On the Unitary Analogues of Certain Totients

By

K. Nageswara Rao, New Delhi, India

(Received June 1, 1965)

1. Introduction and Notation

In this paper we establish the unitary (*Cohen* [3]) analogues of certain well known arithmetical identities (cf [5]). In particular we shall establish the unitary analogue of the identity of the *Cohen*, *Jordan* and *Von-Sterneck* totients (cf Theorem 2.4 below).

We shall now establish some notation. Let M be a positive integer. A divisor d of M is said to be unitary (*Cohen* [3]; see also *Vaidyanathaswamy* [6, p. 606] where it is referred to as a "block factor") if $(d, M/d) = 1$. We write $d \,||\, M$ to say that d is a unitary divisor of M.

Let k be any fixed positive integer. For any positive integer N, let $(N, M^k)_k{}^*$ denote the largest unitary divisor of M^k, that divides N and is a k th power. Denote by $\Phi_k{}^*(M)$, the number of integers N in a complete residue system mod M^k such that $(N, M^k)_k{}^* = 1$. $\Phi_k{}^*(M)$ is the unitary analogue of *Cohen*'s totient (cf [2]).

We now consider the unitary analogue $J_k^*(M)$ of *Jordan*'s totient (*Dickson* [1; p 147]). An ordered k-tuple $(a_1, \ldots, a_k)$ of integers, where $1 \leq a_i \leq M$ for every $i = 1, \ldots, k$, is said to be *a* vector (mod k, M). Let $(a_i) = (a_1 \ldots, a_k)$ denote the greatest common divisor of $a_1, \ldots, a_k$. The number of vectors (mod k, M) such that $((a_i), M)^* \equiv ((a_i), M)_1{}^* = 1$ is denoted by $J_k{}^*(M)$.

The unitary analogue $H_k{}^*(M)$ of *Von-Sterneck*'s totient (*Dickson* [1; p 151]) is defined thus:

$$H_k{}^*(M) \equiv \sum_{[d_1, \ldots, d_k] = M} \Phi^*(d_1) \ldots \Phi^*(d_k)$$

where the sum is extended over all vectors $(d_1, \ldots d_k)$ (mod k, M) such that the least common multiple (l. c. m.) of $d_1, \ldots d_k$ is M and $d_i \,||\, M$ for every $i=1, \ldots, k$. Here and elsewhere in the paper $\Phi^* \equiv \Phi_1{}^*(M)$.

It should be mentioned that $\Phi^*(M)$ was discussed by *Cohen* [3] and *Jager* [4].

We shall also need the unitary analogue $\mu^*(M)$ (*Cohen* [3]) of the Möbius function defined thus: $\mu^*(n) = \pm 1$ according as the number of distinct prime factors of n is even or odd. μ^* has the important property of unitary inversion, namely that if g is an arithmetic function and

$$f(n) = \sum_{d \,||\, n} g(d), \text{ then } g(n) = \sum_{d \,||\, n} f(d)\, \mu^*(n/d).$$

We also need the concept of the unitary convolute $g^* h$ of two arithmetic functions g and h defined by

$$g^*h(n) = \sum_{d \,||\, n} g(d)\, h(n/d).$$

We note that if g and h are multiplicative then so is g^*h.

2. The Unitary Totients Φ_k^*, J_k^*, H_k^*

We shall first establish

Theorem 2.1

$$\Phi_k^*(M) = \sum_{d \,||\, M} d^k\, \mu^*(M/d)$$

the summation running over all unitary divisors d of M.

Proof: — By the definition of $\Phi_k^*(M)$, the number of integers less than or equal to M^k, each of which has d^k the greatest k th power common divisor of M^k which is unitary divisor of M^k, is $\Phi_k^*(M/d)$ for, if a is any integer satisfying $0 < a \leq M^k$ and $(a, M^k)_k^* = d^k$, then $0 < \frac{a}{d^k} \leq (M/d)^k$ and $(\frac{a}{d^k}, \frac{M^k}{d^k})_k^* = 1$. Thus to each integer a with $(a, M^k)_k^* = d^k$, there corresponds a unique integer $\frac{a}{d^k}$ with the property that $\left(\frac{a}{d^k}, \left(\frac{M}{d}\right)^k\right)_k^* = 1$ and the correspondence is reciprocal.

$$\therefore \sum_{d \,||\, M} \Phi_k^*(M/d) = M^k.$$

Hence by unitary inversion

$$\Phi_k^*(M) = \sum_{d \,||\, M} d^k\, \mu^*(M/d).$$

Corollary 2.1.1

$$\Phi_k^*(M) = M^k(1 - \frac{1}{p_1^{d_1 k}}) \ldots (1 - \frac{1}{p_n^{d_r k}})$$

where $M = p_1^{d_1} \ldots p_r^{dr}$ is the canonical representation of M.

Corollary 2.1.2

The function $\Phi_k^*(M)$ is multiplicative i. e.

$$\Phi_k^*(M_1 M_2) = \Phi_k^*(M_1)\, \Phi_k^*(M_2)$$

whenever $(M_1, M_2) = 1$.

We note that the function $\Phi_k^*(M)$ appeared in [4; p. 512] in an equivalent form.

Theorem 2.2

$$J_k^*(M) = \sum_{d \| M} d^k \mu^*(M/d).$$

Proof: — The number of vectors (a_i) (mod k, M) which are such that $((a_i), M)^* = d$ is $J_k^*(M/d)$. On the same lines of the proof of Theorem 2.1, it can be shown that

$$\sum_{d \| M} J_k^*(M/d) = M^k$$

and

$$J_k^*(M) = \sum_{d \| M} d^k \mu^*(M/d)$$

Theorem 2.3

$$H_k^*(M) = \sum_{d \| M} d^k \mu^*(M/d).$$

Proof: — From the definition of $H_k^*(M)$, we have

$$\sum_{d \| M} H_k^*(d) = \Sigma\, \Phi^*(d_1) \ldots \Phi^*(d_k)$$

where the summation on the right runs over all ordered sets of k unitary divisors $d_1, \ldots, d_k$ of M, for the least common multiple of unitary divisors of M is again an unitary divisor of M.

$$\sum_{d \| M} H_k^*(d) = (\sum_{\delta_1 \| M} \Phi_r^*(\delta_1)) \ldots (\sum_{\delta_k \| M} \Phi^*(\delta_k))$$

where each δ_i ranges over all the unitary divisors of M.

i.e. $$\sum_{d \| M} H_k^*(d) = M^k.$$

Hence by unitary inversion

$$H_k^*(M) = \sum_{d \| M} d^k \mu^*(M/d).$$

Theorem 2.4

The functions $\Phi_k^*(M)$, $J_k^*(M)$ and $H_k^*(M)$ are equal.

Theorems 2.1, 2.2 and 2.3 establish the truth of Theorem 2.4.

3. Some more Arithmetical Identities

Let $\tau^*(M)$ denote the number of unitary divisors of M and let $\sigma_k^*(M) = \sum_{d \| M} d^k$. We shall now prove some identities involving Φ_k^*, σ_k^* and τ^*.

Theorem 3.1

$$\sum_{d \| M} \Phi_k^*(M/d)\, \tau^*(d) = \sigma_k^*(M).$$

Proof: — The functions on both the sides are multiplicativa and hence we will verify the truth of the statement of the result for $M = p^a$, a prime power.

$$\sum_{d \| p^a} \Phi_k^*(p^a/d)\, \tau^*(d) = \Phi_k^*(p^a)\, \tau^*(1) + \Phi_k^*(1)\, \tau^*(p^a)$$
$$= p^{ak} - 1 + 2 = p^{ak} + 1 = \sigma_k^*(p^a)$$

Theorem 3.2

$$\sum_{d \| M} \Phi^*(M/d)\, d^k = \sum_{\delta \| M} \Phi_k^*(M/\delta)\, \delta.$$

The result can be obtained by using the multiplicative property of $\Phi_k^*(M)$ and $\Phi^*(M)$ and using Theorem 2.1 on the lines of Theorem 3.1.

Theorem 3.3

$$\sum_{d \| M} \sigma_{s+k}^*(d)\, \Phi_k^*(M/d) = M^k\, \sigma_s^*(M) \quad s, k \text{ being integers.}$$

Proof: — Functions on both the sides are multiplicative and hence $\sum_{d \| M} \sigma_{s+k}(d)\, \Phi_k^*(M/d)$, which is the unitary convolute of $\sigma_{s+k}^*(M)$ and $\Phi_k^*(M)$, is equal to $\prod_p \left[\sum_{d \| p^a} \sigma_{s+k}^*(d)\, \Phi_k^*(p^a/d) \right]$ where $\prod_p$ denotes the product over all distinct prime factors of M. But this is equal to

$$\prod_p [\sigma_{s+k}^*(1)\, \Phi_k^*(p^a) + \sigma_{s+k}^*(p^a)\, \Phi_k^*(1)]$$
$$= \prod_p [\Phi_k^*(p^a) + \sigma_{s+k}^*(p^a)] = \prod_p [p^{ak} - 1 + p^{a(k+s)} + 1] = \prod_p [p^{ak}(1 + p^{as})]$$
$$= \prod_p [p^{ak}\, \sigma_s^*(p^a)] = M^k\, \sigma_s^*(M).$$

Theorem 3.4

$$\Phi_k^*(M) = \sum_{d \,||\, M} d^{k-1}\, \Phi^*_{k-1}(M/d)\, \Phi^*(d) \qquad \text{for } k > 1.$$

Since the functions on both the sides are multiplicative the result can easily be verified for M, a prime power.

Theorem 3.5

$$\mu^*(M) = \sum_{d \,||\, M} d^{k-1}\, \Phi^*_{k-1}\,(M/d)\; \mu^*(d).$$

The result follows from the multiplicative property of the functions involved in it. The same can be verified for any prime power.

Theorem 3.6

$$\Phi_k^*(M) = \sum_{[d_1, \ldots, d_n] = M} \Phi_{k_1}^*(d_1) \ldots \Phi^*_{k_n}(d_n)$$

where the summation extends over all ordered sets of n unitary divisors $d_1, \ldots, d_n$ of M such that their l. c. m. is M and $k_1, \ldots, k_n$ are positive integers such that their sum is equal to k.

Proof: — If the required sum on the right be denoted by $R_k(M)$, then

$$\sum_{d \,||\, M} R_k(d) = \sum \Phi^*_{k_1}(\delta_1) \ldots \Phi^*_{k_n}(\delta_n)$$

summed on the right over all ordered sets of n unitary divisors $\delta_1, \ldots \delta_n$ of M.

$$\sum_{d \,||\, M} R_k(d) = [\sum_{\delta_1 \,||\, M} \Phi^*_{k_1}(\delta_1)] \ldots [\sum_{\delta_n \,||\, M} \Phi^*_{k_n}(\delta_n)]$$

$$= M^{k_1} \ldots M^{k_n} = M^{k_1 + \cdots + k_n} = M^k.$$

Hence by the unitary inversion

$$R_k(M) = \sum_{d \,||\, M} d^k\, \mu^*(M/d).$$

But the latter is

$$\Phi_k^*(M).$$

All the results of § 3 can be obtained by changing summation orders and using the definition of unitary convolution. These results also can be obtained by using generating series of the arithmetic functions involved and the calculus of multiplicative arithmetic functions in particular the compounding operation of *Vaidyanathaswamy* [6].

References

[1] *L. E. Dickson*: History of the theory of numbers, Vol. I (New York), 1934.

[2] *Eckford Cohen*: Some totient functions, Duke Math. J. **23** (1956) pp. 515—522.

[3] *Eckford Cohen*: Arithmetical functions associated with the unitary divisors of an integer, Math. Zeit. **74** (1960) pp. 66—80.

[4] *H. Jager*: The unitary analogues of some identities for certain arithmetical functions, Kon. Nederl. Akad. Wet. Proc. Ser. A, Math. Sciences **64** (1961) pp. 508—515.

[5] *K. Nageswara Rao*: On three extensions of Ramanujan's sum, M. Phil. research thesis (1960), Sri Venkateswara University, Tirupati, India.

[6] *R. Vaidyanathaswamy*: The theory of multiplicative arithmetic functions, Trans. Amer. Math. Soc. **33** (1931) pp. 579—662.

Department of Mathematics,
Sri Venkateswara College,
(University of Delhi),
1, Rouse Avenue,
New Delhi - 1,
India.