Mh. Math. 120, 179–188 (1995)

Monatshefte für
Mathematik

Multipliers for a Distinguished Laplacean on Solvable Extensions of *H*-Type Groups

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(Received 31 January 1994; in revised form 23 September 1994)

Abstract. Let Δ be a distinguished Laplacean on a solvable extension S of an H-type group. We give sufficient conditions on the multiplier m so that the operator $m(\Delta)$ is of type (p,p) for $1 < p < \infty$ and is of weak type $(1,1)$.

1. Introduction and Preliminaries

An H-type Lie algebra $\mathfrak{n}$ is a two-step nilpotent Lie algebra equipped with an inner product satisfying the following property [14]:

Let $\mathfrak{z}$ be the centre of $\mathfrak{n}$ and $\mathfrak{v}$ its orthogonal complement with respect to the inner product; then for every unitary Z in $\mathfrak{z}$ the map $J_Z\colon \mathfrak{v} \to \mathfrak{v}$ defined by the relation

$$\langle J_Z X, Y \rangle = \langle Z, [X, Y] \rangle$$

is orthogonal.

An H-type group N is a connected, simply connected Lie group whose Lie algebra $\mathfrak{n}$ is H-type. Let S be a one-dimensional extension of the group N obtained by making $A = \mathbb{R}^+$ act on N by homogeneous dilations; let H denote the vector acting on $\mathfrak{n}$ with eigenvalues $1/2$ and 1; we can extend the original metric on $\mathfrak{n}$ to the Lie algebra $\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}$ of the group S by asking $\mathfrak{n}$ and $\mathfrak{a}$ to be orthogonal and H to be unitary.

M. Cowling, A. Dooley, A. Korányi and F. Ricci [4, 5], E. Damek [8, 9], E. Damek and F. Ricci [11] studied geometric

1991 Mathematics Subject Classification: Primary 43A22, secondary 22E30, 43A80, 22E25.
Key words: Multiplier, solvable group, exponential volume growth, heat kernel.

properties of these groups, which provide examples of nonsymmetric harmonic manifolds [10].

If $\{E_0,\ldots,E_{m+k}\}$ is an orthonormal basis of $\mathfrak{s}$ such that $E_0=H$, $E_1,\ldots,E_m$ span $\mathfrak{v}$ and $E_{m+1},\ldots,E_{m+k}$ span $\mathfrak{z}$, then the Laplace–Beltrami operator can be written as

$$\mathscr{L}=\sum_{j=0}^{m+k}E_j^2-QE_0$$

where $Q=\dfrac{m}{2}+k$ is the homogeneous dimension of N and $\{E_0,\ldots,E_{m+k}\}$ are regarded as left-invariant vector fields (see [8]).

A radial function on S is a function that depends only on the distance from the identity. A radial function Φ is spherical if

(1) $\Phi(e)=1$;

(2) Φ is an eigenfunction of the Laplace–Beltrami operator $\mathscr{L}$.

Let π be the orthogonal projector of $L^2(S)$ onto $L_r^2(S)$, the closed subspace of $L^2(S)$ consisting of radial functions. Applying π on S corresponds to averaging over geodesic spheres centered at the identity with respect to the surface measure; the operator π extends to L^p for all p, $1\leqslant p\leqslant\infty$, to L^1_{loc} and preserves regularity.

Let δ be the modular function of S; as proved in [11], all spherical functions are of the form $\Phi_s=\pi(\delta^{s/Q-1/2})$, $s\in\mathbb{C}$; the corresponding eigenvalue is $s^2-Q^2/4$ and $\Phi_s=\Phi_{-s}$.

If f is a radial function, its spherical transform is defined by

$$\tilde{f}(s)=\int_S f(x)\Phi_s(x)dx$$

for all values of s for which the integral converges.

There exists a measure $d\mu(\lambda)=|\mathbf{c}(i\lambda)|^{-2}d\lambda$ on $[0,+\infty)$ such that the following Plancherel formula holds

$$\int_S|f(x)|^2dx=C\int_0^{+\infty}|\tilde{f}(i\lambda)|^2d\mu(\lambda)$$

where the constant C does not depend on f and where

$$\mathbf{c}(s)=2^{Q-2s}\frac{\Gamma\left(\dfrac{m+k+1}{2}\right)\Gamma(2s)}{\Gamma\left(\dfrac{m}{4}+\dfrac{1}{2}+s\right)\Gamma\left(\dfrac{Q}{2}+s\right)},\,s\in\mathbb{C}.$$

By Stirling's formula the function $\mathbf{c}$ satisfies the estimate

$$|\mathbf{c}(i\lambda)|^{-2} \leqslant C \begin{cases} |\lambda|^2 & \text{if } |\lambda| \leqslant 1 \\ |\lambda|^{m+k} & \text{if } |\lambda| > 1. \end{cases} \tag{1}$$

A distinguished right-invariant Laplacean on S is

$$\Delta = -\sum_{j=0}^{m+k} \tilde{E}_j^2,$$

where $\tilde{E}_0, \ldots, \tilde{E}_{m+k}$ are right-invariant vector fields agreeing with $E_0, \ldots, E_{m+k}$ respectively at the identity.

The operators $\mathscr{L}_Q = -\mathscr{L} + Q^2/4$ and Δ are nonnegative essentially self-adjoint operators on $C_c^\infty(S)$ with respect to left Haar measure; via functional calculus we can define for every bounded Borel function m on $[0, +\infty)$ the operators $m(\mathscr{L}_Q)$ and $m(\Delta)$, which are bounded operators on $L^2(S)$. These operators are strongly related as the following proposition shows:

Proposition 1. *If $f \in C_c^\infty(S)$ is a radial function, then*

$$\delta^{1/2} \Delta \delta^{-1/2} f = \mathscr{L}_Q f.$$

Moreover, if k and κ are the distributional kernels of the operators $m(\Delta)$ and $m(\mathscr{L}_Q)$ respectively, then $k = \delta^{-1/2}\kappa$.

See [6, 12, 16, 1] for a proof.

The aim of this paper is to show that, under suitable hypotheses on the function $m, m(\Delta)$ is a bounded operator on $L^p(S)$, $1 < p < \infty$, and is of weak type $(1, 1)$; our proof is a simple extension of the work of Cowling, Giulini, Hulanicki and Mauceri [7] to the case of these solvable groups.

Problems of this kind have been studied by a great number of authors; we refer to [7] for a bibliography.

2. Result

Fix a function ψ in $C^\infty(\mathbb{R}^+)$, compactly supported in $(1/2, 2)$, and such that for every ξ in $\mathbb{R}^+$

$$\sum_{-\infty}^{+\infty} \psi(2^{-j}\xi) = 1.$$

Let $H^s(\mathbb{R})$ be the L^2-Sobolev space of order s, i.e.,

$$H^s(\mathbb{R}) = \left\{ f \in \mathscr{S}'(\mathbb{R}) : \|f\|_{H^s} = \int_{\mathbb{R}} (1 + |\xi|^2)^s |\hat{f}(\xi)|^2 d\xi < \infty \right\}.$$

If m is a function on $[0, +\infty)$ which is locally in $H^s(\mathbb{R})$ on $(1, +\infty)$, then define $\|m\|_{(s)}$ as follows:

$$\|m\|_{(s)} = \sup_{t \geqslant 1} \|\psi(\cdot)m(t\cdot)\|_{H^s}.$$

Theorem 2. *Fix* s_0 *and* s *in* $(2, +\infty)$ *such that* $s > \frac{m+k}{2} + 1$. *Let* m *be a function on* $[0, +\infty)$ *such that*

i) *on the interval* $[0, 2]$, m *coincides with a function in* $H^{s_0}(\mathbb{R})$;
ii) m *is locally in* $H^s(\mathbb{R})$ *on* $(1, \infty)$ *and* $\|m\|_{(s)} < \infty$.

Then $m(\Delta)$ *is bounded on* $L^p(S)$, $1 < p < \infty$, *and is of weak type* $(1, 1)$.

3. Proof

The proof follows the work of COWLING, GIULINI, HULANICKI and MAUCERI [7], where they solve the same problem in the case of noncompact symmetric spaces of arbitrary rank; we outline their method and prove that the estimates they use are still valid in the present case. The groups we are studying are still of exponential growth. In fact, DAMEK and RICCI [10, 11] prove that in geodesic polar coordinates the left Haar measure dx of the group S can be written as

$$dx = (\sinh \rho/2)^m (\sinh \rho)^k d\rho\, d\sigma,$$

where ρ is the distance of the point $x \in S$ from the identity $e \in S$ and $d\sigma$ is the surface measure on the unitary ball; as an immediate consequence, if $|B_r|$ denotes the volume of the ball of radius r centered at the identity e, then

$$|B_r| \leqslant C \begin{cases} r^{m+k+1} & \text{if } r \leqslant 1 \\ e^{Qr} & \text{if } r > 1. \end{cases} \tag{2}$$

Moreover, as the spherical function Φ_0 satisfies the estimate

$$\Phi_0(\rho) = c|\rho| e^{-\rho Q/2} + O(|\rho| e^{-\rho Q/2})$$

(see [11, 1]), then trivially

$$\int_{B_r} |\Phi_0(x)|^2 dx \leqslant C \begin{cases} r^{m+k+1} & \text{if } r \leqslant 1 \\ r^3 & \text{if } r > 1. \end{cases} \tag{3}$$

To control the size of nonradial kernels, one can use the following

Lemma 3. *Let E be a radial measurable subset of S, and f a function in $L^2(S)$, such that $\delta^{1/2}f$ is radial. Then*

$$\|\chi_E f\|_2 = \|\chi_E \delta^{1/2} f\|_2. \tag{4}$$

Moreover

$$\|\chi_E f\|_1 \leqslant \|\chi_E \Phi_0\|_2 \|\chi_E f\|_2. \tag{5}$$

Proof. Let g be the function $\delta^{1/2}f$ and $d_r x$ the right Haar measure of S; then, as $\chi_E g$ is radial,

$$\begin{aligned}\|\chi_E f\|_2^2 &= \int_S |\chi_E f(x)|^2 dx = \int_S |\chi_E f(x)|^2 \delta(x) d_r x = \\ &= \int_S |\chi_E g(x)|^2 d_r x = \int_S |(\chi_E g)(x^{-1})|^2 dx = \\ &= \int_S |\chi_E g(x)|^2 dx.\end{aligned}$$

As π is an orthogonal projection, by the Cauchy–Schwarz inequality and (4)

$$\begin{aligned}\|\chi_E f\|_1 &= \int_E \delta^{-1/2} |\delta^{1/2} f| dx = \langle \delta^{-1/2}, \chi_E \delta^{1/2} |f| \rangle = \\ &= \langle \pi(\delta^{-1/2}), \chi_E \delta^{1/2} |f| \rangle = \int_E \Phi_0(x) \delta^{1/2} |f|(x) dx \leqslant \\ &\leqslant \|\chi_E \Phi_0\|_2 \|\chi_E g\|_2 = \|\chi_E \Phi_0\|_2 \|\chi_E f\|_2. \quad \Box\end{aligned}$$

Let τ be a smooth cut-off function defined on $[0, +\infty)$, equal to 1 on $[0, 1]$ and supported in $[0, 2]$. Let m_0 denote $m\tau$ and m_∞ denote $m(1-\tau)$, so that $m = m_0 + m_\infty$; let k_0 and k_∞ be the distributional kernels associated with the operators $m_0(\Delta)$ and $m_\infty(\Delta)$ respectively.

Using Lemma 3 and the estimates (1), (2), (3), as in [7], §3, we can conclude that k_0 is in $L^1(S)$, so $m_0(\Delta)$ is of strong type (p, p) for every p in $[1, +\infty]$.

Now decompose the kernel k_∞ into the sum $k_\infty = k_\infty^1 + k_\infty^\infty$, where $k_\infty^1 = k_\infty \chi_{B_1}$ and $k_\infty^\infty = k_\infty(1 - \chi_{B_1})$; it suffices to control the L^1 norm of k_∞^∞ and, in view of Coifman and Weiss' [3], Théorème III.2.4, integrals of the form

$$\int_{A(y)} |k^1_\infty(xy) - k^1_\infty(x)|\,dx, \qquad (6)$$

where $A(y) = \{x \in S: 2|y| \leqslant |x| \leqslant 1\}$. In order to do this, let h_j be the function on $\mathbb{R}$ defined by

$$h_j(\tau) = m_\infty(\tau^2)\psi(2^{-j}\tau^2)e^{2^{-j}\tau^2}\tau \in \mathbb{R}, j \in \mathbb{Z};$$

then $m_\infty(\Delta) = \sum_{j=0}^\infty h_j(\Delta^{1/2})e^{-2^{-j}\Delta}$ and the kernel k_∞ is $\sum_{j=0}^\infty h_j(\Delta^{1/2})p_{2^{-j}}$, where p_t denotes the heat kernel associated to Δ. Moreover, the following lemma holds.

Lemma 4. *Let $j \geqslant 0$, then*

$$\|\hat{h}_j\|_1 \leqslant C\|m\|_{(s)}$$

$$\||\cdot|^s\hat{h}_j\|_2 \leqslant C2^{j\frac{1-2s}{4}}\|m\|_{(s)}.$$

If $0 < r < R < \infty$, then for every function u in $L^2(S)$,

$$\|\chi_{B^c_R}h_j(\Delta^{1/2})u\|_2 \leqslant C(2^{j\frac{1-2s}{4}}(R-r)^{\frac{1-2s}{2}}\|\chi_{B_r}u\|_2 + \|\chi_{B^c_R}u\|_2)\|m\|_{(s)}.$$

See [7] for a proof, which is based on the property of finite propagation speed of the operator $\cos(t\Delta^{1/2})$ [2].

To estimate the L^1 norm of k^∞_∞ and integrals of the form (6), decompose $S\backslash\{e\}$ into the disjoint union of dyadic annuli; by Lemma 4, it is enough to obtain small time estimates of L^2 norms outside balls of the heat kernel p_t and of its gradient. The first estimates can be achieved using VAROPOULOS' estimate [15], formula (4.1); by Lemma 3 and 4 and Proposition 1, the estimates of $\|\chi_{B^c_R}|\nabla p_t|\|_2$ can be obtained from pointwise estimates of the gradient of the heat kernel associated to the Laplace–Beltrami operator. These estimates will be the subject of the next section.

4. Small Time Estimate of the Gradient of the Heat Kernel

Let q_t be the heat kernel associated to the Laplace–Beltrami operator $\mathscr{L}$; in this section we exploit the method in [16], to obtain pointwise estimates of the gradient of the heat kernel q_t, from Varopoulos' estimate of the heat kernel itself.

Lemma 5. *There exists a constant $h > 0$ such that, for every $\alpha \geqslant 0$ and for every $0 < t \leqslant 1$,*

$$\|q_t(\cdot)\exp(\alpha|\cdot|)\|_2 \leqslant Ct^{-n/4}e^{h\alpha^2 t} \qquad (7)$$

where $n = m + k + 1$. *Moreover,*

$$\left\| \frac{d}{dt} q_t(\cdot) \right\|_2 \leqslant C t^{-1-n/4}. \tag{8}$$

Proof. From VAROPOULOS' estimate of the heat kernel for small time (see [15]), one can deduce the existence of a constant h such that

$$q_t(x)\exp(\alpha|x|) \leqslant C e^{h\alpha^2 t} t^{-n/2} \exp(-|x|^2/Ct)$$

Then

$$\int_S |q_t(x)|^2 \exp(2\alpha|x|)dx =$$

$$= \int_{|x|^2 \leqslant t} + \sum_{j=1}^{\infty} \int_{2^{j-1}t \leqslant |x|^2 \leqslant 2^j t} \leqslant$$

$$\leqslant C e^{2h\alpha^2 t} t^{-n} \left[e^{-2/C} |B_{t^{1/2}}| + \sum_{j=1}^{\infty} e^{-2^j/C} |B_{(2^j t)^{1/2}}| \right] =$$

$$= C e^{2h\alpha^2 t} t^{-n/2} \left[e^{-2/C} + \sum_{j=1}^{\infty} e^{-2^j/C} 2^{nj/2} e^{Q 2^{j/2}} \right].$$

As the last series is convergent, this concludes the proof of the first inequality; the second inequality is an application of the Plancherel formula together with the estimate of the $\mathbf{c}$ function (1):

$$\left\| \frac{d}{dt} q_t(\cdot) \right\|_2^2 = \left\| \frac{d}{dt} \tilde{q}_t(\cdot) \right\|_2^2 =$$

$$= \int_0^{\infty} \lambda^4 e^{-2\lambda^2 t} |\mathbf{c}(\lambda)|^{-2} d\lambda \leqslant$$

$$\leqslant \int_0^1 \lambda^6 e^{-2\lambda^2 t} d\lambda + \int_1^{\infty} \lambda^{3+n} e^{-2\lambda^2 t} d\lambda =$$

$$= (t^{-3-1/2} + t^{-(3+n)/2-1/2}) \int_0^{\infty} e^{-2\mu^2} d\mu \leqslant$$

$$\leqslant C t^{-n/2-2}$$

which is the desired estimate. □

Theorem 6. *If* $i = 0, \ldots, m + k$, *then there exists a positive constant* c *such that, for every* x *in* S *and for every* $0 < t \leqslant 1$,

$$|E_i q_t(x)| \leqslant C t^{-(n+1)/2} \exp(-|x|^2/ct) \tag{9}$$

Proof. As E_i is a left-invariant vector field,

$$E_i q_t(x) = E_i(q_{t/2} * q_{t/2})(x) = q_{t/2} * E_i q_{t/2}(x) = \\ = \int_S q_{t/2}(y) E_i q_{t/2}(y^{-1}x)dx$$

so, for every $\alpha > 0$, since $|y| + |y^{-1}x| \geqslant |x|$, one obtains

$$|e^{\alpha|x|} E_i q_t(x)| \leqslant \int_S q_{t/2}(y) e^{\alpha|y|} |E_i q_{t/2}(y^{-1}x)| e^{\alpha|y^{-1}x|} dy \leqslant \\ \leqslant \| q_{t/2} e^{\alpha|\cdot|} \|_2 \| E_i q_{t/2} e^{\alpha|\cdot|} \|_2. \tag{10}$$

By Lemma 5, formula (7)

$$\| q_{t/2} e^{\alpha|\cdot|} \|_2 \leqslant C t^{-n/4} e^{h\alpha^2 t/2}, \tag{11}$$

so the problem now is to find an estimate for $\| E_i q_{t/2} e^{\alpha|\cdot|} \|_2$; as the distance function satisfies $||x| - |y|| \leqslant |y^{-1}x|$, it can be approximated by positive functions ϕ_n in C_0^∞, such that $|\nabla \phi_n| \leqslant 1$ and

$$\| E_i q_{t/2} e^{\alpha|\cdot|} \|_2 = \lim_n \| E_i q_{t/2} e^{\alpha \phi_n} \|_2.$$

Let ϕ be such a function; then

$$\| E_i q_{t/2} e^{\alpha\phi} \|_2^2 = \int_S E_i q_{t/2} e^{2\alpha\phi} E_i q_{t/2} dx.$$

Remembering that

$$\langle E_i f, g \rangle = - \langle f, E_i g \rangle \quad \text{if } i = 1, \ldots, m+k \\ \langle E_0 f, g \rangle = - \langle f, E_0 g \rangle + Q \langle f, g \rangle$$

it follows that

$$\sum_{i=0}^{m+k} \| E_i q_{t/2} e^{\alpha\phi} \|_2^2 = \int_S q_{t/2}(x) e^{2\alpha\phi(x)} \mathscr{L} q_{t/2}(x) dx - \\ - 2\alpha \sum_{i=0}^{m+k} \int_S q_{t/2}(x) E_i q_{t/2}(x) E_i \phi(x) e^{2\alpha\phi(x)} dx$$

so for every $i = 0, \ldots, m+k$,

$$\| E_i q_{t/2} e^{\alpha\phi} \|_2^2 \leqslant \| q_{t/2} e^{2\alpha\phi} \|_2 (\| \mathscr{L} q_{t/2} \|_2 + 2\alpha C \| \nabla q_{t/2} \|_2).$$

As $\mathscr{L} q_t = \dfrac{d}{dt} q_t$ and $\| \nabla q_t \|_2^2 = \| \mathscr{L}^{1/2} q_t \|_2^2 \leqslant \| \mathscr{L} q_t \|_2 \| q_t \|_2$, putting together the estimates (7), (8), (10) and (11), one obtains

$$|e^{\alpha|x|}E_i q_t(x)| \leqslant Ct^{-(n+1)/2}e^{3/2k\alpha^2 t}$$

where k is $h+\varepsilon$, $\varepsilon>0$. So

$$|E_i q_t(x)| \leqslant Ct^{-(n+1)/2}\exp\left(\frac{3}{2}k\alpha^2 t-\alpha|x|\right).$$

Now, for fixed x and t, choose $\alpha=|x|/3kt$; squaring, adding over i and taking the square root, one obtains

$$|\nabla q_t(x)| \leqslant Ct^{-(n+1)/2}\exp\left(\frac{3k|x|^2 t}{18k^2t^2}-\frac{|x|^2}{3kt}\right)=$$
$$=Ct^{-(n+1)/2}\exp(-|x|^2/6kt).$$

This ends the proof of the theorem. □

Remark. One can also prove that the best constant c in Theorem 6 is $6+\varepsilon$, because in Lemma 5 the constant h in formula (7) is at best $1+\eta$, $\eta>0$. These estimates are enough to prove Lemma 4.4 in [7].

Acknowledgement. I would like to thank Prof. G. Mauceri for valuable conversations on the subject of this paper and for his encouragement.

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