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Stochastic McKean-Vlasov equations *

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Abstract

We prove the existence and uniqueness of solution to the nonlinear local martingale problems for a large class of infinite systems of interacting diffusions. These systems, which we call the stochastic McKean-Vlasov limits for the approximating finite systems, are described as stochastic evolutions in a space of probability measures on $\mathcal{R}^d$ and are obtained as weak limits of the sequence of empirical measures for the finite systems, which are highly correlated and driven by dependent Brownian motions. Existence is shown to hold under a weak growth condition, while uniqueness is proved using only a weak monotonicity condition on the coefficients. The proof of the latter involves a coupling argument carried out in the context of associated stochastic evolution equations in Hilbert spaces. As a side result, these evolution equations are shown to be positivity preserving. In the case where a dual process exists, uniqueness is proved under continuity of the coefficients alone. Finally, we prove that strong continuity of paths holds with respect to various Sobolev norms, provided the appropriate stronger growth condition is verified. Strong solutions are obtained when a coercivity condition is added on to the growth condition guaranteeing existence.

1 Introduction

Limit theorems for systems of exchangeable diffusions in $\mathcal{R}^d$ have been extensively investigated since the original work of McKean [23] on the propagation of chaos in physical systems of interacting particles related to Boltzmann's model for the statistical mechanics of rarefied gases. In the classical case the limit is described as the solution of a nonlinear deterministic evolution equation known as

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the McKean-Vlasov equation. In this paper we investigate a related class of exchangeable diffusions for which propagation of chaos fails and the limiting equation is a nonlinear stochastic evolution equation .

In each of these limit theorems, the particles are assumed to exhibit exchangeability of motion, which is built in the structure of their respective drift and diffusion coefficients and allows for both convergence and characterization. The reader is referred to Gärtner [15] for some historical background and a bibliographical survey. A very general version of the weak law of large numbers in the case of interacting diffusions is found there, as well as the corresponding propagation of chaos. Fluctuations and large deviation results as well are the subject of active research (for example, see Dawson [10], Dawson and Gärtner [11], [12], Brunaud [7], Ben Arous and Brunaud [2]).

In the present paper, we are interested in extending another classical result of probability theory to systems of exchangeable diffusions: the law of large numbers for strongly dependent exchangeable triangular arrays. More precisely, under weak growth and monotonicity conditions on the operator valued coefficients, the sequence of laws of the empirical measures associated with a triangular array of exchangeable diffusions converges weakly (on the appropriate space of trajectories) to a measure-valued diffusion, referred to as the stochastic McKean-Vlasov limit for the sequence. The law of this stochastic process is characterized as the unique solution to the local martingale problem associated with a nonlinear evolution operator of McKean-Vlasov type.

The class of diffusions considered encompasses many of those in [15], where the limits are deterministic evolutions in the space of probability measures on $\mathcal{R}^d$. In many cases here, these solutions are not deterministic. The extension will readily be seen as a natural one.

Recently, Baldwin et al. [1] and Chiang et al. [9] have considered some McKean-Vlasov limits of systems of interacting diffusions driven by a noise valued in the dual of a nuclear space, whereby obtaining stochastic limits as well. The limiting process there exhibits propagation of chaos, a consequence of the weak correlation between the particles. In the present framework, the driving Wiener processes take their values in much smaller spaces and the stochasticity of the limit arises through the strong correlation built in the system of diffusion equations, not from the loudness of the noise. There is no propagation of chaos here, just propagation of the mixture.

The reader should consult Graham [17] for an alternate generalization of Gärtner's work, to nonlinear diffusions with jumps; the stochasticity of the limits there ensues from the non-degeneracy of the jumps in the limit and again propagation of chaos obtains.

Building on the work of Bismut [3] and Kunita [20] on stochastic flows of diffeomorphisms on a manifold (about which more will be said in section 3), Borkar [5] characterized the measure-valued diffusions associated with the stochastic evolution of infinitely many particles in a Brownian medium, for the special case where the evolution operator has polynomial coefficients. His construction yields strong Lipschitz continuity of the trajectories of a single tagged particle driven by the stochastic measure valued solution, a property unlikely to hold for the more general class considered in the present paper.

An alternative approach to Borkar's, which also allows for a detailed analysis of infinitely many strongly interacting diffusions, this time by constructing their paths directly on a Brownian sheet, was presented in Walsh [32] and has been considerably extended by Kotelenez [19]. Here again the strong construction requires Lipschitz conditions on the coefficients of the evolution operator.

The choice we make here to define our processes as solutions to local martingale problems allows for existence and uniqueness results to emerge under considerably weaker conditions than Lipschitz continuity, in parallel with the theory of finite-dimensional diffusions, as presented for instance in the book by Stroock and Varadhan [27].

We next sketch a brief outline of the paper.

The frequently used notation has been gathered in section 2.

In section 3, we review and extend some of the results from Gärtner [15] and Vaillancourt [31] pertinent to the tightness of the laws of the ensembles of N particles, as their size N grows unboundedly, thereby obtaining the existence of stochastic McKean-Vlasov limits. The only conditions required of the coefficients governing the evolution of the finite systems at this point are ellipticity, continuity and a weak growth condition. Passing references are made to the two degenerate cases, partly excluded from our presentation as they are already covered in great detail in Gärtner [15] and Kunita [20].

In section 4, we prove the uniqueness of solution to the limiting local martingale problem, for the special class of McKean-Vlasov limits possessing a function valued dual, in the sense of Dawson and Kurtz [13]. An interesting feature of these particular McKean-Vlasov limits is that, as infinite particle systems, they are in duality with none other than the family of Feller semigroups associated with the sequence of finite particle systems which generated them in the first place. Since existence of a dual process ensures uniqueness of solution, no additional assumptions are required in this special case, other than those ensuring existence.

In section 5, we show that any (probability measure valued) McKean-Vlasov limit constructed in section 3 takes its values in the dual of a Sobolev space. In fact it is shown there that the weak topology on the space of probability measures $\mathcal{M}(\mathcal{R}^d)$ is generated by a scalar product. This suggests the possibility of writing the McKean-Vlasov limits as Hilbert space valued solutions to stochastic evolution equations. We also show in section 5 that any solution to the local martingale problem of section 3 arises as the law of a weak solution to the stochastic evolution equation

$$\langle \mu_t, \phi \rangle = \int_0^t \langle \mu_s, L(\mu_s)\phi \rangle \, ds + \int_0^t \langle \int \sigma^T(\cdot, y, \mu_s)\mu_s(\cdot) W(ds, dy), \nabla\phi(\cdot) \rangle \,, \quad (1.1)$$

where $\{L(\mu) : \mu \in \mathcal{M}_2\}$ is a class of second order differential operators, ϕ belongs to the space of test functions $\mathcal{S}$ and σ is matrix valued and reasonably smooth.

The meaning of equation (1.1) is made precise in section 5 and involves the construction of Hilbert space valued cylindrical Wiener processes W_s of Bojdecki and Jakubowski [4].

In section 6, we use a coupling argument to obtain the distributional uniqueness of solution to the above stochastic evolution equation. The local martingale problems associated with McKean-Vlasov limits are therefore well-posed by the argument of Yamada and Watanabe [33]. Both results are obtained using a rather weak monotonicity condition. As a corollary, we derive the preservation of positivity for a wide class of evolution equations. Section 7 contains additional information about the smoothness of the values taken by this solution, as well as about the continuity of its trajectories with respect to certain Sobolev norms, under stronger growth conditions.

An alternative construction of the McKean-Vlasov limits as strong solutions to stochastic evolution equations in rigged Hilbert spaces is provided in section 8. We show how to use the Krylov-Rozovskii [22] results to obtain that the unique solution to the local martingale problem of sections 3 and 6 arises as the law of the unique strong solution to equation

$$\mu_t = \mu_0 + \int_0^t L^*(\mu_s)\mu_s \, ds + \int_0^t \sum_{i=1}^{d} \sum_{\ell=1}^{f} \partial_i \big(\beta_{i\ell}(\cdot, \mu_s)\mu_s \, dW_\ell(s)\big) , \tag{1.2}$$

(where superscript $*$ indicates the adjoint to a linear operator) under coercivity (which ensures existence) and monotonicity (for uniqueness), plus some smoothness conditions on a and b. The Wiener process appearing here must take values in a space of strongly differentiable functions — conditions on its covariance kernel will ensure this. Bear in mind that such strong solutions will always live inside a Sobolev space and not its dual, so they take their values amongst absolutely continuous probability measures and require very strong conditions indeed. However we give a natural example (8.1) of a McKean-Vlasov limit for which explicit calculations are readily carried out to illustrate the growth, monotonicity and coercivity conditions in sections 6 to 8.

2 Notation

We will often need the following spaces:

$C(X_1, X_2)$, the space of bounded continuous functions $X_1 \to X_2$ with the uniform topology, where X_1 is any topological space and X_2 is any normed space — we write $C(X_1)$ in the case $X_2 = \mathcal{R}$;

$C_k^2(\mathcal{R}^d)$, the space of real-valued continuous functions with compact support which are twice continuously differentiable;

$C^2(\mathcal{R}^d)$, the space of real-valued bounded continuous functions, twice continuously differentiable with first and second derivatives all bounded;

$\mathcal{S}$, the Schwartz space of smooth functions $\mathcal{R}^d \to \mathcal{R}$ which together with all their derivatives are rapidly decreasing;

$\mathcal{S}'$, the Schwartz space of tempered distributions, dual to $\mathcal{S}$;

$C([0,\infty) : X)$, the space of continuous paths $[0,\infty) \to X$ with the topology of uniform convergence on compact sets, where X is a metric space;

$\mathcal{M}(X)$, the space of probability measures on a metric space X, with the weak topology defined by the Prohorov metric (see Ethier and Kurtz [14]) — weak convergence being denoted as usual by $\Rightarrow$;

$\mathcal{M}_2 = \{\mu \in \mathcal{M}(\mathcal{R}^d) : \langle \mu, \varphi_2 \rangle < \infty\}$, where $\varphi_2(z) := 1 + |z|^2 = 1 + z_1^2 + \cdots + z_d^2$, with the strongest topology coinciding on $\{\mu \in \mathcal{M}(\mathcal{R}^d) : \langle \mu, \varphi_2 \rangle < R\}$ with the weak topology, for every $R > 1$ — note that this topology turns $\mathcal{M}_2$ into a Lusin space, for details see Gärtner [15];

$\mathcal{D}$, the set of all continuous functions on $\mathcal{M}_2$ of the form $\psi(\langle \mu, \phi \rangle)$ where $\psi : \mathcal{R} \to \mathcal{R}$ is smooth and $\phi \in C_k^2(\mathcal{R}^d)$;

$L_2(X_1, X_2)$, the Hilbert space of Hilbert-Schmidt operators from a Hilbert space X_1 into another X_2 (When $X_2 = \mathcal{R}$, it is omitted.);

$L_Q(X_1, X_2)$, the space of linear operators $B : Q^{1/2}X_1 \to X_2$ such that $BQ^{1/2} \in L_2(X_1, X_2)$, where $Q \in L_2(X_1, X_1)$ has a square-root;

$H^j \subset H^0 = L_2(\mathcal{R}^d) \subset H^{-j}$ and $(H^j)^d \subset (H^0)^d \subset (H^{-j})^d$, the rigged Hilbert spaces defined in Section 5 for any $j \geq 0$;

$\mathcal{R}^d \otimes \mathcal{R}^f$, the space of real $d \times f$ matrices, with the euclidean norm $|\cdot|$;

$(\mathcal{R}^d \otimes \mathcal{R}^d)_+$, its subspace of symmetric positive definite $d \times d$ real matrices (in the wide sense).

We will also need the following notation:

δ_z, the unit measure at $z \in \mathcal{R}^d$;
$\epsilon_x := 1/N \sum_{k=1}^N \delta_{x_k}$ when $x = (x_1, x_2, \ldots, x_N) \in (\mathcal{R}^d)^N$;
$\langle \mu, \phi \rangle := \int \phi(z)\mu(dz)$ for $\phi \in C(X)$ and $\mu \in \mathcal{M}(X)$;
$\mu^{\times 2}(dz_1\, dz_2) := \mu(dz_1)\mu(dz_2)$;
$\{\mu(t)\}$ or $\{\mu_t\}$, the canonical process on $C([0,\infty) : \mathcal{M}(\mathcal{R}^d))$;
$(\cdot,\cdot)_j$, the scalar product on the Hilbert spaces H^j and $(H^j)^d$, defined in Section 5, with $||\cdot||_j$ the associated norm;
$A[\phi]$, the canonical bilinear form pairing $A \in H^{-j}$ $(\mathcal{S}')$ with $\phi \in H^j$ $(\mathcal{S})$;
$\partial_i : \mathcal{S}' \to \mathcal{S}'$, the weak partial derivative with respect to the i^{th} coordinate;
∇, the gradient operator acting on $C^2((\mathcal{R}^d)^N)$ for any positive integer N, written as $\nabla^T = (\nabla_i^T)_{i=1}^N$, where ∇_i is the gradient on $C^2(\mathcal{R}^d)$ acting on the i^{th} coordinate. All vectors are written in column form.

3 Exchangeable diffusions

The evolution of the system of N particles will be described using the martingale problem formulation of Stroock and Varadhan [27]. Under the appropriate conditions, the system will form a $(\mathcal{R}^d)^N$-valued diffusion with the particles exhibiting exchangeability of motion.

Explicitly, for mappings $b : \mathcal{R}^d \times \mathcal{M}_2 \to \mathcal{R}^d$, $a : \mathcal{R}^d \times \mathcal{M}_2 \to (\mathcal{R}^d \otimes \mathcal{R}^d)_+$ and $c : \mathcal{R}^d \times \mathcal{R}^d \times \mathcal{M}_2 \to (\mathcal{R}^d \otimes \mathcal{R}^d)_+$, the (diffusion) operator for the system of N (exchangeable) particles $L^{(N)} : C_k^2((\mathcal{R}^d)^N) \to C((\mathcal{R}^d)^N)$ may be written in the form

$$
\begin{aligned}
L^{(N)}\phi(x) &= \sum_{i=1}^{N} b^T(x_i, \epsilon_x)\nabla_i\phi(x) + \frac{1}{2}\sum_{i=1}^{N}(a(x_i, \epsilon_x)\nabla_i)^T\nabla_i\phi(x) \\
&\quad + \frac{1}{2}\sum_{\substack{i,j=1 \\ i\neq j}}^{N}(c(x_i, x_j, \epsilon_x)\nabla_j)^T\nabla_i\phi(x).
\end{aligned}
\tag{3.1}
$$

Equip space $\mathcal{M}_2 = \{\mu \in \mathcal{M}(\mathcal{R}^d) : \langle\mu, \varphi_2\rangle < \infty\}$, where we denote $\varphi_2(z) := 1 + |z|^2 = 1 + z_1^2 + \cdots + z_d^2$, with the strongest topology coinciding on $\{\mu \in \mathcal{M}(\mathcal{R}^d) : \langle\mu, \varphi_2\rangle < R\}$ with the weak topology, for every $R > 1$ — note that we have $\mu_n \to \mu$ in $\mathcal{M}_2$ if and only if both $\mu_n \Rightarrow \mu$ in $\mathcal{M}(\mathcal{R}^d)$ and $\sup_n\langle\mu_n, \varphi_2\rangle < \infty$ hold.

A probability measure $P_\pi^{(N)}$ on the Borel subsets of $C([0,\infty) : (\mathcal{R}^d)^N)$ solves the martingale problem started at $\pi \in \mathcal{M}((\mathcal{R}^d)^N)$ for operator $L^{(N)}$, if we have $P_\pi^{(N)} \circ x(0)^{-1} = \pi$ and, for every choice of ϕ in the domain of $L^{(N)}$, the process $\phi(x(t)) - \int_0^t L^{(N)}\phi(x(s))\,ds$ is a $P_\pi^{(N)}$-martingale with respect to the natural (Borel) right continuous filtration on $C([0,\infty) : (\mathcal{R}^d)^N)$. If a solution exists and is unique for every $\pi \in \mathcal{M}((\mathcal{R}^d)^N)$, we say that the martingale problem for $L^{(N)}$ is well-posed.

A similar definition holds with $(\mathcal{R}^d)^N$ replaced by any metric space X — see chapter 4 of Ethier and Kurtz [14] for more details. We will also need the concept of a martingale problem on more abstract topological spaces below but there is no added difficulty there.

The family $\{P_\pi^{(N)} : \pi \in \mathcal{M}((\mathcal{R}^d)^N)\}$ will be called (N-) exchangeable if it satisfies $P_{\pi\circ\sigma^{-1}}^{(N)} = P_\pi^{(N)} \circ \tilde{\sigma}^{-1}$ for every $\pi \in \mathcal{M}((\mathcal{R}^d)^N)$ and every permutation σ of the N coordinates in $(\mathcal{R}^d)^N$, with $\tilde{\sigma}(x(\cdot))(t) := \sigma(x(t))$ for each $x(\cdot) \in C([0,\infty) : (\mathcal{R}^d)^N)$. Since $L^{(N)}(\phi \circ \sigma)(x) = L^{(N)}(\phi)(\sigma(x))$ for every choice of σ, ϕ and x, the following is a particular instance of theorem 10.2.2 of Stroock and Varadhan [27] (see also Gärtner [15]).

Theorem 3.1 *Assume that the mappings a, b and c above are continuous, with $a(z,\mu) - c(z,z,\mu)$ strictly positive definite for every choice of z and μ, and c such that all matrices of the form $(c(x_i,x_j,\epsilon_x))_{i,j=1}^N$ are positive definite for every choice of x and N. Assume also that there exists $K > 0$ such that Trace$(a(z,\mu)) + 2b^T(z,\mu)\, z \leq K(1+|z|^2)$ holds for all z and all measures μ with finite support. The martingale problem for operator $L^{(N)}$ on $C_k^2((\mathcal{R}^d)^N)$ is then well-posed on $C([0,\infty) : (\mathcal{R}^d)^N)$ for each $N \geq 1$ and its family of unique solutions is exchangeable.*

The N particles system may also be viewed in terms of its associated empirical process $\mathcal{P}^{(N)} := P_{\pi_N}^{(N)} \circ e_N^{-1}$, defined for each probability measure $\pi_N \in \mathcal{M}((\mathcal{R}^d)^N)$ with a finite second moment, by the canonical mapping $e_N : C([0,\infty) : (\mathcal{R}^d)^N) \to C([0,\infty) : \mathcal{M}_2)$, with $e_N(x(\cdot))(t) := \epsilon_{x(t)}$ — this measurable mapping is well-defined because of equation (3.2) below. No information on the trajectories is lost as a result of the exchangeability of motion of the N particles.

The next theorem characterizes the limit points of $\{\mathcal{P}^{(N)} : N \geq 1\}$ in the sense of weak convergence in the space of probability measures on $C([0,\infty) : \mathcal{M}_2)$. This requires some explanation as $\mathcal{M}_2$ is a Lusin space but not a Polish space. The following topological results are taken from Gärtner [15], where the reader will find detailed proofs of all statements.

Equip space $C([0,\infty) : \mathcal{M}_2)$ with the strongest topology which coincides on $C([0,\infty) : \{\mu \in \mathcal{M}(\mathcal{R}^d) : \langle \mu, \varphi_2 \rangle < R\})$ with the weak topology for each $R > 1$. In this topology, we have $\mu_n(\cdot) \to \mu(\cdot)$ in $C([0,\infty) : \mathcal{M}_2)$ if and only if both weak convergence $\mu_n(\cdot) \Rightarrow \mu(\cdot)$ in $C([0,\infty) : \mathcal{M}(\mathcal{R}^d))$ and, for every $T \geq 1$, $\sup_n \sup_{t\in[0,T]} \langle \mu_n(t), \varphi_2 \rangle < \infty$, hold.

Since the Borel subsets of $C([0,\infty) : \mathcal{M}_2)$ (resp. $\mathcal{M}_2$) are simply the intersections of Borel subsets of $C([0,\infty) : \mathcal{M}(\mathcal{R}^d))$ (resp. $\mathcal{M}(\mathcal{R}^d)$) with $C([0,\infty) : \mathcal{M}_2)$ (resp. $\mathcal{M}_2$), every probability measure on $C([0,\infty) : \mathcal{M}_2)$ (resp. $\mathcal{M}_2$) has a unique extension to $C([0,\infty) : \mathcal{M}(\mathcal{R}^d))$ (resp. $\mathcal{M}(\mathcal{R}^d)$). We do not distinguish one from the other in the notation.

The definition of weak convergence on the space of probability measures on $C([0,\infty) : \mathcal{M}_2)$ (resp. $\mathcal{M}_2$) is the usual one, i.e., $\{\mathcal{P}_N\}$ converges to $\mathcal{P}_\infty$ in the weak sense if $\{\int f\, d\mathcal{P}_N\}$ converges to $\int f\, d\mathcal{P}_\infty$ for all real valued bounded continuous functions f on $C([0,\infty) : \mathcal{M}_2)$ (resp. $\mathcal{M}_2$).

Given probability measures $\{\mathcal{P}_N\}$ and $\mathcal{P}_\infty$ on $C([0,\infty) : \mathcal{M}_2)$, $\{\mathcal{P}_N\}$ is sequentially weakly compact (resp. converges weakly to $\mathcal{P}_\infty$) if and only if both sequential weak compactness of $\{\mathcal{P}_N\}$ (resp. weak convergence to $\mathcal{P}_\infty$) in $\mathcal{M}(C([0,\infty) : \mathcal{M}(\mathcal{R}^d)))$ and, for every positive T, the following limit $\lim_{R\to\infty} \sup_N \mathcal{P}_N\big(\sup_{t\in[0,T]} \langle \mu(t), \varphi_2 \rangle > R\big) = 0$, hold. A similar statement holds for sequences of probability measures on $\mathcal{M}_2$, with the supremum over time removed from the last condition.

The operator $\mathcal{L}^{(\infty)} : \mathcal{D} \to C(\mathcal{M}_2)$ which generates the process describing the evolution of our infinite particle system — the stochastic McKean-Vlasov limit for the sequence of finite systems — is defined by

$$\begin{aligned} \mathcal{L}^{(\infty)}\psi(\langle\mu,\phi\rangle) &:= \psi'(\langle\mu,\phi\rangle)\langle\mu, L(\mu)\phi\rangle + \frac{\psi''(\langle\mu,\phi\rangle)}{2}\langle\mu^{\times 2}, R(\mu)\phi\rangle\,, \\ \text{with}\quad L(\mu)\phi(z) &:= b^T(z,\mu)\nabla\phi(z) + \frac{1}{2}(a(z,\mu)\nabla)^T\nabla\phi(z) \\ \text{and}\quad R(\mu)\phi(z_1,z_2) &:= (\nabla\phi(z_1))^T c(z_1,z_2,\mu)\nabla\phi(z_2)\,. \end{aligned}$$

Denote by $\epsilon_t^N := 1/N\sum_{k=1}^N \delta_{x_k^N(t)}$ the empirical process associated with the solution to the martingale problem of theorem (3.1) and started with distribution $\{\pi_N \in \mathcal{M}(\mathcal{M}_2)\}$; by $\mathcal{P}^{(N)}$, its law on $C([0,\infty) : \mathcal{M}_2)$; and by $E^{(N)}$, the corresponding expectation. Note that $t \mapsto \langle\epsilon_t^N, \varphi_2\rangle$ is a continuous map for each N though not necessarily in the limit.

Theorem 3.2 *Assume that the conditions in theorem* (3.1) *hold. Provided that the sequence of starting measures* $\{\pi_N \in \mathcal{M}(\mathcal{M}_2)\}$ *is sequentially weakly compact and verifies* $\sup_N E^{(N)}\langle\epsilon_0^N, \varphi_2\rangle < \infty$, *the sequence* $\{\mathcal{P}^{(N)}\}$ *is sequentially compact in* $\mathcal{M}\big(C([0,\infty) : \mathcal{M}_2)\big)$ *and all its limit points are solutions to the local martingale problem on* $C([0,\infty) : \mathcal{M}_2)$ *for* $\mathcal{L}^{(\infty)} : \mathcal{D} \to C(\mathcal{M}_2)$.

For any starting measure $\pi_\infty \in \mathcal{M}(\mathcal{M}_2)$, there exists at least one solution to the local martingale problem on $C([0,\infty) : \mathcal{M}_2)$ for $\mathcal{L}^{(\infty)}$.

In particular, for any $\phi \in C_k^2(\mathcal{R}^d)$, the process $M(\phi)$ associated with $\langle\cdot,\phi\rangle$, given by $M_t(\phi) := \langle\mu(t),\phi\rangle - \int_0^t\langle\mu(s), L(\mu(s))\phi\rangle\, ds$, is a continuous locally square integrable martingale, with quadratic variational process

$$< M(\phi) >_t = \int_0^t \langle\mu(s)^{\times 2}, R(\mu(s))\phi\rangle\, ds\,.$$

The proof is broken into a series of lemmas.

Lemma 3.1 *Assume that the conditions in theorem* (3.1) *hold.*
Provided $\sup_N E^{(N)}\langle\epsilon_0^N, \varphi_2\rangle < \infty$, *we have*

$$\lim_{R\to\infty}\sup_N \mathcal{P}^{(N)}\Big(\sup_{t\in[0,T]}\langle\epsilon_t^N, \varphi_2\rangle \geq R\Big) = 0\,.$$

Proof: The stopping times $\tau_R^N := \inf\{t > 0 : \langle\epsilon_t^N, \varphi_2\rangle > R\}$ for $R \geq 1$ ensure that $M^N_{t\wedge\tau_R^N}(\varphi_2) := \langle\epsilon^N_{t\wedge\tau_R^N}, \varphi_2\rangle - \int_0^{t\wedge\tau_R^N}\langle\epsilon_s^N, L(\epsilon_s^N)\varphi_2\rangle\, ds$ is a bounded martingale, and the growth condition implies, for every $t \geq 0$,

$$0 \leq E^{(N)}\langle\epsilon^N_{t\wedge\tau_R^N}, \varphi_2\rangle \leq E^{(N)}\langle\epsilon_0^N, \varphi_2\rangle + K\int_0^t E^{(N)}\langle\epsilon^N_{s\wedge\tau_R^N}, \varphi_2\rangle\, ds\,.$$

Gronwall's inequality (see Ethier and Kurtz[14]) yields, for every positive T, N and $R > 1$,

$$\sup_{t\in[0,T]} E^{(N)}\langle \epsilon^N_{t\wedge\tau^N_R}, \varphi_2\rangle \le e^{KT} E^{(N)}\langle \epsilon^N_0, \varphi_2\rangle . \tag{3.2}$$

This implies both $\mathcal{P}^{(N)}(\lim_{R\to\infty}\tau^N_R = \infty) = 1$, for each N, and, for every positive T and N,

$$\sup_{t\in[0,T]} E^{(N)}\langle \epsilon^N_t, \varphi_2\rangle \le e^{KT} E^{(N)}\langle \epsilon^N_0, \varphi_2\rangle . \tag{3.3}$$

On the set $\{\tau^N_R < \infty\}$, we have $\langle \epsilon^N_{\tau^N_R}, \varphi_2\rangle = R$ for every $N \ge 1$ and $R > 1$, because of the continuity of trajectories of the empirical process $\{\epsilon^N_t : t \ge 0\}$.

This implies, for every positive T, N and $R > 1$,

$$R\cdot \mathcal{P}^{(N)}(\tau^N_R \le T) \le E^{(N)}\langle \epsilon^N_{T\wedge\tau^N_R}, \varphi_2\rangle .$$

Making use of equation (3.2), we get

$$\mathcal{P}^{(N)}\Big(\sup_{t\in[0,T]}\langle \epsilon^N_t, \varphi_2\rangle \ge R\Big) \le \frac{1}{R} e^{KT} E^{(N)}\langle \epsilon^N_0, \varphi_2\rangle ,$$

from which the conclusion follows. □

Lemma 3.2 *Assume that the conditions in lemma* (3.1) *hold. Provided that the sequence of starting measures* $\{\pi_N\}$ *is sequentially weakly compact, so is the sequence* $\{\mathcal{P}^{(N)}\}$.

Proof: Denote by $\mathcal{P}^{(N)}_{\phi,R,T}$ the law on $C([0,T] : [0,R])$ of the stopped process $\langle \epsilon^N_{t\wedge\tau^N_R}, \phi\rangle$. By lemma (1.4) of Gärtner [15] and the present lemma (3.1), it suffices to show that the sequence $\{\mathcal{P}^{(N)}_{\phi,R,T}\}$ is sequentially weakly compact for every choice of $\phi \in C^2_k(\mathcal{R}^d)$, $R > 1$ and $T > 0$. Using theorem (1.4.6) of Stroock and Varadhan [27], all we need to prove is that, for every $\psi \in C^2(\mathcal{R}^d)$, there is a positive constant $A = A(\psi,\phi,T,R)$, invariant under translation of ψ and independent of N, such that $\psi(\langle \epsilon^N_{t\wedge\tau^N_R}, \phi\rangle) + At$ is a submartingale for all values of N.

Treating $\psi(\langle \epsilon_x, \phi\rangle)$ as a function of x, the martingale problem for $L^{(N)}$ (theorem 3.1) implies that, for every $s,t \in [0,T]$,

$$\psi(\langle \epsilon^N_{t\wedge\tau^N_R}, \phi\rangle) - \psi(\langle \epsilon^N_{s\wedge\tau^N_R}, \phi\rangle) + (\Psi^N(t) - \Psi^N(s)) = \int_{s\wedge\tau^N_R}^{t\wedge\tau^N_R} \mathcal{L}^{(N)}\psi(\langle \epsilon_u, \phi\rangle)\, du ,$$

where

$$\begin{aligned}\mathcal{L}^{(N)}\psi(\langle\mu,\phi\rangle) &:= \mathcal{L}^{(\infty)}\psi(\langle\mu,\phi\rangle)\\ &+\frac{1}{2N}\psi''(\langle\mu,\phi\rangle)\langle\mu, (\nabla\phi(\cdot))^T\big(a(\cdot,\mu) - c(\cdot,\cdot,\mu)\big)\nabla\phi(\cdot)\rangle\end{aligned} \tag{3.4}$$

and $\Psi^N(t)$ is a continuous local $\mathcal{P}^{(N)}_{\phi,R,T}$-martingale.

Since $\mathcal{M}^R := \{\mu \in \mathcal{M}(\mathcal{R}^d) : \langle \mu, \varphi_2 \rangle \leq R\}$ is a sequentially weakly compact subset of $\mathcal{M}_2$, the continuity of a, b and c plus the compactness of support of ϕ ensures that $A = \sup_N \sup_{\mu \in \mathcal{M}^R} |\mathcal{L}^{(N)} \psi(\langle \mu, \phi \rangle)|$ is finite, and translation invariant in ψ since $\mathcal{L}^{(N)}$ is. Therefore, $\Psi^N(t)$ is actually a bounded $\mathcal{P}^{(N)}_{\phi,R,T}$-martingale and we are done, since we can infer

$$\psi(\langle \epsilon^N_{t\wedge\tau^N_R}, \phi \rangle) - \psi(\langle \epsilon^N_{s\wedge\tau^N_R}, \phi \rangle) + (\Psi^N(t) - \Psi^N(s)) \geq -A(t-s) . \qquad \square$$

Lemma 3.3 *Assume that the conditions in lemma* (3.2) *hold. Any weak limit of the sequence* $\{\mathcal{P}^{(N)}\}$ *solves the local martingale problem for* $\mathcal{L}^{(\infty)}$.

Proof: For every choice of $\mu \in C([0,\infty) : \mathcal{M}_2)$ and $R \geq 1$, define

$$\tau_R := \inf\{t > 0 : \langle \mu(t), \varphi_2 \rangle > R\} ,$$

with $\tau_R = \infty$ whenever the set is empty. For every choice of $F \in \mathcal{D}$, $N = 1, 2, \ldots, \infty$ and $t \geq 0$, the mapping $\Xi^N(\cdot, t) : C([0,\infty) : \mathcal{M}_2) \to \mathcal{R}$ defined by

$$\mu \to \Xi^N(\mu, t) := F(\mu(t \wedge \tau_R)) - \int_0^{t\wedge\tau_R} \mathcal{L}^{(N)} F(\mu(s))\, ds$$

is bounded and continuous in view of the argumentation in the preceding lemma (3.2). Equation (3.4) yields $\sup_{\mu \in C([0,\infty):\mathcal{M}_2)} |\Xi^N(\mu,t) - \Xi^\infty(\mu,t)| \leq Ct/N$for some positive constant C, so that for any weakly convergent subsequence of $\{\mathcal{P}^{(N)}\}$ with limit $\mathcal{P}$, we have

$$\lim_{m\to\infty} \int G(\cdot) \Xi^{N_m}(\cdot, t)\, d\mathcal{P}^{(N_m)}(\cdot) = \int G(\cdot) \Xi^\infty(\cdot, t)\, d\mathcal{P}(\cdot)$$

for every $\mathcal{F}_s$-measurable $G \in C\big(C([0,\infty) : \mathcal{M}_2)\big)$, with $\mathcal{F}_s$the usual filtration at s on $C([0,\infty) : \mathcal{M}_2)$. Since Ξ^N is a $\mathcal{P}^{(N)}$-martingale for each $N \geq 1$, then Ξ^∞ is a $\mathcal{P}$-martingale as well. The argument in the proof of lemma (3.1) may be duplicated here to conclude that $\mathcal{P}(\lim_{R\to\infty} \tau_R = \infty) = 1$, which completes the proof. $\square$

The last two lemmas together imply the existence of at least one solution to the local martingale problem for operator $\mathcal{L}^{(\infty)}$, for every starting distribution $\pi_\infty \in \mathcal{M}(\mathcal{M}_2)$. The last statement of theorem (3.2) is an application of Itô's formula and the proof of the theorem is complete. $\square$

Special attention should be given at this point to the two possible interpretations afforded by stochastic McKean-Vlasov limits arising respectively from the following two degenerate classes of possible limits. In the case $c = 0$, Gärtner also proved that, under some mild coercivity and monotonicity conditions on operator $\mathcal{L}^{(\infty)}$, the above local martingale problem is actually a martingale problem. It is well-posed and its solution is the unit mass along the trajectory of the unique solution $\mu(\cdot)$ to the classical McKean-Vlasov evolution equation

$$\frac{d}{dt} \langle \mu(t), \phi \rangle = \langle \mu(t), L(\mu(t))\phi \rangle ,$$

for every $\phi \in C^2_k(\mathcal{R}^d)$, given any starting point $\mu(0) \in \mathcal{M}_2$.

All solutions to the local martingale problem of theorem (3.2) can therefore be viewed as classical, deterministic McKean-Vlasov systems which are randomly perturbed by immersion into a Brownian medium, which is highly correlated locally and can exhibit relatively slow decay when particles move apart. This physical interpretation is made mathematically explicit in the form of operator $\mathcal{L}^{(\infty)}$ by the addition of a tagged noise term (with continuous diffusion coefficient c) to each particle, in such away that particles lying close together are affected by highly correlated noise sources.

In the degenerate case where $a(z,\mu)-c(z,z,\mu)$ is positive definite everywhere, but not strictly positive definite for some choices of z and μ, the uniqueness of solution to the martingale problems of theorem (3.1) still holds, provided we assume an additional, Lipschitz continuity condition on both b and some square-root of each matrix of the form $(c^*(x_i,x_j,\epsilon_x))_{i,j=1}^N$, with $c^*(x_i,x_j,\epsilon_x) = c(x_i,x_j,\epsilon_x)$ when $i \neq j$ and $c^*(x_i,x_j,\epsilon_x) = a(x_i,\epsilon_x)$ when $i = j$ — see section 5.2 in Stroock and Varadhan [27].

In particular, when representation $c(z_1,z_2,\mu) = \sum_{j=1}^K \sigma(z_1,j,\mu)\sigma^T(z_2,j,\mu)$ holds for some Lipschitz continuous mapping $\sigma : \mathcal{R}^d \times \{1,\dots,K\} \times \mathcal{M}_2 \to \mathcal{R}^d \otimes \mathcal{R}^f$ and $a(z,\mu) = c(z,z,\mu)$ holds for every choice of z and μ, the martingale problem of theorem (3.1) can be seen to arise from the following N-point motion in a stochastic flow in $(\mathcal{R}^d)$, driven by K independent $(\mathcal{R}^f)$-valued Wiener process $\{W_j : j = 1,\dots K\}$ (cf. Kunita, 1984):

$$x_i(t) = x_i(0) + \int_0^t b(x_i(s),\epsilon_{x(s)})\,ds + \sum_{j=1}^K \int_0^t \sigma(x_i(s),j,\epsilon_{x(s)})\,dW_j(s).$$

We can therefore provide a second interpretation of the solutions of the local martingale problem of theorem (3.2) under these additional Lipschitz conditions on the coefficients, namely as stochastic flows enhanced by an ambient diffusion source which acts locally on each particle, solely as a function a of that particle's position and the position of the entire cloud of particles.

4 Uniqueness using duality arguments

If we make the restriction that the coefficients a, b and c of operator $\mathcal{L}^{(\infty)}$ do not depend on μ, then uniqueness of solution for the local martingale problem of theorem (3.2) may be obtained using a duality argument, without any additional condition to those ensuring existence of a solution. Monotonicity is not needed in that case: continuity of the coefficients plus a growth condition will suffice. This class of martingale problems is akin to that studied by Dawson and Kurtz [13], where difference operators play the role of our second order differential operator $R(\mu)$ in generating randomness in the measure-valued trajectories.

Theorem 4.1 *Assume that the conditions in theorem (3.1) hold and that the coefficientsa, b and c do not depend on* μ. *The local martingale problem on* $C([0,\infty):\mathcal{M}_2)$ *for* $\mathcal{L}^{(\infty)}$ *is then a martingale problem and it is well-posed.*

Proof: We actually prove the unicity of solution on the larger, Polish space $C([0,\infty):\mathcal{M}(\mathcal{R}^d))$, for every starting distribution supported by $\mathcal{M}_2$. Because of the independence in μ, the processes

$$F(\mu(t)) - \int_0^t \mathcal{L}^{(\infty)} F(\mu(s))\, ds$$

are actually bounded $\mathcal{P}$-martingales, for every $F \in \mathcal{D}$ and every solution $\mathcal{P}$ to the local martingale problem for $\mathcal{L}^{(\infty)}$. This is the first statement. Consider the set of all continuous functions on $\mathcal{M}(\mathcal{R}^d)$ of the form $F_f(\mu) := \int_{(\mathcal{R}^d)^\ell} f\, d\mu^{\times \ell}$, where f belongs to $C^2((\mathcal{R}^d)^\ell)$ for some $\ell \geq 1$ and $\mu^{\times\ell}$ is the ℓ-fold product of measure $\mu \in \mathcal{M}(\mathcal{R}^d)$ by itself. For any such F_f, operator $\mathcal{L}^{(\infty)}$ may be rewritten in the (dual) form

$$\mathcal{L}^{(\infty)} F_f(\mu) = F_{L^{(\ell)} f}(\mu)$$

where $L^{(\ell)}$ is defined in theorem (3.1). This means that the (deterministic) function valued process started at $f \in C_k^2((\mathcal{R}^d)^\ell)$ which is dual to the limiting empirical measure process is precisely $\mathcal{S}_t^\ell(f)$, where $\{\mathcal{S}_t^\ell\}$ stands for the Feller semigroup for the system of ℓ particles which generated the sequence of empirical processes in the first place! By theorem (3.1), the martingale problem on $C([0,\infty): \cup_{\ell\geq 1} C^2((\mathcal{R}^d)^\ell))$ associated with the generator given by the family of operators $\{L^{(\ell)} : \ell \geq 1\}$ has a unique solution for every starting point $f \in \cup_{\ell \geq 1} C_k^2((\mathcal{R}^d)^\ell)$. The result then follows from the duality equation for $\mathcal{L}^{(\infty)}$ (see Ethier and Kurtz [14]), provided the following integrability condition is satisfied: for every positive T and every $f \in \cup_{\ell\geq 1} C_k^2((\mathcal{R}^d)^\ell)$, there exists an integrable random variable $\Gamma = \Gamma(T,f)$ such that $\sup_{s,t\leq T}(|F_{\mathcal{S}_t^\ell(f)}(\mu(s))| + |\mathcal{L}^{(\infty)} F_{\mathcal{S}_t^\ell(f)}(\mu(s))|) \leq \Gamma$. Since $\{\mathcal{S}_t^\ell\}$ is a contraction semigroup, we may select

$$\Gamma = \sup_x(|f(x)| + |L^{(\ell)} f(x)|) < \infty . \quad \square$$

5 McKean-Vlasov limits in Hilbert spaces

From now on we immerse $\mathcal{M}(\mathcal{R}^d)$ continuously into $\mathcal{S}'$, the Schwartz space of tempered distributions (see Trèves [30]; Holley and Stroock [18]), in order to analyze the smoothness of the sample paths of our stochastic McKean-Vlasov limits more closely. Let $h_n(t) := (\pi^{1/2} 2^n n!)^{-1/2}(-1)^n e^{t^2/2} D_t^n(e^{-t^2})$ for $n = 0, 1, \ldots$ and $t \in \mathcal{R}$, where D_t^n denotes the n^{th} derivative with respect to t. Define the Hermite function of index $n = (n_1, n_2, \ldots, n_d)$ by $h_n(z) := h_{n_1}(z_1) \cdot \ldots \cdot h_{n_d}(z_d)$ for each $z = (z_1, z_2, \ldots, z_d) \in \mathcal{R}^d$. The set $\{h_n\}$ forms a complete orthonormal system in $L_2(\mathcal{R}^d)$ and satisfies the Charlier-Cramér [8] upper bound $\sup_{n,z} |h_n(z)| < 1$.

Szász [28] later proved $\sup_{n,z} |h_n(z)| = h_0(0) = \pi^{-d/4}$ — see chapters 7, 8 and 9 of Szegö [29]. The separable Hilbert spaces

$$H^j := \left\{ X \in \mathcal{S}' : ||X||_j^2 = \sum_n (2|n|_1 + d)^j X[h_n]^2 < \infty \right\},$$

with $|n|_1 := \sum_{i=1}^d n_i$, provide the dense continuous inclusions

$$\mathcal{S} \subset H^j \subset H^0 = L_2(\mathcal{R}^d) \subset H^{-j} \subset \mathcal{S}'$$

for any real valued $j \geq 0$. Remember that the elements of H^j have all their first order partial derivatives belonging to H^{j-1} for every $j \in \mathcal{R}$ and that the inclusion $H^j \subset H^i$ is of Hilbert-Schmidt type as soon as $j - i > d$. We denote by $(H^j)^f$ (resp. $(H^j)^{d \otimes f}$) the Hilbert space of mappings from $\mathcal{R}^d$ into $\mathcal{R}^f$ (resp. $\mathcal{R}^d \otimes \mathcal{R}^f$) such that each coordinate belongs to H^j. The Charlier-Cramér bound implies both $\sup_z |\nabla h_n(z)|^2 \leq (2|n|_1 + d)$ and $\sup_z |\nabla \nabla^T h_n(z)|^2 \leq (2|n|_1 + d)^2$, using the well-known representation of the derivative of Hermite functions ($d = 1$):

$$D_t^1 h_n(t) = (\frac{n}{2})^{1/2} h_{n-1}(t) - (\frac{n+1}{2})^{1/2} h_{n+1}(t).$$

The Charlier-Cramér bound also implies the elementary observation $\mathcal{M}(\mathcal{R}^d) \subset H^{-d-1}$: indeed, for all probability measures $\mu \in \mathcal{M}(\mathcal{R}^d)$, we have $||\mu||_{-d-1}^2 \leq \sum_n (2|n|_1 + d)^{-d-1} \langle \mu, h_n^2 \rangle \leq C_{-d-1}$, with $C_j := \sum_n (2|n|_1 + d)^j$ finite whenever $j < -d$. Thus every solution to the local martingale problem of theorem (3.2) starts and remains inside the dual space H^{-d-1} with probability one. Further, we have continuity with respect to the norm $|| \cdot ||_{-d-1}$ for every solution as a consequence of the following proposition.

Proposition 5.1 *The weak topology on $\mathcal{M}(\mathcal{R}^d)$ is the restriction of the norm topology of H^{-d-1} to its subset $\mathcal{M}(\mathcal{R}^d)$.*

Proof: Weak convergence of a sequence $\{\mu_m\} \in \mathcal{M}(\mathcal{R}^d)$ to $\mu \in \mathcal{M}(\mathcal{R}^d)$ implies, for every n, $\lim_{m\to\infty} \langle \mu_m, h_n \rangle = \langle \mu, h_n \rangle$. This last statement is equivalent to $\lim_{m\to\infty} ||\mu_m - \mu||_{-d-1} = 0$ because of the Charlier-Cramér bound and the finiteness of C_{-d-1}. Convergence in the norm $|| \cdot ||_{-d-1}$ of a sequence $\{\mu_m\} \in \mathcal{M}(\mathcal{R}^d)$ to $\mu \in \mathcal{M}(\mathcal{R}^d)$ is also easy. Since $\mathcal{S}$ is convergence-determining — $C_k^2(\mathcal{R}^d)$ is actually in the uniform closure of $\mathcal{S}$, see Trèves [30], lemma 15.2 — it suffices to show that $\mathcal{S}$ is in the uniform closure of the linear span of $\{h_n\}$. Since $(\sum_n |(\phi, h_n)_0|)^2 \leq C_{-j} \sum_n (2|n|_1 + d)^j (\phi, h_n)_0^2 < \infty$ holds for any $\phi \in H^j$ with $j > d$ by the Cauchy-Schwarz inequality, the sequence of continuous functions $\sum_{|n|_1 \leq \ell} (\phi, h_n)_0 h_n \in \mathcal{S}$ converges uniformly to ϕ for every choice of $\phi \in \mathcal{S}$. □

In fact, a bit more work yields convergence in the stronger, inductive topology on $\mathcal{S}$ of $\sum_{|n|_1 \leq \ell} (\phi, h_n)_0 h_n$ to ϕ — see formula (A.15) in the appendix of Holley and Stroock [18].

Beware: $\mathcal{M}(\mathcal{R}^d)$ is not closed with respect to the norm $||\cdot||_{-d-1}$, since convergence in this norm does not imply tightness in $\mathcal{M}(\mathcal{R}^d)$.

An immediate consequence of proposition (5.1) is that all solutions to the martingale problem of theorem (3.2) are in fact probability measures on $C([0,\infty) : H^{-d-1})$. The rest of the paper is devoted to proving uniqueness and regularity of solution on that space.

Proposition 5.2 *In addition to the conditions of theorem* (3.1)*, assume that, for every choice of $R > 1$, there exists $K_R > 0$ such that*

$$|a(z,\mu)| + |b(z,\mu)| \leq K_R(1+|z|^2)$$

holds for all choices of $z \in \mathcal{R}^d$ and $\mu \in \mathcal{M}^R = \{\mu \in \mathcal{M}(\mathcal{R}^d) : \langle \mu, \varphi_2 \rangle \leq R\}$. For any solution $\mathcal{P}$ to the local martingale problem for $\mathcal{L}^{(\infty)}$, started at a random point μ_0 satisfying $E\langle \mu_0, \varphi_2\rangle < \infty$,

$$M_t := \sum_n (2|n|_1 + d)^{-d-2} M_t(f_n) f_n$$

where $f_n = (2|n|_1 + d)^{\frac{d+2}{2}} h_n$, defines an H^{-d-2}-valued continuous square-integrable local martingale.

Proof: Recall from the proof of lemma (3.3) the localizing sequence of stopping times $\{\tau_R : R > 1\}$, defined by $\tau_R := \inf\{t > 0 : \langle \mu(t), \varphi_2 \rangle > R\}$. For each $\phi \in C_k^2(\mathcal{R}^d)$ and any solution $\mathcal{P}$, define $M_t(\phi)$ and $Q_t(\phi)$, the local $\mathcal{P}$-martingales associated with the process $\langle \mu_t, \phi\rangle$ and its square, by $M_t(\phi) := \langle \mu_t, \phi \rangle - \int_0^t \langle \mu_s, L(\mu_s)\phi\rangle\, ds$ and $Q_t(\phi) := M_t^2(\phi) - < M(\phi) >_t$. The first observation is that the same property holds for each $\phi \in \mathcal{S}$ under the additional growth condition on a and b — we need not assume the compactness of support of ϕ for the local martingale property to remain, because any smooth function with rapidly decreasing derivatives can be approximated uniformly, together with a finite number of its derivatives, by a sequence in $C_k^2(\mathcal{R}^d)$.

Indeed, let $\{\phi_n\} \in C_k^2(\mathcal{R}^d)$ be such an approximating sequence for some $\phi \in \mathcal{S}$, so that both $\{M_{t\wedge\tau_R}(\phi_n)\}$ and$\{Q_{t\wedge\tau_R}(\phi_n)\}$ are square-integrable martingales for every $n \geq 1$ and $R > 1$.

Writing $|\cdot|_\infty$ for the sup-norm on $C(\mathcal{R}^d)$, and by extension on the spaces of vectors and matrices with coordinates valued in $C(\mathcal{R}^d)$, we get

$$\begin{aligned}\sup_{t\in[0,T]} |M_{t\wedge\tau_R}(\phi_n) & - M_{t\wedge\tau_R}(\phi)| \\ & \leq |\phi_n - \phi|_\infty + \int_0^{T\wedge\tau_R} \langle \mu_s, |L(\mu_s)(\phi_n - \phi)|\rangle\, ds \\ & \leq |\phi_n - \phi|_\infty + K_R R T\big(|\nabla(\phi_n - \phi)|_\infty + |\nabla\nabla^T(\phi_n - \phi)|_\infty\big),\end{aligned}$$

this last inequality the result of the definition of τ_R. The martingale property for $M_{t\wedge\tau_R}(\cdot)$ and the fact that it remains well defined and square-integrable in the limit follow at once.

The conditions of theorem (3.1) implicitly ensure that any bound on a carries over to c, since $|c(z_1, z_2, \mu)|^2 \leq \text{Trace}(a(z_1, \mu))\ \text{Trace}(a(z_2, \mu))$ holds for all $z_1, z_2 \in \mathcal{R}^d$ and $\mu \in \mathcal{M}(\mathcal{R}^d)$. The additional growth condition on a implies

$$|c(z_1, z_2, \mu)|^2 \leq dK_R^2(1 + |z_1|^2)(1 + |z_2|^2)$$

for all choices of $z_1, z_2 \in \mathcal{R}^d$ and $\mu \in \mathcal{M}^R$.

Therefore, the martingale property for Q remains valid in the limit, with $< M(\phi) >_t = \int_0^t \langle \mu(s)^{\times 2}, R(\mu(s))\phi \rangle \, ds$, in view of

$$\begin{aligned}
\sup_{t\in[0,T]} |Q_{t\wedge\tau_R}(\phi_n) & - Q_{t\wedge\tau_R}(\phi)| \\
& \leq \sup_{t\in[0,T]} |M_{t\wedge\tau_R}(\phi_n) - M_{t\wedge\tau_R}(\phi)|^2 \\
& \quad + 2 \sup_{t\in[0,T]} |M_{t\wedge\tau_R}(\phi)| \sup_{t\in[0,T]} |M_{t\wedge\tau_R}(\phi_n) - M_{t\wedge\tau_R}(\phi)| \\
& \quad + \sqrt{d} K_R R^2 T |\nabla(\phi_n - \phi)|_\infty \left(|\nabla(\phi_n)|_\infty + |\nabla(\phi)|_\infty \right).
\end{aligned}$$

The Charlier-Cramér bound plus Doob's martingale inequality applied to $M_{t\wedge\tau_R}(\phi)$ together yield, for every $R > 1$ and $T \geq 0$,

$$\begin{aligned}
E \sup_{t\in[0,T]} & ||M_{t\wedge\tau_R}||_{-d-2}^2 \\
& \leq 4E||M_{T\wedge\tau_R}||_{-d-2}^2 \\
& = 4\sum_n (2|n|_1 + d)^{-d-2} E < M(h_n) >_{T\wedge\tau_R} + 4||\mu_0||_{-d-2}^2 \\
& \leq 4\sqrt{d} K_R \sum_n (2|n|_1 + d)^{-d-2} |\nabla h_n|_\infty^2 E \int_0^{T\wedge\tau_R} \langle \mu_s, \varphi_2 \rangle^2 \, ds + 4C_{-d-2} \\
& \leq 4\sqrt{d} K_R R^2 T C_{-d-1} + 4C_{-d-2} < \infty.
\end{aligned}$$

Writing $\sup_\delta$ for the supremum over all choices of $s, t \in [0, T]$ satisfying $|t - s| \leq \delta$, we also get an upper bound for $E \sup_\delta ||M_{t\wedge\tau_R} - M_{s\wedge\tau_R}||_{-d-2}^2$:

$$\begin{aligned}
& \sum_{|n|_1 \leq i} (2|n|_1 + d)^{-d-2} E \sup_\delta M_{t\wedge\tau_R}(h_n) - M_{s\wedge\tau_R}(h_n))^2 \\
& + E \sup_\delta \sum_{|n|_1 > i} (2|n|_1 + d)^{-d-2} (M_{t\wedge\tau_R}(h_n) - M_{s\wedge\tau_R}(h_n))^2.
\end{aligned}$$

Since the second term is bounded above by

$$\begin{aligned}
& 4E \sup_{t\in[0,T]} \sum_{|n|_1 > i} (2|n|_1 + d)^{-d-2} M_{t\wedge\tau_R}^2(h_n) \\
& \leq 16(\sqrt{d} K_R R^2 T + 1) \sum_{|n|_1 > i} (2|n|_1 + d)^{-d-1},
\end{aligned}$$

which goes to 0 as $i \to \infty$, it suffices to show that the first term can be made arbitrarily small for each fixed i. The weak continuity of $M_{t\wedge\tau_R}$, i.e., the continuity of the real-valued martingale $M_{t\wedge\tau_R}(h_n)$ for each n, is immediate from the continuity conditions on the coefficients of operator $L(\cdot)$ — see the beginning of this proof. By a theorem of Paul Lévy, the continuous martingale $M_{t\wedge\tau_R}(h_n)$ possesses a continuous time change which turns it into a Brownian motion stopped at random time $< M(h_n) >_{\tau_R}$. Lévy's modulus of continuity for Brownian motion(see, e.g., McKean [24]) guarantees the uniform continuity of $M_{t\wedge\tau_R}(h_n)$ and completes the proof. □

We now state the particular form which coefficient c must take in order to obtain a stochastic integral representation for our processes. Recall that a cylindrical Wiener process W in $(H^j)^f$ is defined as a mapping $W : \Omega \times [0,\infty) \times (H^j)^f \to \mathcal{R}$ such that $W_t(h) := W(\cdot,t,h)$ is a real-valued standard Brownian motion for every h in the unit ball of $(H^j)^f$ and $W(\omega,t,\cdot)$ is ω-almost surely linear on $(H^j)^f$ for every $t \in [0,\infty)$. Here Ω stands for a (possible) enlargement of $C([0,\infty) : \mathcal{M}_2)$. The weak continuity of W in $(H^j)^f$ is immediate.

The stochastic integral $N = \int X\, dW$ of a progressively measurable process X valued in the space of Hilbert-Schmidt operators $L_2((H^j)^f, H^k)$ with respect to a cylindrical Wiener process W in $(H^j)^f$ is written as follows (see Métivier and Pellaumail [25]): $N_t := \sum_n (2|n|_1 + d)^j \int_0^t X_s(h_n)\, dW_s(h_n)$ or equivalently $N_t[\phi] := \sum_n (2|n|_1 + d)^j \int_0^t (\phi, X_s(h_n))_k\, dW_s(h_n)$.

Theorem 5.1 *Assume that the mappings $a : \mathcal{R}^d \times \mathcal{M}_2 \to (\mathcal{R}^d \otimes \mathcal{R}^d)_+, b : \mathcal{R}^d \times \mathcal{M}_2 \to \mathcal{R}^d$ and $\sigma : \mathcal{R}^d \times \mathcal{R}^d \times \mathcal{M}_2 \to \mathcal{R}^d \otimes \mathcal{R}^f$ are continuous and that a and b satisfy the growth conditions of theorem (3.1) and proposition (5.2). Assume that the mapping $c(z_1,z_2,\mu) = (\sigma(z_1,\cdot,\mu), \sigma^T(z_2,\cdot,\mu))_{-d-2}$ is such that $a(z,\mu) - c(z,z,\mu)$ is strictly positive definite for each $z \in \mathcal{R}^d$ and $\mu \in \mathcal{M}_2$ — here we mean that element c_{ij} of matrix c is given by $c_{ij}(z_1,z_2,\mu) = \sum_{\ell=1}^f (\sigma_{i,\ell}(z_1,\cdot,\mu), \sigma_{j,\ell}(z_2,\cdot,\mu))_{-d-2}$. For any solution $\mathcal{P}$ to the local martingale problem of theorem (3.2), started at a random point μ_0 satisfying $E\langle\mu_0,\varphi_2\rangle < \infty$, there exists on some enlargement of the ambient probability space a progressively measurable process $X_s \in L_2((H^{-d-2})^f, H^{-d-2})$ and a cylindrical Wiener process W in $(H^{-d-2})^f$ such that $M_t(\phi) - M_0(\phi) = \left(\int_0^t X_s\, dW_s\right)[\phi]$ holds for every $\phi \in \mathcal{S}$.*

Remark 5.1 *Notice that the positivity condition on c in theorem (3.1) is automatically satisfied here.*

Proof: First define process $U_s \in L_2(H^{-d-2}, (H^{-d-2})^f)$ densely on its domain by $U_s(\phi) := \int \sigma^T(z,\cdot,\mu_s)\nabla\phi(z)\, \mu_s(dz)$ for all $\phi \in \mathcal{S}$. It is clearly progressively measurable; so is its adjoint $X_s = U_s^* \in L_2((H^{-d-2})^f, H^{-d-2})$. We first show that U_s and X_s are indeed valued in the appropriate space of Hilbert-Schmidt operators prior to time τ_R, i.e., as long as μ_s belongs to $\mathcal{M}^R$.

Standard operator theory (see,e.g., Gelfand and Vilenkin [16]) yields that the Hilbert-Schmidt norm of U_s (and therefore also that of X_s) is given by:

$$
\begin{aligned}
& \sum_n (2|n|_1 + d)^{-d-2} ||U_s(h_n)||^2_{-d-2} \\
= \ & \sum_{m,n} (2|n|_1 + d)^{-d-2} (2|m|_1 + d)^{-d-2} (U_s(h_n), h_m)_0^2 \\
\leq \ & \sum_n (2|n|_1 + d)^{-d-2} \int |\nabla h_n(z)|^2 ||\sigma(z, \cdot, \mu_s)||^2_{-d-2} \, \mu_s(dz) \\
\leq \ & C_{-d-1} \int ||\sigma(z, \cdot, \mu_s)||^2_{-d-2} \, \mu_s(dz) \\
\leq \ & C_{-d-1} \int \mathrm{Trace}(a(z, \mu_s)) \, \mu_s(dz) \\
\leq \ & C_{-d-1} K_R R \sqrt{d}
\end{aligned}
$$

where we obtain the first inequality via Fubini's theorem, the second one with the Charlier-Cramér bound, the third one via the assumption of positive definiteness, the fourth one because of the growth condition of proposition (5.2) on a, together with the assumption $s \leq \tau_R$.

Notice that we have just proved that $\sigma(z, \cdot, \mu_s)$ belongs to $(H^{-d-2})^{d \otimes f}$ for every z and $s \leq \tau_R$, so that c is indeed well defined. Note also that U_s as a compact linear operator has a unique extension to all of H^{-d-2} which remains of Hibert-Schmidt type.

Finally we prove that the tensor cross-variation process verifies

$$
<< M >>_{t \wedge \tau_R} = \int_0^{t \wedge \tau_R} X_s U_s \, ds \, .
$$

Let $\{f_n = (2|n|_1 + d)^{(d+2)/2} h_n\}$ denote a complete orthonormal basis of H^{-d-2}. The scalar cross-variation $< M(f_m), M(f_n) >_{t \wedge \tau_R}$ is given by

$$
\begin{aligned}
& \int_0^{t \wedge \tau_R} \langle \mu_s^{\times 2}, R(\mu_s)(f_m \otimes f_n) \rangle \, ds \\
= \ & \int_0^{t \wedge \tau_R} \int \int (\nabla f_m(z_1))^T c(z_1, z_2, \mu_s) \nabla f_n(z_2) \, \mu_s^{\times 2}(dz_1 \, dz_2) \, ds \\
= \ & \int_0^{t \wedge \tau_R} (U_s(f_m), U_s(f_n))_{-d-2} \, ds \\
= \ & \int_0^{t \wedge \tau_R} (X_s U_s(f_m), f_n)_{-d-2} \, ds \, .
\end{aligned}
$$

Therefore $M_{t \wedge \tau_R} \otimes M_{t \wedge \tau_R} - << M >>_{t \wedge \tau_R}$ is an $L_2(H^{-d-2}, H^{-d-2})$-valued martingale, with

$$
M_{t \wedge \tau_R} \otimes M_{t \wedge \tau_R} = \sum_{m,n} M_{t \wedge \tau_R}(f_m) M_{t \wedge \tau_R}(f_n) \, f_m \otimes f_n
$$

and

$$\begin{aligned}<< M >>_{t\wedge\tau_R} &= \sum_{m,n} < M(f_m), M(f_n) >_{t\wedge\tau_R} \; f_m \otimes f_n \\ &= \sum_{m,n} \int_0^{t\wedge\tau_R} (X_s U_s(f_m), f_n)_{-d-2}\, ds \quad f_m \otimes f_n \\ &= \int_0^{t\wedge\tau_R} X_s U_s \, ds\,.\end{aligned}$$

Summarizing: X is a progressively measurable process with values in $L_2((H^{-d-2})^f, H^{-d-2})$ and M a continuous square integrable local martingale valued in H^{-d-2}, with tensor cross-variation $<< M >>_t = \int_0^t X_s X_s^* \, ds$. Bojdecki and Jakubowski [4] (generalizing Yor [34]) construct a cylindrical Wiener process W in $(H^{-d-2})^f$, on a (possibly extended) probability space supporting both M and X, such that $M = \int X \, dW$. □

Corollary 5.1 *Any solution to the local martingale problem of theorem* (3.2) *is a "weak solution" of the stochastic evolution equation*

$$\langle \mu_t, \phi \rangle = \int_0^t \langle \mu_s, L(\mu_s)\phi \rangle \, ds + \int_0^t \langle \int \sigma^T(\cdot, y, \mu_s)\mu_s(\cdot) W(ds, dy), \nabla\phi(\cdot) \rangle.$$

6 Uniqueness

Denote by $\Pi^\ell : \mathcal{S}' \to \mathcal{S}$ the projection operator defined by

$$\Pi^\ell(X) := \sum_{|n|_1 \le \ell} X[h_n] h_n$$

and define the seminorms $|| \cdot ||_{j,\ell}$ on $\mathcal{S}'$ with $j \in \mathcal{R}$ and $\ell \geq 0$ by

$$||X||_{j,\ell}^2 := ||\Pi^\ell(X)||_j^2 = \sum_{|n|_1 \le \ell} (2|n|_1 + d)^j X[h_n]^2\,.$$

Consider the following condition on the operator $\mathcal{L}^{(\infty)}$.

6.1 Weak Monotonicity Condition

For some $j \in \mathcal{R}$, there exists positive constants k_j and ℓ_j such that, for every integer $\ell \geq \ell_j$ and every probabilities $\mu, \nu \in \mathcal{M}_2$, we have

$$\begin{aligned} &\sum_{|n|_1 < \ell} (2|n|_1 + d)^j \Big(2\langle \mu - \nu, h_n \rangle [\langle \mu, L(\mu) h_n \rangle - \langle \nu, L(\nu) h_n \rangle] \\ &\quad + || \int \sigma^T(z, \cdot, \mu) \nabla h_n(z)\, \mu(dz) - \int \sigma^T(z, \cdot, \nu) \nabla h_n(z)\, \nu(dz) ||_{-d-2}^2 \Big) \\ \le\;& k_j ||\mu - \nu||_{j,\ell}^2\,.\end{aligned}$$

The purpose of this section is to prove the following.

Theorem 6.1 *In addition to the conditions of theorem* (5.1), *assume that monotonicity condition* (6.1) *holds. The martingale problem of theorem* (3.2) *has at most one* H^{-d-1}*-valued solution (and therefore a unique* $\mathcal{M}_2$*-valued solution), for every random starting point* μ_0 *satisfying* $E\langle\mu_0, \varphi_2\rangle < \infty$.

Remark 6.1 *The theorem remains true under condition* (6.1) *with* $\ell_j = \infty$, *provided both series involved are convergent.*

Proof: Let $(\Omega_i, (\mathcal{F}_t^i)_{t\geq 0}, P_i)$, i=1,2, denote two complete filtered probability spaces, each respectively supporting an $\mathcal{M}_2$-valued continuous process μ^i and an $(H^{-d-2})^f$-valued cylindrical Wiener process W^i (independent of μ_0^i), such that, for each pair (W^i, μ^i), the law of the second coordinate solves the martingale problem of theorem (3.2) while the first coordinate indicates the cylindrical Wiener process in $(H^{-d-2})^f$ driving the basic martingale M^i associated with μ^i and exhibited in theorem (5.1).

Any H^j-valued cylindrical Wiener process W is H^{j-d-1}-continuous almost surely — in particular in the case of interest here $j = -d-2$ — the proof is exactly along the lines of that of proposition (5.2). We can therefore restrict ourselves to the case $\Omega_1 = \Omega_2 = \Omega_f^{(-2d-3)} \times \Omega_1^{(-d-1)}$, $(\mathcal{F}_t^1)_{t\geq 0} = (\mathcal{F}_t^2)_{t\geq 0} = (\mathcal{F}_t)_{t\geq 0}$, this last the canonical Borel filtration on $\Omega_f^{(-2d-3)} \times \Omega_1^{(-d-1)}$, with the shorthand notation $\Omega_f^{(j)} := C([0,\infty) : (H^j)^f)$.

Denote by $\{B_i^m : i \geq 1\} \subset \mathcal{B}(\Omega_f^{(-2d-3)} \times H^{-d-1})$ a sequence of countable partitions of $\Omega_f^{(-2d-3)} \times H^{-d-1}$ into Borel sets satisfying both the conditions: $\lim_{m\to\infty} \sup_i \operatorname{diam}(B_i^m) = 0$ holds and $\{B_i^{m+1} : i \geq 1\}$ refines $\{B_i^m : i \geq 1\}$ for every $m \geq 1$.

Given any two such solutions (W^i, μ^i), i=1,2, to equation (1.1) with respective law P_i on $(\mathcal{F}_t)_{t\geq 0}$, define the maps $\Xi^i : \Omega_i \to \Omega_f^{(-2d-3)} \times H^{-d-1}$ by $\Xi^i(W^i, \mu^i) := (W^i, \mu_0^i)$ and then define a sequence of approximate couplings $P^m \in \mathcal{M}(\Omega_1 \times \Omega_2)$ by

$$P^m(C) := \sum_i E^{P_1\times P_2}\left[\delta_C \delta_{B_i^m}(\Xi^1)\delta_{B_i^m}(\Xi^2)\right] / P_1 \circ (\Xi^1)^{-1}(B_i^m).$$

With the notation $P_1 \circ (\Xi^1)^{-1} = P_2 \circ (\Xi^2)^{-1} = \nu$, lemma (5.15) of Ethier and Kurtz [14], which extends to abstract spaces the corresponding result of Yamada and Watanabe [33], yields the weak convergence of probabilities $P^m \Rightarrow P_\nu \in \mathcal{M}(\Omega_1 \times \Omega_2)$, where the coupling measure P_ν is such that, for all $A_1, A_2 \in \mathcal{B}(\Omega_1 \times \Omega_2)$, we have

$$P_\nu(A_1 \times A_2) = \int E^{P_1}[\delta_{A_1} | \Xi^1 = x] E^{P_2}[\delta_{A_2} | \Xi^2 = x]\, \nu(dx).$$

In particular, $P_\nu(A_1 \times \Omega_2) = P_1(A_1)$ and $P_\nu(\Omega_1 \times A_2) = P_2(A_2)$ confirm that P_ν is indeed a coupling measure.

The above coupling also yields, for all $B_1, B_2 \in \mathcal{B}(\Omega_f^{(-2d-3)})$,

$$P_\nu(B_1 \times \Omega_1^{(-d-1)} \times B_2 \times \Omega_1^{(-d-1)}) = \nu((B_1 \cap B_2) \times H^{-d-1}),$$

so the coupling measure is such that a common Wiener process is driving this new (coupled) process.

Denote by $\tilde{W}$, $\tilde{\mu}^1$ and $\tilde{\mu}^2$ the first (and third), second and fourth coordinates of the canonical process on $\Omega_1 \times \Omega_2$ with law P_ν. Then clearly, for any bounded measurable functions $f, g : \Omega_1 \times \Omega_2 \to \mathcal{R}$ we have:

$$E^{P_\nu}[f(\tilde{W}, \tilde{\mu}^1), g(\tilde{W}, \tilde{\mu}^2)] =$$
$$\int E^{P_1}[f(W^1, \mu^1)|(W^1, \mu_0^1) = x] E^{P_2}[g(W^2, \mu^2)|(W^2, \mu_0^2) = x]\, \nu(dx).$$

It is immediate from this equation that both pairs $(\tilde{W}, \tilde{\mu}^1)$ and $(\tilde{W}, \tilde{\mu}^2)$ solve the stochastic evolution equation (1.1). Moreover, for every $R > 1$, $t \geq 0$ and $\phi \in \mathcal{S}$, we have (since $P_\nu(\tilde{\mu}_0^1 = \tilde{\mu}_0^2) = 1$ and with $\tau_R = \tau_R^1 \wedge \tau_R^2$)

$$\langle \tilde{\mu}^1_{t\wedge\tau_R} - \tilde{\mu}^2_{t\wedge\tau_R}, \phi \rangle = \int_0^{t\wedge\tau_R} \left[\langle \tilde{\mu}_s^1, L(\tilde{\mu}_s^1)\phi \rangle - \langle \tilde{\mu}_s^2, L(\tilde{\mu}_s^2)\phi \rangle \right] ds$$
$$+ \int_0^{t\wedge\tau_R} \left[\int \sigma^T(z, \cdot, \tilde{\mu}_s^1) \nabla\phi(z)\, \tilde{\mu}_s^1(dz) - \int \sigma^T(z, \cdot, \tilde{\mu}_s^2) \nabla\phi(z)\, \tilde{\mu}_s^2(dz) \right]^* d\tilde{W}_s(\phi).$$

As in the proof of theorem (5.1), superscript $*$ here again indicates the adjoint to the operator that sends ϕ to the mapping in brackets.

Itô's formula for the semimartingale $e^{-k_j t}\langle \tilde{\mu}^1_{t\wedge\tau_R} - \tilde{\mu}^2_{t\wedge\tau_R}, \phi \rangle^2$ reads

$$\begin{aligned}
& e^{-k_j t}\langle \tilde{\mu}^1_{t\wedge\tau_R} - \tilde{\mu}^2_{t\wedge\tau_R}, \phi \rangle^2 \\
= \; & 2\int_0^{t\wedge\tau_R} e^{-k_j s}\langle \tilde{\mu}_s^1 - \tilde{\mu}_s^2, \phi \rangle \left[\langle \tilde{\mu}_s^1, L(\tilde{\mu}_s^1)\phi \rangle - \langle \tilde{\mu}_s^2, L(\tilde{\mu}_s^2)\phi \rangle \right] ds \\
+ \; & \int_0^{t\wedge\tau_R} e^{-k_j s} || \int \sigma^T(z, \cdot, \tilde{\mu}_s^1) \nabla\phi(z)\, \tilde{\mu}_s^1(dz) - \int \sigma^T(z, \cdot, \tilde{\mu}_s^2) \nabla\phi(z)\, \tilde{\mu}_s^2(dz) ||^2_{-d-2}\, ds \\
- \; & \int_0^{t\wedge\tau_R} k_j e^{-k_j s} \langle \tilde{\mu}_s^1 - \tilde{\mu}_s^2, \phi \rangle^2 \\
+ \; & 2\int_0^{t\wedge\tau_R} e^{-k_j s}\langle \tilde{\mu}_s^1 - \tilde{\mu}_s^2, \phi \rangle \cdot \\
& \left[\int \sigma^T(z, \cdot, \tilde{\mu}_s^1) \nabla\phi(z)\, \tilde{\mu}_s^1(dz) - \int \sigma^T(z, \cdot, \tilde{\mu}_s^2) \nabla\phi(z)\, \tilde{\mu}_s^2(dz) \right]^* d\tilde{W}_s(\phi),
\end{aligned}$$

which, after an application of condition (6.1), yields that the positive process $e^{-k_j t}||\tilde{\mu}^1_{t\wedge\tau_R} - \tilde{\mu}^2_{t\wedge\tau_R}||^2_{j,\ell}$ is bounded above by a (positive) martingale. Hence, for every $t \geq 0$, every $\ell > \ell_j$ and every $R > 1$, there holds successively $||\tilde{\mu}^1_{t\wedge\tau_R} - \tilde{\mu}^2_{t\wedge\tau_R}||_{j,\ell} = 0$ and then $\tilde{\mu}_t^1 = \tilde{\mu}_t^2$ P_ν-almost surely. □

Consider (1.1) as a stochastic evolution equation over H^{-d-1}. This means, in particular, that a, b and σ should now be defined over H^{-d-1} instead of $\mathcal{M}_2$: assume that the mappings $a : \mathcal{R}^d \times H^{-d-1} \to (\mathcal{R}^d \otimes \mathcal{R}^d)_+$, $b : \mathcal{R}^d \times H^{-d-1} \to \mathcal{R}^d$ and $\sigma : \mathcal{R}^d \times \mathcal{R}^d \times H^{-d-1} \to \mathcal{R}^d \otimes \mathcal{R}^f$ are continuous and denote their respective restrictions from H^{-d-1} to $\mathcal{M}_2$ by $a_{|\mathcal{M}_2}$, $b_{|\mathcal{M}_2}$ and $\sigma_{|\mathcal{M}_2}$.

We have the following positivity principle, which extends that in section 7 of Pardoux [26] — see his pp. 152ff.

Corollary 6.1 *Assume that $a_{|\mathcal{M}_2}$, $b_{|\mathcal{M}_2}$ and $\sigma_{|\mathcal{M}_2}$ satisfy all the conditions of theorem (3.1). Provided distribution uniqueness holds for equation (1.1) over H^{-d-1}, the unique solution started in the positive cone $\mathcal{M}_2$ remains inside that cone for all positive times.*

7 Solutions with smooth values and regularity of paths

Consider the following growth condition on operators L and R.

7.1

For some $j \in \mathcal{R}$, there exists constants $0 \le k_j < \infty$ and $0 < \ell_j \le \infty$ such that, for every integer $\ell \ge \ell_j$ and every probability $\mu \in \mathcal{M}_2$, we have

$$\sum_{|n|_1 \le \ell} (2|n|_1 + d)^j \Big(2\langle \mu, h_n \rangle \langle \mu, L(\mu) h_n \rangle + \langle \mu^{\times 2}, R(\mu) h_n \rangle \Big) \le k_j (1 + ||\mu||_{j,\ell}^2) \, .$$

Proposition 7.1 *Under condition* (7.1) *and the conditions of proposition* (5.2), *every solution to the local martingale problem of theorem* (3.2), *started at a random point μ_0 satisfying both $E||\mu_0||_j^2 < \infty$ for that value of j in condition (7.1) and $E\langle \mu_0, \varphi_2 \rangle < \infty$, belongs to $\mathcal{M}(C([0,\infty) : H^{j-\epsilon}))$ for every $\epsilon > 0$ and satisfies for every $T > 0$ both $\sup_{t \in [0,T]} E||\mu_t||_j^2 < \infty$ and $P(\sup_{t \in [0,T]} ||\mu_t||_j^2 < \infty) = 1$.*

Proof: The proofs of propositions (5.1) and (5.2) ensure that all processes of the form $\{\langle \mu_t, \phi \rangle : t \ge 0\}$ are continuous semimartingales with common localization sequence for all choices of $\phi \in \mathcal{S}$. With the notation $\theta_{j,\ell}(t) = e^{-k_j(t \wedge \tau_R)}(1 + ||\mu_{t \wedge \tau_R}||_{j,\ell}^2)$, Itô's formula therefore holds for each of the continuous semimartingales $\{\theta_{j,\ell}(t) : t \ge 0\}$: for all choices of $0 \le s \le t \le T < \infty$, $R > 1$ and $\ell \ge 0$, we have

$$\begin{aligned} \theta_{j,\ell}(t) - \theta_{j,\ell}(s) &= \int_{s \wedge \tau_R}^{t \wedge \tau_R} 2e^{-k_j v} \sum_{|n|_1 \le \ell} (2|n|_1 + d)^j \langle \mu_v, h_n \rangle \langle \mu_v, L(\mu_v) h_n \rangle \, dv \\ &+ \int_{s \wedge \tau_R}^{t \wedge \tau_R} e^{-k_j v} \sum_{|n|_1 \le \ell} (2|n|_1 + d)^j \langle \mu_v^{\times 2}, R(\mu_v) h_n \rangle \, dv \end{aligned}$$

$$
\begin{aligned}
&- \int_{s\wedge\tau_R}^{t\wedge\tau_R} k_j e^{-k_j v}(1+||\mu_v||_{j,\ell}^2)\,dv \\
&+ 2\sum_{|n|_1\le \ell}(2|n|_1+d)^j \int_{s\wedge\tau_R}^{t\wedge\tau_R} e^{-k_j v}\langle \mu_v, h_n\rangle\, dM_v(h_n)\,.
\end{aligned}
$$

Condition (7.1) implies that $\theta_{j,\ell}(t)$ is a continuous, positive supermartingale for every ℓ large enough, and subsequently there holds

$$1+E||\mu_{t\wedge\tau_R}||_{j,\ell}^2 \le e^{k_j t}(1+E||\mu_0||_{j,\ell}^2)\,.$$

First let ℓ increase to infinity to get (via monotone convergence)

$$1+\sup_{t\in[0,T]} E||\mu_{t\wedge\tau_R}||_j^2 \le e^{k_j T}(1+E||\mu_0||_j^2)\,;$$

then let R increase to infinity to obtain $\sup_{t\in[0,T]} E||\mu_t||_j^2<\infty$.

Doob's inequality reads, for every $x>0$,

$$P(\sup_{t\in[0,T]} \theta_{j,\ell}(t)>x)\le x^{-1}E\theta_{j,\ell}(0)\,.$$

Since $\{\theta_{j,\ell}:\ell\ge 0\}$ is an increasing sequence, it follows by monotonicity

$$P(\sup_{t\in[0,T]}\lim_{\ell\to\infty}\theta_{j,\ell}(t)=\infty)=P(\lim_{\ell\to\infty}\sup_{t\in[0,T]}\theta_{j,\ell}(t)=\infty)=0\,,$$

provided the underlying probability space is complete.

The unique limit $\theta_j(t)=e^{-k_j(t\wedge\tau_R)}(1+||\mu_{t\wedge\tau_R}||_j^2)$ is then also a positive supermartingale. This implies $P(\sup_{t\in[0,T]}||\mu_{t\wedge\tau_R}||_j^2<\infty)=1$ for every $R>1$ and therefore also $P(\sup_{t\in[0,T]}||\mu_t||_j^2<\infty)=1$.

Finally, let ϵ denote some arbitrary positive number and notice that, by analogy with the proof of proposition (5.2) and using the same notation $\sup_\delta$ for the supremum over all choices of $s,t\in[0,T]$ satisfying $|t-s|\le\delta$, for every i there holds:

$$
\begin{aligned}
\sup_\delta \quad & ||\mu_{t\wedge\tau_R}-\mu_{s\wedge\tau_R}||_{j-\epsilon}^2 \\
&\le \sum_{|n|_1\le i}(2|n|_1+d)^{j-\epsilon}\sup_\delta(\langle\mu_{t\wedge\tau_R},h_n\rangle-\langle\mu_{s\wedge\tau_R},h_n\rangle)^2 \\
&+4(2i+d)^{-\epsilon}\sup_{t\in[0,T]}||\mu_{t\wedge\tau_R}||_j^2\,,
\end{aligned}
$$

so the $||\cdot||_{j-\epsilon}$-continuity of μ_t follows from the weak continuity of μ_t and the finiteness of $||\mu_t||_j$. □

Remark 7.1 *In view of proposition* (5.1), *the case* $j=-d-1$ *does not require the additional condition* (7.1).

8 Strong solutions with square integrable densities

We next show that the well-posed local martingale problem of theorem (6.1) may arise by way of an associated stochastic evolution equation, which has a unique strong solution, under some additional conditions. This equation (8.1 below) should be viewed as the forward Kolmogorov equation corresponding to the backward Kolmogorov formulation provided by the local martingale problem of theorems (3.2) and (6.1).

Given is a symmetric positive definite trace-class (or nuclear) linear operator $Q = (Q_k)_{k=1}^f : (H^w)^f \to (H^w)^f$ — such an operator always possesses a Hilbert-Schmidt square-root $Q^{1/2} \in L_2((H^w)^f, (H^w)^f)$. The trace of Q on $(H^w)^f$ is given explicitly by $\sum_{k,\ell=1}^f \sum_n (2|n|_1 + d)^{-w} ||Q_k^{1/2}(h_n e_\ell)||_w^2 < \infty$, writing $\{e_\ell : \ell = 1, 2, \ldots, f\}$ for the canonical basis in $\mathcal{R}^f$. An $(H^w)^f$-valued Wiener process $W = (W_1, W_2, \ldots, W_f)$ with covariance kernel Q is defined as a (strongly) continuous $(H^w)^f$-valued martingale started at 0 with quadratic variation $< W >_t = (< W_k, W_\ell >_t)_{k,\ell=1}^f$ given explicitly by $< W_k, W_\ell >_t [\phi, \psi] = t(Q_k^{1/2}\phi, Q_\ell^{1/2}\psi)_w$. For details of its elementary properties, consult Yor [34],Métivier and Pellaumail [25], Krylov and Rozovskii [22].

We assume $w \geq 0$ here and for the rest of this section, to ensure that some sense can be made of the diffusion term in equation (8.1), which in our case (equation 1.2) involves pointwise multiplication of its solution μ_t with the driving Wiener process W_t.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a complete filtered probability space with a right-continuous filtration and an $(H^w)^f$-valued Wiener process W on it with covariance kernel Q. Let $V \subset H^0 \subset V^*$ denote rigged separable Hilbert spaces, with the respective norms $|||\cdot|||_V \geq ||\cdot||_0 \geq |||\cdot|||_{V^*}$.

Consider the stochastic evolution equation

$$\mu_s \quad = \quad \mu_0 + \int_0^s A(\mu_v)\,dv + \int_0^s B(\mu_v)\,dW(v)\,, \tag{8.1}$$

where $A : V \to V^*$ and $B : V \to L_Q((H^w)^f, H^0)$ are assumed to be continuous, and W denotes an $(H^w)^f$-valued Wiener process with nuclear covariance kernel $Q : (H^w)^f \to (H^w)^f$. Here, $L_Q((H^w)^f, H^0)$ is the space of linear operators $\Phi : (H^w)^f \to H^0$ such that $\Phi \circ Q^{1/2} \in L_2((H^w)^f, H^0)$ holds.

A strong solution to equation (8.1) is a H^0-valued process μ defined on $[0, \infty) \times \Omega$, strongly continuous in H^0 with respect to t, $\mathcal{F}_t$ adapted, satisfying for every $T > 0$ the inequality $E \int_0^T |||\mu_t|||_V^2\,dt < \infty$, and verifying (8.1) in the sense of equality in V^* for all $t \in [0, \infty)$ on a set of total probability in Ω, i.e., for every $\phi \in V$, we have almost surely

$$\mu_t[\phi] = \mu_0[\phi] + \int_0^t A(\mu_s)[\phi]\,ds + \Big(\int_0^t B(\mu_s)\,dW(s)\Big)[\phi]\,.$$

We need the following set of conditions.

8.1 Coercivity and Monotonicity Conditions
There exists $K > 0$ and $\varpi > 0$ such that, for all $\phi, \phi_1, \phi_2 \in V$, we have

1. $|||A(\phi)|||_{V^*} \leq K(1 + |||\phi|||_V)$;
2. $2A(\phi)[\phi] + ||B(\phi)Q^{1/2}||^2_{L_2((H^w)^f,H^0)} + \varpi|||\phi|||^2_V \leq K(1 + ||\phi||^2_0)$;
3. $2(A(\phi_1) - A(\phi_2))[\phi_1 - \phi_2] + ||(B(\phi_1) - B(\phi_2))Q^{1/2}||^2_{L_2((H^w)^f,H^0)}$ $\leq K||\phi_1 - \phi_2||^2_0$.

Condition (8.1.1) is a growth condition, while conditions (8.1.2) and (8.1.3) are respectively called coercivity and monotonicity for operator $\mathcal{L}^{(\infty)}$ in the literature on partial differential equations. Under all of the above conditions, Krylov and Rozovskii [22], extending results of Pardoux [26], prove that equation (8.1) possesses a (pathwise) unique H^0-valued strong solution μ, for every starting distribution satisfying $E||\mu_0||^2_0 < \infty$; and that, for each $T > 0$ and some constant $L_T > 0$, there holds $E\sup_{t\leq T} ||\mu_t||^2_0 + E\int_0^T |||\mu_t|||^2_V\, dt \leq L_T E||\mu_0||^2_0$. Note also that monotonicity condition (8.1.3) is used solely in the proof of uniqueness; it is not required in the proof of existence of a strong solution.

We will apply this result with $w = 1$, $V = W^{1,2}(\mathcal{R}^d)$ and $V^* = W^{-1,2}(\mathcal{R}^d)$ where $W^{1,2}(\mathcal{R}^d) = W^{1,2}$ is the Sobolev space with inner product

$$(u, v)_{1,2} = (u, v)_0 + \sum_{i=1}^{d} (\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_i})_0$$

and associated norm

$$||u||_{1,2} = \sqrt{(u, u)_{1,2}}$$

Note that $W^{1,2} \supset H^1$. Also $W^{1,2} \cap \mathcal{M}_2 \subset H^1$ (cf. Holley and Stroock [18] (A.17)). Also note that if $u \in W^{1,2}$ and g is bounded and has bounded continuous derivatives then the pointwise product gu also belongs to $W^{1,2}$ and $||gu||_{1,2} \leq K||u||_{1,2}$ for some constant K (cf. Brezis [6] Prop. IX.4).

We first need to specify the class of coefficients a, b and c allowing for a representation such as (8.1) for the unique solution to the local martingale problem of theorems (3.2) and (6.1) — such a representation will be made explicit in the remarks preceding theorem (8.1).

8.2 Coercivity Hypotheses for McKean-Vlasov

1. Assume that mappings $a = (a_{ij})^d_{i,j=1} : \mathcal{R}^d \times H^{-d-1} \to (\mathcal{R}^d \otimes \mathcal{R}^d)_+$, $b = (b_i)^d_{i=1} : \mathcal{R}^d \times H^{-d-1} \to \mathcal{R}^d$ and $\beta = (\beta_{i\ell})_{i=1,\ldots,d;\ell=1,\ldots,f} : \mathcal{R}^d \times H^{-d-1} \to \mathcal{R}^d \otimes \mathcal{R}^f$ are bounded and continuous; and that their restrictions to $\mathcal{R}^d \times W^{1,2}$, denoted respectively by $a_{|_{W^{1,2}}}$, $b_{|_{W^{1,2}}}$ and $\beta_{|_{W^{1,2}}}$, together with their derivatives $\partial_j(a_{|_{W^{1,2}}})_{ij}$, $\partial_j(b_{|_{W^{1,2}}})_j$ and $\partial_j(\beta_{|_{W^{,2}1}})_{i\ell}$,

are continuous in the norm $||\cdot||_{1,2}$ and belong to $C(\mathcal{R}^d \times W^{1,2})$, for every $i,j = 1,2,\ldots,d$ and $\ell = 1,2,\ldots,f$, where $(\partial_j)_{j=1}^d$ denotes the gradient with respect to the first ($\mathcal{R}^d$-valued) coordinate. This implies that $\overline{\beta} = \sup_{z,\mu}(|\beta(z,\mu)| + \max_{i,k}|\partial_i\beta_{ik}(z,\mu)|) < \infty$.

2. Assume that $r = (r_{k\ell})_{k,\ell=1}^f : \mathcal{R}^d \times \mathcal{R}^d \to \mathcal{R}^f \otimes \mathcal{R}^f$ is a mapping of the form $r_{k\ell}(z_1,z_2) = \sum_{m,n} r_{k\ell}^{mn} h_m(z_1)h_n(z_2)$ for some double sequences $\{r_{k\ell}^{mn} \in \mathcal{R} : m,n\}$ satisfying both $\sum_{k,\ell=1}^f \sum_{m,n} |r_{k\ell}^{mn}| < \infty$ and

$$\mathcal{Q} := \sum_{k,\ell=1}^{f} \sum_{n} (2|n|_1 + d)^{-1} \Big(\sum_{m} (2|m|_1 + d)^{1/2} |r_{k\ell}^{mn}| \Big)^2 < \infty \,.$$

3. Assume the following weak ellipticity condition on a: for every $z \in \mathcal{R}^d$ and $\mu \in H^{-d-1}$, the following lower bound holds

$$\mathcal{A} := \inf_{\theta\in\mathcal{R}^d : |\theta|=1} \theta^T a(z,\mu)\theta > |\beta(z,\mu)|^2 ||r(z,\cdot)||_0^2 \,.$$

4. Assume that $c = (c_{ij})_{i,j=1}^d : \mathcal{R}^d \times \mathcal{R}^d \times H^{-d-1} \to (\mathcal{R}^d \otimes \mathcal{R}^d)_+$ can be written both as in theorem (5.1) for some σ (when c is restricted to $\mathcal{M}_2 \subset H^{-d-1}$) and in the form

$$c(z_1,z_2,\mu) := \int_{\mathcal{R}^d} \big(\beta(z_1,\mu)r(z_1,z)\big)\big(\beta(z_2,\mu)r(z_2,z)\big)^T dz \,.$$

Remark 8.1 *The condition* $\sum_{k,\ell=1}^f \sum_{m,n} |r_{k\ell}^{mn}| < \infty$ *in hypothesis* (8.2.2) *serves two purposes. First, it ensures that r is a continuous mapping, by way of the Charlier-Cramér bound — actually, since it also holds that* $\sum_{k,\ell=1}^f \sum_{m,n} |r_{k\ell}^{mn}|^2 < \infty$, *we see that r is a continuous Hilbert-Schmidt kernel over* $(H^0)^d$ *(*$\iint |r(z_1,z_2)|^2\, dz_1 dz_2 < \infty$*). It also implies that* $||r(z,\cdot)||_0^2$ *is finite for every* $z \in \mathcal{R}^d$, *a necessary condition for hypothesis* (8.2.3) *to hold.*

Note that hypotheses (8.2.1) *to* (8.2.4) *together imply all the hypotheses of theorem* (5.1)*: all matrices of the form* $(c(x_i,x_j,\epsilon_x))_{i,j=1}^N$ *are positive definite for every choice of N and* $x \in (\mathcal{R}^d)^N$*; and* $a(z,\mu)-c(z,z,\mu)$ *is strictly positive definite for each* $z \in \mathcal{R}^d$ *and* $\mu \in \mathcal{M}_2$, *because of the bound*

$$\theta^T c(z,z,\mu)\theta = \int dy \sum_{\ell=1}^{f} \Big(\sum_{k=1}^{f} \sum_{i=1}^{d} \theta_i \beta_{ik}(z,\mu) r_{k\ell}(z,y) \Big)^2 \le |\theta|^2 |\beta(z,\mu)|^2 ||r(z,\cdot)||_0^2 \,.$$

The smoothness conditions (8.2.1) on a and b allow us to define the adjoint $L^*(\mu) : W^{1,2} \to W^{-1,2}$ to operator $L(\mu)$ appearing in theorem (3.2):

$$\begin{aligned} L^*(\mu)\phi &= -\nabla^T(\phi b(\cdot,\mu)) + \frac{1}{2}\nabla^T(a^T(\cdot,\mu)\nabla\phi) \\ &= -\sum_{i=1}^{d} \partial_i(\phi b_i(\cdot,\mu)) + \frac{1}{2}\sum_{i,j}^{d} \partial_j(a_{ij}(\cdot,\mu)\partial_i\phi) \,. \end{aligned} \tag{8.2}$$

We set the nonlinear drift operator to $A(\mu) = L^*(\mu)\mu$.

Define $Q^{1/2} \in L_2((H^1)^f, (H^1)^f)$ by $Q^{1/2}W(z) = \int r(z_2, z)W(z_2)\, dz_2$. Finally, define the diffusion operator $B : W^{1,2} \to L_Q((H^1)^f, H^0)$ by

$$B(\mu)W : \;=\; \sum_{i=1}^{d}\sum_{k=1}^{f} \partial_i(\beta_{ik}(\cdot,\mu)\mu W_k) \tag{8.3}$$

for every $W = (W_1, W_2, \ldots, W_f) \in (H^1)^f$.

Note that in this case equation (8.1) takes the form (1.2).

Theorem 8.1 *Assume that hypotheses* (6.1) *and* (8.2.1) *to* (8.2.4) *hold; that operators A and B, defined respectively by expressions* (8.2) *and* (8.3), *are continuous and that the process verifies both $E||\mu_0||_0^2 < \infty$ and $P(\mu_0 \in \mathcal{M}_2) = 1$. Let $\mathcal{B} := 4d^2 f\overline{\beta}^2$. If $\mathcal{A} > \mathcal{B}\mathcal{Q}$, then equation* (8.1) *possesses a strong solution, the law of which solves the well-posed martingale problem on $C([0,\infty) : \mathcal{M}_2)$ of theorem* (6.1); *and the positivity principle of corollary* (6.1) *extends to equation* (8.1).

Proof: One verifies at once the growth condition (8.1.1) as follows. For $\mu \in W^{1,2}$

$$\begin{aligned} ||A(\mu)||_{-1,2} &= \sup_{||\phi||_{1,2}\le 1} |(A(\mu)\mu, \phi)_0| \\ &= \sup_{||\phi||_{1,2}\le 1} |(-\sum_{i=1}^{d} \partial_i(\mu b_i(\cdot,\mu)), \phi)_0 + \frac{1}{2}\sum(\partial_j(a_{i,j}(\cdot,\mu)\partial_i\mu), \phi)_0| \\ &= \sup_{||\phi||_{1,2}\le 1} |\sum_{i=1}^{d}(\mu b_i(\cdot,\mu), \partial_i\phi)_0 - \frac{1}{2}\sum(a_{i,j}(\cdot,\mu)\partial_i\mu, \partial_j\phi)_0| \\ &\le K||\mu||_{1,2} \end{aligned}$$

for some constant K.

Hypothesis (8.2.2) implies that both operators $Q^{1/2}$ and $B(\mu)Q^{1/2}$ are indeed of Hilbert-Schmidt type. For the former we calculate

$$\begin{aligned} ||Q^{1/2}||^2_{L_2((H^1)^f,(H^1)^f)} &= \sum_{k,\ell=1}^{f}\sum_{n}(2|n|_1+d)^{-1}||Q_k^{1/2}(h_n e_\ell)||_1^2 \\ &= \sum_{k,\ell=1}^{f}\sum_{n}(2|n|_1+d)^{-1}||\sum_{m} r_{k\ell}^{mn} h_m||_1^2 \\ &= \sum_{k,\ell=1}^{f}\sum_{m,n}(2|n|_1+d)^{-1}(2|m|_1+d)(r_{k\ell}^{mn})^2 \\ &\le \sum_{k,\ell=1}^{f}\sum_{n}(2|n|_1+d)^{-1}\Big(\sum_{m}(2|m|_1+d)^{1/2}|r_{k\ell}^{mn}|\Big)^2. \end{aligned}$$

For the latter, we get, writing once again $\{e_\ell : \ell = 1, 2, \ldots, f\}$ for the canonical basis in $\mathcal{R}^f$,

$$
\begin{aligned}
&||B(\mu)Q^{1/2}||^2_{L_2((H^1)^f,H^0)}\\
= &\sum_n (2|n|_1 + d)^{-1} \sum_{\ell=1}^{f} ||B(\mu)Q^{1/2}(h_n e_\ell)||_0^2\\
= &\sum_n (2|n|_1 + d)^{-1} \sum_{\ell=1}^{f} ||\sum_{i=1}^{d}\sum_{k=1}^{f}\sum_m r^{mn}_{k\ell}\partial_i(\beta_{ik}(\cdot,\mu)\mu h_m)||_0^2\\
\leq & df \sum_n (2|n|_1 + d)^{-1} \sum_{k,\ell=1}^{f}\sum_{i=1}^{d} ||\sum_m r^{mn}_{k\ell}\partial_i(\beta_{ik}(\cdot,\mu)\mu h_m)||_0^2\\
\leq & 4d^2 f\overline{\beta}^2 ||\mu||^2_{1,2} \sum_{k,\ell=1}^{f}\sum_n (2|n|_1 + d)^{-1} \big(\sum_m (2|m|_1 + d)^{1/2}|r^{mn}_{k\ell}|\big)^2,
\end{aligned}
$$

with the last inequality via the Charlier-Cramér bound.

We next verify the coercivity condition (8.1.2). For $\mu \in W^{1,2}$

$$
\begin{aligned}
A(\mu)[\mu] &= (L^*(\mu)\mu, \mu)_0\\
&= -\sum_{i=1}^{d}(\partial_i(\mu b_i(\cdot,\mu)), \mu)_0 + \frac{1}{2}(\sum_{i,j}\partial_j(a_{i,j}(\cdot,\mu)\partial_i\mu), \mu)_0\\
&= \sum_{i=1}^{d}((\mu b_i(\cdot,\mu)), \partial_i\mu)_0 - \frac{1}{2}(\sum_{i,j}(a_{i,j}(\cdot,\mu)\partial_i\mu), \partial_j\mu)_0\\
&\leq -\mathcal{A}||\mu||^2_{1,2} + \overline{b}||\mu||_0||\mu||_{1,2} + (\mathcal{A}/2)||\mu||_0^2
\end{aligned}
$$

where $\overline{b} = \sup_{z,\mu}|b(z,\mu)| < \infty$. Then

$$
\begin{aligned}
&2A(\mu)[\mu] + ||B(\mu)Q^{1/2}||^2_{L_2((H^w)^f,H^0)} + \varpi||\mu||^2_{1,2}\\
&\leq (-\mathcal{A} + \mathcal{BQ} + \varpi)||\mu||^2_{1,2} + 2\overline{b}||\mu||_0||\mu||_{1,2} + \mathcal{A}]||\mu||_0^2\\
&\leq [(\mathcal{A} - \mathcal{BQ} - \varpi)^{-1}\overline{b}^2 + \mathcal{A}]||\mu||_0^2
\end{aligned}
$$

provided that $0 < \varpi < \mathcal{A} - \mathcal{BQ}$. This yields condition (8.1.2)

The above stated theorem of Krylov and Rozovskii [22] yields the existence of a strong solution μ_s. Since its trajectories are continuous in H^0, its law solves the local martingale problem on $C([0,\infty) : H^0)$ for operator $\mathcal{L}^{(\infty)}$ —Itô's formula actually implies that, for every smooth ψ and every $\phi \in \mathcal{S}$, the real valued process $\psi((\mu_t,\phi)_0) - \int_0^t \mathcal{L}^{(\infty)}\psi((\mu_s,\phi)_0)\,ds$ is a local martingale.

Indeed, for each $\mu \in W^{1,2}$, operator $B(\mu)$ defined above is densely defined on $(H^0)^f$ and therefore possesses an adjoint $B^*(\mu) : \mathcal{S} \to (H^1)^f$ defined by $B^*(\mu)\phi = -\mu\beta^T(\cdot,\mu)\nabla\phi$, in the sense of $(W, B^*(\mu)\phi)_0 = (B(\mu)W, \phi)_0$.

For each $\phi \in \mathcal{S}$, the real-valued local martingale $(\int_0^t B(\mu_s)\,dW(s))[\phi]$ has quadratic variation equal to

$$\int_0^t ||Q^{1/2}B^*(\mu_s)\phi||_0^2\,ds = \int_0^t (\mu_s \otimes \mu_s, R(\mu(s))\phi)_0\,ds\,.$$

Just as in proposition (5.2), the associated $W^{-1,2}$-valued local martingale $M_t = \int_0^t B(\mu_s)\,dW(s)$ is actually a square-integrable martingale here since hypothesis (8.2.1) implies $\sup_R K_R < \infty$.

Since condition (8.1.2) with $w = 1$ and $\varpi = 0$ implies condition (7.1) with $j = 0$ and $\ell_j = \infty$ we have for the unique solution P to the martingale problem of theorem (6.1) and proposition (7.1), that the canonical process on $C([0,\infty) : \mathcal{M}_2)$ then takes its values in the space H^0,P-almost surely for each $t \geq 0$. The one-dimensional marginals of P must therefore necessarily coincide with those of the law of the strong solution. □

Example 8.1 *To complete this section we give a simple example on the real line. We take $L\phi = \alpha^2\Delta\phi$ and $d = f = 1$.*

Consider a map $\beta \in C(H^{-2})$ and a sequence $\{r_n\} \subset \mathcal{R}$ such that $\sum_{n\geq 0}(2n+1)^{3/2}|r_n| < \infty$ holds. Select the diagonal Hilbert-Schmidt kernel

$$r(z_1, z_2) = \sum_{n\geq 0} r_n h_n(z_1) h_n(z_2) \in C(\mathcal{R}^2)$$

and note that Q is here indeed diagonal: $Q(h_n) = r_n^2 h_n$ for all $n \geq 0$. Define

$$\sigma(z_1, z_2, \mu) = \beta(\mu) \sum_{n\geq 0} (2n+1)^{3/2} r_n h_n(z_1) h_n(z_2) \in C(\mathcal{R}^2 \times H^{-2})$$

and note that both the definitions of c coincide:

$$c(z_1, z_2, \mu) = \beta(\mu)^2 \sum_{n\geq 0} r_n^2 h_n(z_1) h_n(z_2)\,.$$

If we make the additional assumption

$$\beta(\mu)^2 \sum_{n\geq 0} r_n^2 < 2\alpha^2\sqrt{\pi}$$

for all $\mu \in \mathcal{M}_2$, then all the hypotheses (8.2) *follow. Hence, proposition* (5.2), *theorem* (3.1) *and theorem* (5.1) *are satisfied. Hypothesis* (6.1) *is satisfied as well, if we add a Lipschitz condition on β: for all μ, ν there holds $|\beta(\mu) - \beta(\nu)|^2 \leq k_j||\mu - \nu||_{-d-3}^2$. Therefore both theorems* (6.1) *and* (8.1) *hold in this case.*

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