii) Take a point x where $\nu(x)$ is defined and take an orthonormal basis $\{e_1,\dots,e_n\}$ of $\mathbb{R}^n$ with $e_1=\nu(x)$, we then have $\nu_1(x)=1$, $\nu_j(x)=0$ for $j=2,\dots,n$ and

$$|\tau(u,x)|^2 = \sum_{i,j=1}^{n}\frac{1}{4}(u^i\nu_j+u^j\nu_i)^2 = u_\nu^2(x) + \frac{1}{2}|u_\tau(x)|^2$$

Integrating over $\partial\Omega$ we get ii) .

q.e.d.

One can also prove the following:

$$\int_\Omega |\varepsilon^D(\alpha f)| = \int_\Omega \left\{\frac{n-1}{n}|\nabla_\alpha f|^2 + \frac{1}{2}|\nabla_{\alpha^\perp} f|^2\right\}^{1/2}$$

$$\int_{\partial\Omega} |\tau^D(u)| = \int_{\partial\Omega} \left\{\frac{n-1}{n}\, u_\nu^2 + \frac{1}{2}|u_\tau|^2\right\}^{1/2}$$

The trace operator $\gamma : BD(\Omega) \longrightarrow L^1(\partial\Omega,\mathbb{R}^n)$ is onto, in fact every function $\varphi \in L^1(\partial\Omega,\mathbb{R}^n)$ can be extended, by Gagliardo's theorem, to a function in $H^{1,1}(\Omega)$ (provided Ω has a Lipschitz boundary). For our purposes,see next section, a more refined extension result is needed and precisely theorem 1.8 below.

Let's first recall a well known fact. Take an open bounded set Ω with a class C^2 boundary and set $d(x)=\operatorname{dist}(x,\partial\Omega)$ for $x\in\Omega$, then there exist a number $a>0$ such that if $0<d(x)<a$ the following is true:

i) there exists a unique point $U(x)\in\partial\Omega$ such that

$$d(x) = \operatorname{dist}(x, U(x))$$

ii) the function $d(x)$ is differentiable at x

iii) $\nabla d(x) = -\nu(U(x))$

THEOREM 1.8 - Let Ω be an open bounded set with a class C^2 boundary, and let φ be a function in $L^1(\partial\Omega, \mathbb{R}^n)$ (or in $L^1(\Gamma, \mathbb{R}^n)$, Γ being the intersection of $\partial\Omega$ with an open set A) such that

$$\nu \cdot \varphi = 0 \qquad \text{on } \partial\Omega \quad (\text{ on } \Gamma\ )$$

then there exists a function $\phi \in BD(\Omega)$ with $\operatorname{div}\phi \in L^2(\Omega)$ such that

$$\phi = \varphi \qquad \text{on } \partial\Omega \quad (\text{ on } \Gamma\ )$$

and

$$\phi(y) \cdot \nabla d(y) = 0$$

for all points $y = x - \nu t$ where $x \in \partial\Omega$ ($x \in \Gamma$) and $0 < t < a$, moreover

$$\operatorname{spt}\phi \subset \{ x \in \Omega \mid d(x) < a \}$$

Proof: set

$$Q = \{ y \in \mathbb{R}^n \mid |y_i| < 1, \ i = 1, \dots, n \}$$

$$Q^+ = \{ y \in Q \mid y_n > 0 \}$$

by a partition of unity argument we reduce to the case of $\operatorname{spt}\varphi \subset V \cap \partial\Omega$ where V is open and there is a diffeomorphism $\sigma : V \longrightarrow Q$ such that

$$\sigma(V \cap \Omega) = Q^+$$

$$d\sigma(x)\,(\nabla d(x)) = e_n \qquad \text{for} \quad d(x) < a$$

and the jacobian of σ is bounded and bounded away from zero in V . Set now

$$\tilde{\varphi}(y_1, \dots, y_{n-1}) = \varphi(\sigma^{-1}(y_1, \dots, y_{n-1}, 0))$$

and use lemma 1.9 below to get a function $\tilde{\phi} \in BD(Q^+)$ with $\mathrm{div}\,\tilde{\phi} \in L^2(Q^+)$, $\tilde{\phi} = \tilde{\varphi}$ on $\{x \in Q \mid x_n = 0\}$, $\tilde{\phi} \cdot e_n = 0$ in Q^+ it is then easy to see that the function

$$\phi(x) = \tilde{\phi}(\sigma(x))$$

is the desired extension of φ .

q.e.d.

LEMMA 1.9 - Let $\varphi \in L^1(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$, then there exists a function $\phi \in BD(\{x \in \mathbb{R}^n \mid x_n > 0\})$ such that

i) $\phi(x_1, \dots, x_{n-1}, 0) = (\varphi(x_1, \dots, x_{n-1}), 0)$

ii) $\phi(x) \cdot e_n = 0 \qquad \forall x \in \mathbb{R}^n$ with $x_n > 0$

iii) $\mathrm{div}\,\phi \in L^2(\{x \in \mathbb{R}^n \mid x_n > 0\})$

moreover, if Ω_1 is an open set in $\mathbb{R}^n$ and $\mathrm{spt}\,\varphi \subset\subset \Omega_1 \cap \{x \in \mathbb{R}^n \mid x_n = 0\}$ we can find ϕ so that $\mathrm{spt}\,\varphi \subset \Omega_1$.

Proof: take a sequence of functions $\psi_h \in C_0^\infty(\mathbb{R}^{n-1}, \mathbb{R}^{n-1})$ with

$$\psi_h \longrightarrow \varphi \qquad \text{in} \qquad L^1(\mathbb{R}^{n-1}, \mathbb{R}^{n-1}) .$$

and let $\{\tau_h\}$ be a decreasing sequence of positive numbers with $\tau_h \to 0$. Set

$$\phi(\xi, x_n) = \begin{cases} 0 & \text{if} \quad x_n > \tau_0 \\ \psi_h(\xi) + \dfrac{x_n - \tau_h}{\tau_{h+1} - \tau_h}\left(\psi_{h+1} - \psi_h\right)(\xi) & \text{if} \quad \tau_h > x_n > \tau_{h+1} \end{cases}$$

where $\xi = (x_1, \dots, x_{n-1})$. It's easy to check that for a suitable choice of the τ_h we have

$$\int_{x_n > 0} |\phi| + \int_{x_n > 0} |\varepsilon(\phi)| + \int_{x_n > 0} (\mathrm{div}\,\phi)^2 < +\infty$$

and i), ii) are also obviously verified.

In case $\operatorname{spt}\varphi \subset \Omega_1 \cap \{x \in \mathbb{R}^n \mid x_n = 0\}$ we can find the functions ψ_h so that $\operatorname{spt}\psi_h \subset W \subset\subset \Omega_1 \cap \{x \in \mathbb{R}^n \mid x_n > 0\}$, hence $\operatorname{spt}\phi \subset W \times [0, \tau_0]$ and $W \times [0, \tau_0] \subset \Omega_1$ if τ_0 is small.

q.e.d.

The last results we are now going to state are Poincaré inequality, a compactness theorem and some corollaries.

Let $\mathcal{T}$ be the space of infinitesimal rigid motions of $\mathbb{R}^n$, i.e.

$$\mathcal{T} = \{ T = Ax + b \mid b \in \mathbb{R}^n,\ A \text{ is a skew-symmetric matrix} \}$$

<u>THEOREM 1.10 - Let Ω be a Lipschitz domain in $\mathbb{R}^n$ and let</u>

$$T : BD(\Omega) \longrightarrow \mathcal{T}$$

<u>be any continuous linear function which fixes the elements of $\mathcal{T}$, then there is a constant $c(\Omega, T)$ such that</u>

$$\| u - Tu \|_{L^{n/n-1}(\Omega, \mathbb{R}^n)} \leq c(\Omega, T) \int_\Omega |\varepsilon(u)|$$

<u>Asuitable function T_0 can be obtained as follows:</u>

$$[(T_0 u)(x)]^j = \frac{1}{2} \sum_{i=1}^n (\rho^{ij}(u) - \rho^{ji}(u))(x - x_0)_i + \sigma^j(u)$$

<u>where x_0 is a fixed point in Ω and</u>

$$\sigma^j(u) = \frac{1}{\alpha(n) R^n} \int_{B_R(x_0)} u^i(y)\, dy$$

$$\rho^{ij}(u) = \frac{n+1}{\alpha(n-1) R^{n+1}} \int_{\{y \in B_R(x_0) \mid y \cdot e_j > 0\}} (u^i - \sigma^i(u))\, dy$$

<u>where $B_R(x_0) = \{x \mid |x - x_0| < R\} \subset\subset \Omega$, $\{e_1, \dots, e_n\}$ is an orthonormal basis in $\mathbb{R}^n$ and $\alpha(n)$ = n-dimensional measure of $\{x \in \mathbb{R}^n \mid |x| \leq 1\}$.</u>

One also has the following

COROLLARY 1.11 - *Let* Γ *be a subset of* $\partial\Omega$ *with positive* $(n-1)$*-dimensional measure, then there is a constant* $c(\Omega,\Gamma)$ *such that*

$$\|u\|_{L^{n/n-1}(\Omega,\mathbb{R}^n)} \leq c(\Omega,\Gamma)\int_\Omega |\varepsilon(u)|$$

for all $u \in BD(\Omega)$ *with* $u|_\Gamma = 0$.

Theorem 1.10 has been proved by Kohn [11].

As for BV functions, see [5], [2], theorem 1.10 , together with the ε-net argument, yields the following compactness theorem, see [11].

THEOREM 1.12 - *Let* Ω *be a Lipschitz domain. Then the inclusion of the space* $BD(\Omega)$ *in* $L^p(\Omega,\mathbb{R}^n)$ *is compact for* $p < \frac{n}{n-1}$.

For a different proof of the compactness theorem see also [17], [20].

A simple consequence of theorems 1.10 , 1.12 is the following (see also [17] for a completely different proof).

PROPOSITION 1.13 - *Let* S *be a* $\mathbb{R}^n$*-valued distribution in* Ω *such that* $\varepsilon(S)$ *is a Radon measure in* Ω , *then* S *is represented by a function* $u \in L^1_{loc}(\Omega,\mathbb{R}^n)$, *that is*

$$\langle S,\varphi\rangle = \int_\Omega u\cdot\varphi \qquad \textit{for all} \quad \varphi \in \mathcal{D}(\Omega,\mathbb{R}^n)$$

Proof: for any given open set $A \subset\subset \Omega$ we shall prove that S is represented in A by a function $u_A \in L^1(A,\mathbb{R}^n)$ and this obviously proves the theorem.

Take an open set A' and a ball $B_R(x_0)$ with $B_R(x_0) \subset\subset A \subset\subset A' \subset\subset \Omega$, take then a sequence of mollifiers ψ_h and set $S_h = S * \psi_h$. Then we have

$$S_h \longrightarrow S \qquad \text{weakly in } A$$

$$\|S_h - T_0(S_h)\|_{L^1(A)} \leq c(A,T_0)\int_A |\varepsilon(S_h)|$$

where T_0 is as in theorem 1.10 . We also have

$$\int_A |\varepsilon(S_h)| \leq \int_{A'} |\varepsilon(S)|$$

for h large enough, this implies that the numbers $\| S_h - T_0(S_h) \|_{BD(A)}$ are bounded independently of h and, by compactness theorem 1.12 (possibly taking a subsequence), we have

$$S_h - T_0(S_h) \longrightarrow v_A \in L^1(A, \mathbb{R}^n) \qquad \text{in } L^1(A, \mathbb{R}^n)$$

We shall now prove that $T_0(S_h)$ $\quad R \in \mathcal{J}$, that is S_h $\quad v_A + R$ and $u_A = v_A + R$ represents S in A .

Consider a test function $\varphi = (\varphi_1, \ldots, \varphi_n)$ with $\varphi_2 = \ldots = \varphi_n = 0$ $\varphi_1 \in D(B_R(x_0))$, $\int \varphi \neq 0$, $\varphi(x) = \xi(|x - x_0|)$ we then have

$$\langle S_h, \varphi \rangle \to \langle S, \varphi \rangle$$

$$\langle S_h, \varphi \rangle - \langle T_0(S_h), \varphi \rangle \longrightarrow \int_A v_A \varphi$$

hence

$$\sigma^1(S_h) \int_A \varphi_1 = \int_A [T_0(S_h)]^1 \varphi_1 \longrightarrow \int_A v_A \varphi_1 - \langle S, \varphi \rangle$$

and

$$\sigma^1(S_h) \longrightarrow \left(\int_A v_A^1 \varphi_1 - \langle S, \varphi \rangle \right) \left(\int_A \varphi_1 \right)^{-1} = \bar{\sigma}^1$$

In a similar way one can show that

$$\lim_{h \to \infty} \sigma(S_h) = \bar{\sigma} \in \mathbb{R}^n$$

We shall now prove that there exist numbers $t_{ij} \in \mathbb{R}$ such that

$$(*) \qquad \lim_{h \to \infty} \frac{1}{2} \left(\rho^{ij}(S_h) - \rho^{ji}(S_h) \right) = t_{ij}$$

In fact, take a test function $\varphi = (0, \ldots, \varphi_j, \ldots, 0) \in \mathcal{D}(B_R(x_0), \mathbb{R}^n)$ with $\int |\varphi_j| > 0$ and such that $\varphi_j(x - x_0)$ is odd in the variable $(x - x_0)^i$ and even in the remaining variables, then

$$\langle T_0(S_h), \varphi \rangle = \frac{1}{2} \left(\rho^{ij}(S_h) - \rho^{ji}(S_h) \right) \int (x - x_0)^i \varphi_j \longrightarrow \int v_A^j \varphi_j - \langle S, \varphi \rangle$$

where $\int (x-x_0)^i \varphi_j \neq 0$ and $(*)$ follows .

2. Existence for the displacements field

We are now going to discuss the existence of a solution to problem (P_1) .

Unless otherwise stated, Ω will be a bounded connected open set in $\mathbb{R}^n$ with Lipschitz boundary and $\nu(x)$ will be the outward unit normal vector to $\partial\Omega$ at x .

Let A_1 be a bounded open set and call

$$T_D = A_1 \cap \partial\Omega \quad , \quad T_N = \partial\Omega \setminus T_D$$

we shall suppose that the set $\Omega_1 = \Omega \cup A_1$ is connected, that $\mathcal{H}^{n-1}(\overline{T}_D \cap \overline{T}_N) = 0$ (where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure) and that T_N coincides with the closure of its interior.

The function $\phi : M_n \longrightarrow [0,\infty)$, defined on the space M_n of the nxn matrices as

$$\phi(a) = \begin{cases} \frac{1}{2}|a|^2 & \text{if } |a| \leq 1 \\ |a| - \frac{1}{2} & \text{if } |a| \geq 1 \end{cases}$$

where $a = \{a_{ij}\}_{ij=1,\dots,n}$, and $|a|^2 = \sum_{ij=1}^{n} a_{ij}^2$, is obviously a convex function of a .

As we said in the introduction, we shall look for a solution u to problem (P_1) in the space $P(\Omega)$ defined as follows

$$P(\Omega) = \left\{ u \in BD(\Omega) \;\middle|\; \operatorname{div} u \in L^2(\Omega) \right\}$$

Clearly, the term $\int_\Omega (\operatorname{div} u)^2 dx$ is well defined for $u \in P(\Omega)$, not so obvious is the meaning of $\int_\Omega \phi(\varepsilon^D(u))$ as the ε_{ij}^D are measures and not functions. To get rid of this difficulty we proceed in a similar way to [7].

First we define a new function $\phi : M_n \times [0,\infty) \quad [0,\infty)$ setting

$$\bar\phi(t,a) = \begin{cases} \phi\left(\frac{a}{t}\right)t & \text{if } t > 0 \\ \lim\limits_{t\downarrow 0} \phi\left(\frac{a}{t}\right)t & \text{if } t = 0 \end{cases}$$

As it is easy to check (see [7]), $\bar\phi$ is convex and positively homogeneous in (t,a). Now, for any M_n-valued measure $\mu = \{\mu_{ij}\}$ in Ω we consider the $(\mathbb{R}\times M_n)$-valued measure $\alpha = (\alpha_0, \{\alpha_{ij}\})$ where

$$\alpha_0 = \mathcal{L}^n = \text{Lebesgue measure in } \Omega \quad , \qquad \alpha_{ij} = \mu_{ij}$$

and we define

$$\int_\Omega \phi(\mu) = \int_\Omega \bar\phi\left(\frac{d\alpha_0}{d|\alpha|}, \frac{d\alpha_{ij}}{d|\alpha|}\right) d|\alpha|$$

where the positive measure $|\alpha|$ is the total variation of α and the functions $\frac{d\alpha_0}{d|\alpha|}$, $\frac{d\alpha_{ij}}{d|\alpha|}$ are the Radon-Nicodym derivatives.

Using this definition, as a corollary of a theorem by Reschetnyak, see [16],[7], we have the following semicontinuity result.

<u>PROPOSITION 2.1</u> - <u>Let</u> u, $u_h \in BD(\Omega)$ <u>and set</u>

$$\alpha = (\mathcal{L}^n, \varepsilon^D_{ij}(u)) \quad , \quad \alpha_h = (\mathcal{L}^n, \varepsilon^D_{ij}(u_h))$$

<u>Suppose that</u> $u_h \longrightarrow u$ <u>weakly</u>, <u>then</u> $\alpha_h \longrightarrow \alpha$ <u>weakly and</u>

$$\int_\Omega \phi(\varepsilon^D(u)) \le \liminf_{h\to\infty} \int_\Omega \phi(\varepsilon^D(u_h))$$

As far as the force terms are concerned, let us remark by now that they are certainly defined if

$$f = f_1 + \nabla q \qquad , \quad f_1 \in L^n(\Omega,\mathbb{R}^n) \quad , \quad q \in L^2(\Omega) \quad , \quad \operatorname{spt} q \subset \Omega$$

$$F \in L^\infty(T_N, \mathbb{R}^n)$$

in which case we have

$$\left| \int_\Omega f u \right| \leq \|f_1\|_{L^n} \|u\|_{BD} + \|q\|_{L^2} \|\operatorname{div} u\|_{L^2}$$

$$\left| \int_{T_N} F u \right| \leq c(n,\Omega) \|u\|_{BD(\Omega)} \|F\|_{L^\infty(T_N)}$$

where $c(n,\Omega)$ is the constant in the trace estimate (1.1).

Actually, it can be physically reasonable and formally useful to suppose that the force $F(x)$ depends on the normal $\nu(x)$ to $\partial\Omega$ in the following way

$$F_j(x) = K_{ij}(x)\, \nu_i(x)$$

where $K_{ij}(x)$ is a symmetric tensor; in this case we have

$$F_j u^j = K_{ij}\, \tau_{ij}(u)$$

because K is symmetric, and

$$K_{ij}\, \tau_{ij}(u) = \left(K^D_{ij} + \frac{\operatorname{trace} K}{n} \delta_{ij} \right)\left(\tau^D_{ij} + \frac{u \cdot \nu}{n} \delta_{ij} \right) = \tag{2.1}$$

$$= K^D_{ij}\, \tau^D_{ij} + \mu \cdot \nu \, \frac{\operatorname{trace} K}{n}$$

because the matrices of zero trace are orthogonal to the identity matrix.

We shall need the following facts.

<u>PROPOSITION 2.2</u> - <u>Let</u> $u \in P(\Omega_1)$ <u>and set</u>

$$u^- = \text{trace of } u|_\Omega \text{ on } T_D$$

$$u^+ = \text{trace of } u|_{\Omega_1 \setminus \Omega} \text{ on } T_D$$

<u>then we have</u>

i) $u^-(x) \cdot \nu(x) = u^+(x) \cdot \nu(x)$ <u>for</u> $\mathcal{H}^{n-1}$<u>-almost all</u> $x \in T_D$

ii) $\displaystyle\int_{T_D} \phi(\varepsilon^D(u)) = \int_{T_D} |\tau^D(u^+ - u^-)|\, d\mathcal{H}^{n-1}$

<u>Proof</u>: i) recall that u^+ and u^- are L^1- funtions on T_D

by the trace theorem 1.4 , and that Green's formulae hold, hence, for all $\varphi \in C_0^1(\Omega_1)$ we have

$$\int_{T_D} \varphi u^+ \cdot \nu - \int_{T_D} \varphi u^- \cdot \nu = \int_{\Omega} \operatorname{div}(u\varphi) + \int_{\Omega_1 \setminus \Omega} \operatorname{div}(u\varphi) +$$

$$+ \int_{T_D} \operatorname{div}(u\varphi) = \int_{\Omega_1} \operatorname{div}(u\varphi) = 0$$

ii) By definition we have

$$\int_{T_D} \phi(\varepsilon^D(u)) = \int_{T_D} \bar{\phi}\left(\frac{d\mathcal{L}^n}{d|\alpha|}, \frac{d\varepsilon^D_{ij}(u)}{d|\alpha|}\right) d|\alpha|$$

where $\alpha = (\mathcal{L}^n, \varepsilon^D_{ij}(u))$ is a $(\mathbb{R} \times M_n)$-valued measure, but

$$\mathcal{L}^n\big|_{T_D} = 0 \qquad \text{and} \qquad |\alpha|\big|_{T_D} = |\varepsilon^D(u)|\big|_{T_D}$$

On the other hand, by Green's formulae

$$\int_{T_D} \varepsilon^D_{ij}(u)\varphi = - \int_{T_D} \tau^D_{ij}(u^+ - u^-)\varphi \, d\mathcal{H}^{n-1}$$

that is

$$\varepsilon^D_{ij}(u)\big|_{T_D} = - \tau^D_{ij}(u^+ - u^-)\, d\mathcal{H}^{n-1}$$

and finally we get

$$\int_{T_D} \phi(\varepsilon^D(u)) = \int_{T_D} \bar{\phi}\left(0, \frac{\tau^D_{ij}(u^+ - u^-)}{|\tau^D(u^+ - u^-)|}\right) |\tau^D(u^+ - u^-)|\, d\mathcal{H}^{n-1} =$$

$$= \int_{T_D} |\tau^D(u^+ - u^-)|\, d\mathcal{H}^{n-1}$$

because $\bar{\phi}(0, a) = |a|$

q.e.d.

We shall now relax the Dirichlet boundary condition in a similar way to what has been done in [8],[7] for the minimum area problem.

$$(P_2)\quad \begin{cases} \text{minimize the functional} \\ \mathcal{F}'(u) = \int_\Omega \phi(\varepsilon^D(u)) + \frac{K_0}{2}\int_\Omega (\operatorname{div} u)^2 dx + \int_\Omega fu\, dx + \\ \qquad + \int_{T_N} Fu\, d\mathcal{H}^{n-1} + \int_{T_D} |\tau^D(u-g)|\, d\mathcal{H}^{n-1} \\ \text{in the class of functions } u \in P(\Omega) \text{ such that} \\ u\cdot\nu = g\cdot\nu \qquad \text{on } T_D \\ \text{where } g \text{ is a fixed function in } P(\Omega_1)\cap H^{1,1}(\Omega_1) \end{cases}$$

If moreover we introduce the problem

$$(P_3)\quad \begin{cases} \mathcal{F}''(u) = \int_{\Omega_1} \phi(\varepsilon^D(v)) + \frac{K_0}{2}\int_{\Omega_1} (\operatorname{div} v)^2 dx + \int_{\Omega_1} \tilde{f} v\, dx + \int_{T_N} Fv \longrightarrow \inf \\ v \in P(\Omega_1)\quad , \qquad v = g \qquad \text{in } A_1\setminus\bar{\Omega} \\ \tilde{f} = \begin{cases} f & \text{in } \Omega \\ 0 & \text{in } A_1\setminus\bar{\Omega} \end{cases} \end{cases}$$

then, by proposition 2.2 ii) , problem (P_3) is equivalent to problem (P_2) . In fact, if $v \in P(\Omega_1)$ with $v=g$ in $\Omega_1\setminus\bar{\Omega}$ (or $u \in P(\Omega)$ with $u\cdot\nu = g\cdot\nu$ on T_D and set $v=u$ in Ω , $v=g$ in $\Omega_1\setminus\bar{\Omega}$) we have

$$\mathcal{F}''(v) = \mathcal{F}'(v) + \frac{K_0}{2}\int_{\Omega_1\setminus\bar{\Omega}} (\operatorname{div} g)^2 dx + \int_{\Omega_1\setminus\Omega} \phi(\varepsilon^D(u))$$

We want to emphasize the fact that the possible solution to problem (P_2) needs not take the prescribed boundary value on T_D , more precisely, the normal component

$u \cdot \nu$ of u will take the value $g \cdot \nu$ while, in general, the tangential component u_τ of u will not be equal to g_τ .

The following theorem justifies our relaxed form (P_2) or (P_3) of problem (P_1) .

<u>THEOREM 2.3 - Let the boundary of Ω be of class C^2 , then we have</u>

$$\inf_{\substack{u \in P(\Omega) \\ u = g \text{ on } \Gamma_D}} \mathcal{F}(u) = \inf_{\substack{u \in P(\Omega) \\ u\cdot\nu = g\cdot\nu \text{ on } \Gamma_D}} \mathcal{F}'(u)$$

<u>Proof</u>: obviously we have $\inf \mathcal{F}' \leq \inf \mathcal{F}$ so let us prove the converse. Let $\delta > 0$ and take $u \in P(\Omega)$ with $u \cdot \nu = g \cdot \nu$ on Γ_D and such that

$$\mathcal{F}'(u) \leq \delta + \inf \mathcal{F}'$$

We have $(u-g) \in L^1(\Gamma_D, \mathbb{R}^n)$ and $(u-g)\cdot\nu = 0$ on Γ_D so, by theorem 1.7 , we can find a function $\psi \in P(\Omega)$ such that

$$\psi \cdot \nu = 0 \quad \text{in } \Omega \quad , \quad \psi = g - u \quad \text{on } \Gamma_D$$

Set then

$$w = u + \eta_K \psi$$

where

$$\eta_K(x) = \max\left(0, 1 - K \operatorname{dist}(x, \partial\Omega)\right)$$

We have $w \in P(\Omega)$, in fact

$$\int_\Omega |\varepsilon(w)| \leq \int_\Omega |\varepsilon(u)| + \eta_K \int_\Omega |\varepsilon(\psi)| + \frac{1}{2}\int_\Omega |\psi^j \nabla_i \eta_K + \psi^i \nabla_j \eta_K| \, dx$$

$$\int_\Omega |\operatorname{div} w|^2 \leq \text{cost} \left\{ \int_\Omega (\operatorname{div} u)^2 + \int_\Omega |\psi \cdot \nabla \eta_K|^2 dx + \int_\Omega \eta_K^2 (\operatorname{div} \psi)^2 dx \right\}$$

where $\psi \cdot \nabla \eta_K = \psi \cdot \nu K = 0$. We also have

$$\mathcal{F}(w) \leq \mathcal{F}(u) + \frac{1}{2} \int_{(T_D)_{1/K}} |\nu_i \psi^j + \nu_j \psi^i| \, dx + \sigma(\delta)$$

where $(T_D)_{1/K} = \{ x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) < 1/K \}$ and

$$\lim_{\delta \to 0^+} \sigma(\delta) = 0$$

$$\lim_{K \to \infty} \frac{1}{2} K \int_{(T_D)_{1/K}} |\nu_i \psi^j + \nu_j \psi^i| \, dx = \int_{T_D} |\tau^D(u-g)| \, d\mathcal{H}^{n-1}$$

that is, for k large enough

$$\mathcal{F}(w) \leq \inf \mathcal{F}' + \sigma(\delta) + 2\delta$$

and, this holding for all positive δ , the theorem is proved.

q.e.d.

We shall now give a sufficient condition on the forces f , F in order problem (P_2) (or equivalently (P_3)) be defined and have a solution, this condition is also almost necessary in view of proposition 2.5 . We shall suppose that F is expressed in the form

$$F_j = -(H_{ij} + p\delta_{ij})\nu_i$$

with $H_{ij} = H_{ji}$, $\sum_{i=1}^{n} H_{ii} = 0$ (see (2.1)) , hence, if V_0 denotes the space of the traces of functions $u \in P(\Omega)$, we could more precisely allow F to be only a function in the dual space of V_0 instead of being in $L^\infty(T_N, \mathbb{R}^n)$.

Then we have

THEOREM 2.4 - *Assume that the functions H_{ij} and p can be extended to functions still called H_{ij} , p on $\bar{\Omega}$ in such a way that*

1. $H_{ij} = H_{ji}$, $\sum_{i=1}^{n} H_{ii} = 0$

2. $|H| = \left(\sum_{i,j=1}^{n} H_{ij}^2\right)^{1/2} \leq 1-\varepsilon_0$ <u>for some</u> $\varepsilon_0 > 0$

3. $\|p\|_{L^2(\Omega)} < +\infty$

4. $$\begin{cases} \nabla_i (H_{ij} + p\delta_{ij}) = f_j & \text{in } \Omega \\ (H_{ij} + p\delta_{ij})\nu_i = -F_j & \text{on } \Gamma_N \end{cases}$$

<u>then there exists a solution</u> u <u>to problem</u> (P_2) <u>or equivalently</u> (P_3) .

<u>Proof</u>: we have

$$\int_\Omega fu + \int_{\Gamma_N} Fu = \int_{\Gamma_D} (H_{ij}+p\delta_{ij})\tau_{ij}(u) - \int_\Omega (H_{ij}+p\delta_{ij})\varepsilon_{ij}(u) =$$

$$= -\int_\Omega H_{ij}\,\varepsilon_{ij}^D(u) - \int_\Omega p\,\mathrm{div}\,u - \int_{\Gamma_D} H_{ij}\,\tau_{ij}^D(u-g) - \int_{\Gamma_D}(H_{ij}+p\delta_{ij})\tau_{ij}(g)$$

hence

$$\mathcal{F}''(u) = \int_{\Omega_1} \phi(\varepsilon^D(u)) + \frac{K_0}{2}\int_{\Omega_1}(\mathrm{div}\,u)^2 - \int_\Omega p\,\mathrm{div}\,u -$$

$$- \int_\Omega H_{ij}\,\varepsilon_{ij}^D(u) - \int_{\Gamma_D} H_{ij}\,\tau_{ij}^D(u-g) - \int_{\Gamma_D}(H_{ij}+p\delta_{ij})\tau_{ij}(g)$$

If we now extend H_{ij} and p to Ω_1 with no 'discontinuity' on Γ_D ,still calling them H_{ij} and p , we get

$$\mathcal{F}''(u) = \mathcal{J}(u) + \int_{\Omega_1\setminus\bar\Omega} H_{ij}\,\tau_{ij}^D(g) + \int_{\Omega_1} p\,\mathrm{div}\,g - \int_{\Gamma_D}(H_{ij}+p\,\delta_{ij})\tau_{ij}(g)$$

where

$$\mathcal{J}(u) = \int_{\Omega_1}\phi(\varepsilon^D(u)) - \int_{\Omega_1} H_{ij}\,\tau_{ij}^D(u) + \frac{K_0}{2}\int_{\Omega_1}(\mathrm{div}\,u)^2 - \int_{\Omega_1} p\,\mathrm{div}\,u$$

therefore problem (P_2) is equivalent to problem

$$(P_3)' \quad \begin{cases} \text{minimize } \mathcal{J}(u) \text{ in the class of the functions} \\ u \in P(\Omega_1) \text{ such that } u=g \text{ in } \Omega_1 \setminus \bar{\Omega} \end{cases}$$

Now we have

$$(2.2) \qquad \mathcal{J}(g) < +\infty$$

and, for all $v \in P(\Omega_1)$ with $v=g$ on $\Omega_1 \setminus \bar{\Omega}$

$$(2.3) \qquad \mathcal{J}(v) \geq \varepsilon_0 \left\{ \int_{\Omega_1} |\varepsilon^D(v)| + \frac{\kappa_0}{2} \int_{\Omega_1} (\operatorname{div} v)^2 \right\} - c(\Omega, g)$$

Let's now take a minimizing sequence $\{v_h\} \subset P(\Omega_1)$ with $v_h = g$ in $\Omega_1 \setminus \Omega$ then, by proposition 2.2 i) ,we have that $v_h \cdot \nu = g \cdot \nu$ on Γ_D , and, by (2.2) , (2.3) and corollary 1.11 , the quantities $\|v_h\|_{BD(\Omega_1)}$, $\|\operatorname{div} v_h\|_{L^2(\Omega_1)}$ are bounded independently of h . By compactness theorem 1.12 possibly taking a subsequence, we have

$$v_h \longrightarrow v \qquad \text{in } L^1(\Omega_1, \mathbb{R}^n)$$

$$\operatorname{div} v_h \longrightarrow \operatorname{div} v \qquad \text{weakly}$$

where $v \in P(\Omega_1)$ by semicontinuity, $v=g$ in $\Omega_1 \setminus \Omega$ and $v \cdot \nu = g \cdot \nu$ on Γ_D (see proposition 2.2 i)) .

It is now sufficient to remark that the functional $\mathcal{J}(v)$ is lower-semicontinuous with respect to the weak convergence in order to conclude the proof.

q.e.d.

Let us remark that problem (P_2) can always be solved in case $f=0$, $F=0$ as one can see directly minimizing $\mathcal{J}''$ or just taking $H_{ij}=0$, $p=0$ in the preceding theorem. In particular we can always solve (P_2) in case $f=0$ and only Dirichlet boundary conditions are given.

<u>THEOREM 2.5</u> - <u>Suppose</u> u <u>is a solution to problem</u> (P_2) <u>and suppose that</u> $u \in C^2(\Omega) \cap C^0(\Omega \cup \Gamma_N)$, <u>then there exist functions</u> $\{H_{ij}\}_{i,j=1,\dots,n}$, p <u>such that</u>

$$\nabla_i \left(H_{ij} + p\delta_{ij} \right) = f_j \qquad \text{in } \Omega$$

$$\left(H_{ij} + p\delta_{ij} \right) \nu_i = -F_j \qquad \text{on } \Gamma_N$$

and that $\quad p \in L^2(\Omega)$, $|H| \leq 1$.

Proof: we have for every function $\varphi \in C^1(\Omega, \mathbb{R}^n)$ with $\varphi = 0$ on Γ_D

$$\int_\Omega \beta'(|\varepsilon^D(u)|) \sum_{i,j=1}^n \frac{\varepsilon^D_{ij}(u)}{|\varepsilon^D(u)|} \varepsilon^D_{ij}(\varphi) + K_0 \int_\Omega \operatorname{div} u \operatorname{div} \varphi + \int_\Omega f\varphi + \int_{\Gamma_N} F\varphi \, d\mathcal{H}^{n-1} = 0$$

where β is such that $\phi(a) = \beta(|a|)$. Integrating by parts and recalling that $\sum_{i,j} \varepsilon^D_{ij}(u)\delta_{ij} = \operatorname{trace}(\varepsilon^D(u)) = 0$ we get

$$\sum_{i,j=1}^n \left\{ \int_{\partial\Omega} \beta'(|\varepsilon^D(u)|) \frac{\varepsilon^D_{ij}(u)}{|\varepsilon^D(u)|} \nu_j \varphi^i - \int_\Omega \nabla_i \left(\beta'(|\varepsilon^D(u)|) \frac{\varepsilon^D_{ij}(u)}{|\varepsilon^D(u)|} \right) \varphi^j + \right.$$

$$\left. + K_0 \int_{\partial\Omega} \operatorname{div} u \; \delta_{ij} \nu_i \varphi^j - K_0 \int_\Omega \nabla_i (\operatorname{div} u \; \delta_{ij}) \varphi^j + \int_\Omega f_j \varphi^j + \int_{\Gamma_N} F_j \varphi^j \right\} = 0$$

Setting now $p = K_0 \operatorname{div} u$, $H_{ij} = \beta'(|\varepsilon^D(u)|) \dfrac{\varepsilon^D_{ij}(u)}{|\varepsilon^D(u)|}$ and choosing suitable test functions φ we get our result.

q.e.d.

3. Additional remarks

What is essentially needed to solve problem (P_3) by the direct method of calculus of variations, besides the lower-semicontinuity of the functional $\mathcal{F}''$, is a condition on the forces f , F that yield a bound of the type

$$(3.1) \qquad \left| \int_\Omega f u + \int_{\Gamma_N} F u \right| \leq (1-\varepsilon_0) \left(\int_\Omega \phi(\varepsilon^D(u)) + \frac{K_0}{2} \int_\Omega (\operatorname{div} u)^2 \right)$$

We have given such a condition in theorem 2.4 , we shall see now some other sufficient conditions that can be useful in special cases.

a) Suppose $T_N = \emptyset$, or $T_N \neq \emptyset$ and $F = 0$. If $f = f_1 + \nabla q$ $\operatorname{spt} q \subset \Omega$ then

$$\left| \int_\Omega f u \right| \leq \|f_1\|_{L^n(\Omega,\mathbb{R}^n)} \|u\|_{BD(\Omega)} + \|q\|_{L^2(\Omega)} \|\operatorname{div} u\|_{L^2(\Omega)}$$

so that (3.1) is verified in case $\|f_1\|_{L^n(\Omega,\mathbb{R}^n)}$ is sufficiently small, and the functional $\mathcal{F}''$ is obviously lower-semicontinuous.

A sharp estimate on how small $\|f_1\|_{L^n(\Omega,\mathbb{R}^n)}$ has to be depends on the best constant γ in the Sobolev-Poincaré inequality for functions with compact support

$$\left(\int_\Omega |u|^{n/n-1} \right)^{\frac{n-1}{n}} \leq \gamma \int_\Omega |\varepsilon(u)| \tag{3.2}$$

While the best constant γ is the isoperimetric constant in case u is a scalar function, it doesn't seem to be known in the more general case (3.2) .

b) Neumann boundary conditions are much better handled if a term of the type $\int_\Omega |u|^2 dx$ is added to the energy functional (this behaviour is well known for instance for elliptic linear equations or for the capillarity problem). In that case we look for a solution to

$$\begin{cases} \mathcal{F}''(u) + \int_\Omega |u|^2 \longrightarrow \inf \\ u \in P(\Omega) \cap L^2(\Omega) \\ u \cdot \nu = g \cdot \nu \qquad \text{on } \Gamma_D \end{cases}$$

One can still proceed as in theorem 2.4 and find a sufficient condition for the existence of a solution, the only difference is that the functions $(H_{ij} + p\delta_{ij})\nu_i - f_j$ are now only required to be in $L^2(\Omega)$. But in this case another sufficient condition for the existence can be obtained as follows. Suppose

$$f = f_1 + \nabla q \qquad , \qquad F_j = (K_{ij} + \sigma\delta_{ij}) \nu_i$$

where $f_1 \in L^2(\Omega, \mathbb{R}^n)$, $q \in L^2(\Omega)$, $q = 0$ on T_D , $K \in L^\infty(T_N, \mathbb{R}^{n^2})$ $(q+\sigma) \in H^{1/2}(T_N)$, then we have

$$\left| \int_\Omega f u + \int_{T_N} F u \right| \le \|f_1\|_{L^2(\Omega)} \|u\|_{L^2(\Omega)} + \|q\|_{L^2(\Omega)} \|\operatorname{div} u\|_{L^2(\Omega)} +$$

$$+ \|K\|_{L^\infty(T_N)} \|\tau^D\|_{L^1(T_N)} + \|q+\sigma\|_{H^{1/2}(T_N)} \|u_\nu\|_{H^{-1/2}(T_N)}$$

where we used the fact (see for example [21]) that if $u \in L^2(\Omega, \mathbb{R}^n)$ and $\operatorname{div} u \in L^2(\Omega)$ then $u \cdot \nu \in H^{-1/2}(\partial\Omega)$. Recall now that (see (1.1)')

$$\|u_\nu\|_{H^{-\frac{1}{2}}} \le \alpha_3(\Omega) \left\{ \|u\|_{L^2(\Omega, \mathbb{R}^n)} + \|\operatorname{div} u\|_{L^2(\Omega)} \right\}$$

$$\|\tau^D(u)\|_{L^1(T_N)} \le \alpha_1(n, L) \int_\Omega |\varepsilon(u)| + \alpha_2(\Omega) \int_\Omega |u|$$

so that we have

$$\left| \int_\Omega f u + \int_{T_N} F u \right| \le \delta \left\{ \|u\|^2_{L^2(\Omega, \mathbb{R}^n)} + \|\operatorname{div} u\|_{L^2(\Omega)} \right\} +$$

$$+ \alpha_1(n, L) \|K\|_{L^\infty(T_N, \mathbb{R}^n)} \int_\Omega |\varepsilon^D(u)| + \mathrm{const}(\delta)$$

where δ is any positive number, and in case

$$\alpha_1(n, L) \|K\|_{L^\infty(T_N, \mathbb{R}^n)} < 1 - \varepsilon_0$$

we can choose δ so that

$$\mathcal{F}''(u) + \int_\Omega |u|^2 \geq \varepsilon_0 \int_\Omega |\varepsilon^D(u)| + c_1 \int_\Omega |u|^2 + c_2 \int_\Omega (\operatorname{div} u)^2 - c_3$$

with $c_1, c_2 > 0$, $c_3 \geq 0$

Of course, in order to have a good condition on F one needs to know the best value of the constant $\alpha_1(n, L)$ in the trace estimate. More precisely, a sharp condition on F would follow from a trace estimate (on smooth domains) of the type

$$(3.3)\qquad \int_{\partial\Omega} |\tau^D(u)|\, d\mathcal{H}^{n-1} \le \int_{\Omega} |\varepsilon^D(u)| + c(\Omega)\int_{\Omega} |u|$$

Such an estimate seems to be reasonable but we don't know whether it is true or not. Actually, one would only need to have (3.3) with $\varepsilon(u)$ instead of $\varepsilon^D(u)$, notice however that the estimate

$$\int_{\partial\Omega} |u|\, d\mathcal{H}^{n-1} \le \int_{\Omega} |\varepsilon(u)| + c(\Omega)\int_{\Omega} |u|$$

is false in general as we can see taking

$$\Omega = \{ x \in \mathbb{R}^2 \mid x_2 < 0 \} \qquad , \qquad u(x) = \alpha\, \varphi_{T_\alpha}(x)$$

where $\varphi_{T_\alpha}(x)$ is the characteristic function of the set

$$T_\alpha = \left\{ x \in \Omega \;\middle|\; x_2 > -\frac{x_1}{\operatorname{tg}\gamma} \;,\; x_2 > (x_1 - d)\operatorname{tg}\gamma \right\}$$

$$\alpha = (\cos\gamma, \operatorname{sen}\gamma) \in \mathbb{R}^2 \qquad , \qquad \gamma \in (0, \pi/2)$$

and choosing γ such that

$$\cos\gamma > \operatorname{sen}\gamma + \frac{\cos\gamma}{\sqrt{2}}$$

In fact we have

$$\int_{\Omega} |\varepsilon(u)| = \left(\operatorname{sen}\gamma + \frac{\cos\gamma}{\sqrt{2}}\right) d$$

$$\int_{\Omega} |u| = \frac{d^2}{2} \operatorname{sen}\gamma \cos\gamma$$

$$\int_{\partial\Omega} |u| \ge \int_{\partial\Omega} |u_\tau| = d \cos\gamma$$

and it cannot exist a constant C such that for all $d>0$

$$d\cos\gamma \le d\left(\operatorname{sen}\gamma + \frac{\cos\gamma}{\sqrt{2}}\right) + C\,\frac{d^2}{2}\operatorname{sen}\gamma\cos\gamma$$

To conclude, we have to remark that in this case the term $\int_{\Gamma_N} Fu$ is not lower-semicontinuous, anyway, under the condition

$$(3.4) \qquad \alpha_1(n,L)\,\|K\|_{L^\infty} \leq 1$$

the functional $\mathcal{F}''(u)$ on the whole is still lower-semicontinuous with respect to $L^1(\Omega,\mathbb{R}^n)$ convergence (see[9] for the area functional). In fact, for $\delta>0$ set

$$\Omega_\delta = \left\{ x\in\Omega \;\middle|\; \operatorname{dist}(x,\partial\Omega)>\delta \right\}$$

for almost all δ we have

$$\int_{\partial\Omega_\delta} |\varepsilon(u)| = 0$$

Now, by the trace estimate (1.1)' and (3.4) , we have for u , $u_h \in P(\Omega)\cap L^2(\Omega,\mathbb{R}^n)$

$$\int_\Omega \phi(\varepsilon^D(u)) + \frac{K_0}{2}\int_\Omega (\operatorname{div} u)^2 + \int_{\Gamma_N} Fu - \int_\Omega \phi(\varepsilon^D(u_h)) - \frac{K_0}{2}\int_\Omega (\operatorname{div} u_h)^2 - \int_{\Gamma_N} Fu_h \leq$$

$$\leq \int_\Omega \phi(\varepsilon^D(u)) + \frac{K_0}{2}\int_\Omega (\operatorname{div} u)^2 - \int_\Omega \phi(\varepsilon^D(u_h)) - \frac{K_0}{2}\int_\Omega (\operatorname{div} u_h)^2 +$$

$$+ \int_{\Omega\setminus\Omega_\delta} |\varepsilon(u-u_h)| + c\int_{\Omega\setminus\Omega_\delta} |u-u_h| \leq$$

$$\leq \cdots + \int_{\Omega\setminus\Omega_\delta} |\varepsilon(u)| + \int_{\Omega\setminus\Omega_\delta} \phi(\varepsilon^D(u_h)) + \int_{\Omega\setminus\Omega} (\operatorname{div} u_h)^2 \leq$$

$$\leq \int_{\Omega_\delta} \phi(\varepsilon^D(u)) - \int_{\Omega_\delta} \phi(\varepsilon^D(u_h)) + \frac{K_0}{2}\int_{\Omega_\delta} (\operatorname{div} u)^2 - \frac{K_0}{2}\int_{\Omega_\delta} (\operatorname{div} u_h)^2 +$$

$$+ 2\int_{\Omega_\delta} |\varepsilon(u)| + \int_{\Omega_\delta} (\operatorname{div} u)^2 + \frac{2}{K_0}\mathcal{L}^n(\Omega\setminus\Omega_\delta)$$

Supposing now $u_h \longrightarrow u$ in $L^1(\Omega,\mathbb{R}^n)$,going to the limit for $h \to \infty$ and taking δ arbitrarily small we get our result.

c) Finally, by the same methods we have used so far, we can also study the case of a deformation energy of the type

$$\int_\Omega \phi(x, \varepsilon^D(u)) + \int_\Omega (\operatorname{div} u)^2$$

where $\phi : \bar{\Omega} \times M_n \longrightarrow [0,\infty)$ is a continuous function of (x, ε^D) and a convex function of ε^D, and

$$M_0|a| \leq |\phi(x,a)| \leq M_1(1+a)$$

holds. We can again define a function

$$\bar{\phi}(x,t,a) = \begin{cases} \phi(x, \frac{a}{t})\, t & \text{if } t>0 \\ \lim\limits_{t \downarrow 0} \phi(x, \frac{a}{t})\, t & \text{if } t=0 \end{cases}$$

to get a semicontinuous extension of the functional $\int_\Omega \phi(x, \varepsilon^D(u))$ to $BD(\Omega)$, and we can solve a mixed boundary value problem with relaxed Dirichlet conditions:

$$(P_4) \quad \begin{cases} \int_\Omega \phi(x, \varepsilon^D(u)) + \int_\Omega (\operatorname{div} u)^2 + \int_\Omega fu + \int_{T_N} Fu + \int_{\Gamma_D} \bar{\phi}(x, 0, \tau^D(u))\, dH^{n-1} \longrightarrow \inf \\ u \in P(\Omega) \\ u \cdot \nu = g \cdot \nu \quad \text{on } \Gamma_D \quad , \quad g \in P(\Omega_1) \end{cases}$$

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