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Volumes of Complementary Projections of Convex Polytopes

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Abstract: The centrally symmetric convex polytopes whose images under orthogonal projection on to any pair of orthogonal complementary subspaces of $\mathbb{E}^d$ have numerically equal volumes are shown here to be certain cartesian products of polygons and line segments. For $d \geqslant 3$, the general projection property in fact follows from that for pairs of hyperplanes and lines. A conjecture is made about the problem in the non-centrally symmetric case.

1. Introduction

Let P be a (d-dimensional) convex polytope in euclidean space $\mathbb{E}^d$, let L be a linear subspace of $\mathbb{E}^d$, let Φ_L denote orthogonal projection on to L, and let $V = V_L$ denote ordinary ($\dim L$)-dimensional volume in L (if $L = \{o\}$, then $V = 1$). We say that P has the *property* (VP), or is a (VP)*-polytope*, if for each pair L and M of orthogonal complementary subspaces of $\mathbb{E}^d$, $V_L(\Phi_L P) = V_M(\Phi_M P)$.

We have shown in [4] (see also [2]) that a unit regular d-cube has the property (VP). In this paper, we shall investigate the more general problem of determining which d-polytopes are (VP)-polytopes. This problem is easily solved if $d \leqslant 2$; our main object here is to settle the problem for centrally symmetric polytopes. We shall also propose a solution to the problem in general.

An interesting feature of the centrally symmetric case for $d \geqslant 3$ is that it reduces to solving the equation $\Pi P = D P (= 2P)$ relating the projection and difference bodies of a polytope P, which expresses the property (VP) for hyperplanes and lines alone.

2. The Planar Case

As a preliminary to the discussion of the general case, we first describe the d-dimensional (VP)-polytopes for $d \leqslant 2$. An immediate consequence of the definition is:

Lemma 1. *A (VP)-polytope P in $\mathbb{E}^d$ has d-dimensional volume $V(P) = 1$.*

This holds because $\mathbb{E}^d$ is the orthogonal complement of $\{o\}$. As a consequence:

Theorem 1. *The (VP)-polytopes in $\mathbb{E}^1$ are the unit line segments.*

Now let P be a (VP)-polygon in $\mathbb{E}^2$. Apart from having area 1 by Lemma 1, the property (VP) implies that P has equal widths in perpendicular directions. An easy way of expressing this is in terms of the *difference body*

$$DP = P - P = \{x - y \mid x, y \in P\} .$$

Theorem 2. *A polygon P in $\mathbb{E}^2$ has property (VP) if and only if P has area 1, and DP has 4-fold rotational symmetry.*

It suffices to remark that DP (which is centrally symmetric about o) has twice the width of P in any direction.

3. The Higher Dimensional Case: the Main Theorem

From now on, we suppose that $d \geqslant 3$. It is of great help in the discussion to introduce another auxiliary body associated with a polytope P. This is the *projection body* ΠP of P, whose support function $h(\Pi P, \cdot)$ satisfies

$$h(\Pi P, u) = V_M(\Phi_M P) ,$$

if u is a unit vector and M is the hyperplane orthogonal to u. Thus ΠP is centrally symmetric about the origin o. In fact, ΠP is actually a *zonotope*, or vector sum of line segments; for more details, see [9] or the survey article [6].

We can now state our main theorem. This is confined to centrally symmetric polytopes, whose centres of symmetry we shall tacitly assume to be o.

Theorem 3. *Let P be a centrally symmetric d-polytope in $\mathbb{E}^d$ $(d \geqslant 3)$. Then the following conditions are equivalent.*

a) *P is a (VP)-polytope;*

b) *$\Pi P = DP (= 2P)$;*

c) *P is a cartesian (orthogonal) product of (VP)-polygons and unit line segments.*

Since (b) is just that part of the definition of a (VP)-polytope P corresponding to the projection of P on to pairs of orthogonal lines and hyperplanes (compare the definitions of ΠP and DP, and the discussion above), we shall see that (b) follows trivially from (a). It is interesting that the apparently weaker condition (b) is in fact equivalent to (a), at least in the special case of centrally symmetric polytopes. We shall complete the proof of Theorem 3 in the next two sections.

4. Necessity of the Condition

Here we shall show that condition (c) follows from condition (b). The proof will follow from a result of WEIL [9], together with an examination of particular projections.

The condition $\Pi P = 2P$ implies, by [9], that P is a cartesian product of certain (centrally symmetric) polygons and line segments, say

$$P = P_1 \times \ldots \times P_r \times I_1 \times \ldots \times I_s,$$

where $P_1, \ldots, P_r$ are polygons and $I_1, \ldots, I_s$ are line segments so that $2r + s = d$. We choose an orthonormal basis $\{e_1, \ldots, e_d\}$ in $\mathbb{E}$ so that

$$P_j \subseteq E_j = \operatorname{lin}\{e_{2j-1}, e_{2j}\} \quad (j = 1, \ldots, r),$$
$$I_k \subseteq \operatorname{lin}\{e_{2r+k}\} \quad (k = 1, \ldots, s).$$

(We thus suppose, as we may, that the P_j and I_k also have centre o).

We first look at the polygons P_j. For arbitrary $u \in E_j$ $(j = 1, \ldots, r)$, we have

$$h(DP, u) = h(DP_j, u),$$
$$h(\Pi P, u) = (\Pi_j P_j, u) \prod_{i \neq j} V_2(P_i) \prod_{k=1}^{s} V_1(I_k),$$

where $\Pi_j p_j$ is the projection body of P_j in E_j. If u' is a unit vector in E_j orthogonal to u, we have

$$h(\Pi_j P_j, u) = h(DP_j, u'),$$

and so replacing u by u' in the above shows first

$$\prod_{i \neq j} V_2(P_i) \prod_{k=1}^{s} V_1(I_k) = 1,$$

and second

$$\Pi_j P_j = DP_j (= 2P_j).$$

On the other hand, if $u = e_{2r+k}$ $(k = 1, \ldots, s)$, the condition $h(\Pi P, u) = h(D P, u)$ implies

$$V_1(I_k) = \prod_{j=1}^{r} V_2(P_j) \prod_{i \neq k} V_1(I_i) .$$

Bearing mind that $d \geqslant 3$ (so that $r + s \geqslant 2$), easy algebraic manipulations now show that

$$V_2(P_j) = 1 = V_1(I_k)$$

for $j = 1, \ldots, r$ and $k = 1, \ldots, s$. In view of Theorem 2, this establishes the necessity of condition (c).

5. Sufficiency of the Condition

We now show that condition (a) follows from condition (c). If P satisfies the latter condition, then P and all its projections are zonotopes, or Minkowski sums of line segments. Our first step is to generalize a result of Shephard [7] on zonotopes.

We recall a consequence of the Lifting Theorem of Walkup and Wets [8], which we have used in a number of contexts (see, for example, [5]).

Lemma 2. *A Minkowski sum $Q_1 + \ldots + Q_t$ of polytopes admits a dissection into direct sums of polytopes $G_1 + \ldots + G_t$, where G_i is a face of Q_i for $i = 1, \ldots, t$.*

In the special case of zonotopes, this result can be somewhat strengthened. The faces of a zonotope Q fall into *face-classes*, in which the faces are translates of one another. Corresponding to the face-classes of the zonotope Q_1 $(i = 1, \ldots, t)$ the cells $G_1 + \ldots + G_t$ of a dissection of $Q_1 + \ldots + Q_t$ fall into *cell-classes*, according to the face-classes of the G_i; so, we regard $G_1 + \ldots + G_t$ and $G'_1 + \ldots + G'_t$ as being in the same cell-class if and only if G'_1 is a translate of G_i $(i = 1, \ldots, t)$. We then have:

Lemma 3. *Every dissection of a sum $Q_1 + \ldots + Q_t$ of zonotopes into cells $G_1 + \ldots + G_t$ (G_i a face of Q_i, $i = 1, \ldots, t$) contains exactly one cell from each possible cell-class (of full-dimensional cells).*

The result of Shephard [7] is the particular case when each Q_i is itself a line segment. Lemma 3 follows directly from this special case, for if we express each zonotope Q_i (and hence each of its faces G_i) as a sum of line segments, then the essential uniqueness of the resulting subdissection implies the statement of the lemma.

We are now in a position to prove the sufficiency. The crucial result which led to the special case of the cube is the following. If L and M are two linear subspaces of $\mathbb{E}^d$ of the same dimension, then the number

$$\langle L, M\rangle = V_L(\Phi_L P)/V_M(P)$$

is obviously independent of the choice of full-dimensional polytope P in M. We then have ([1]):

Lemma 4. *If L, M are two subspaces of $\mathbb{E}^d$ of the same dimension, then $\langle L, M\rangle = \langle L^\perp, M^\perp\rangle$.*

Now let $P = P_1 \times \ldots \times P_r \times I_1 \times \ldots \times I_s$ be as in the statement of Theorem 2, and let L be a linear subspace of $\mathbb{E}^d$. We can suppose that L is neither $\{o\}$ nor $\mathbb{E}^d$ itself, since the (VP)-property clearly holds for these subspaces. Let us write

$$Q_j = \Phi_L P_j, \quad J_k = \Phi_L I_k,$$

so that the Q_j and J_k are (possibly degenerate) centrally symmetric polygons and line segments, respectively, and hence zonotopes. Let

$$G = G_1 + \ldots + G_r + H_1 + \ldots + H_s$$

be a full-dimensional cell in a dissection of

$$\Phi_L P = Q_1 + \ldots + Q_r + J_1 + \ldots + J_s \,.$$

We can suppose (and here we are really referring back to the proof of Lemma 2 using [8]) that there are faces F_j of P_j and E_k of I_k, with

$$G_j = \Phi_L F_j, \quad H_k = \Phi_L E_k,$$

and $F_1 + \ldots + F_r + E_1 + \ldots + E_s = F$ (say) is a face of P of dimension $\dim L$. We now define faces $F'_1, \ldots, F'_r, E'_1, \ldots, E'_s$ as follows: If $F_j =$ vertex (P_j), then $F'_j = P_j$ (vertex); if F_j is an edge of P_j, then F'_j is a perpendicular edge of P_j; if $E_k =$ vertex (I_k), then $E'_k = I_k$ (vertex). Now let

$$F' = F'_1 + \ldots + F'_r + E'_1 + \ldots + E'_s,$$

$$G'_j = \Phi_{L^\perp} F'_j, \quad H'_k = \Phi_{L^\perp} E'_k;$$

this determines an element

$$G' = G'_1 + \dots + G'_r + H'_1 + \dots + H'_s$$

of a cell-class of a dissection of $\Phi_{L^\perp} P$. Now, if M is a linear subspace of $\mathbb{E}^d$ parallel to F, then $M^\perp$ is that parallel to F'. Moreover, since each P_j has 4-fold symmetry and unit area, and each I_k has unit length, we see that

$$\mathrm{vol}_M(F) = \mathrm{vol}_{M^\perp}(F') \;.$$

In view of Lemma 4 and the preceding remarks, there at once follows

$$\mathrm{vol}_L(G) = \mathrm{vol}_{L^\perp}(G') \;.$$

In the course of this proof, we have constructed a one-to-one correspondence between full-dimensional cell-classes, induced by $G \leftrightarrow G'$, and using Lemma 3 this correspondence and the equality above completes the proof of the sufficiency, and hence of Theorem 3.

6. The General Case

We shall now slightly extend Theorem 3 in one direction, and adduce some evidence that this extension describes the general state of affairs.

Theorem 4. *Let the d-polytope P be a cartesian product of* (VP)-*polygons and unit line segments, of which at most one polygon is not centrally symmetric. Then P is a* (VP)-*polytope.*

We prove this as a consequence of (the method of) Theorem 3. Let us denote, as before,

$$P = P_1 + \dots + P_r + I_1 + \dots + I_s;$$

we assume that P_1 (above) is not centrally symmetric. Again, as before, if L is a linear subspace of $\mathbb{E}^d$, we write

$$Q_j = \Phi_L P_j, \quad J_k = \Phi_L I_k \;.$$

Now, apart from the fact that

$$V_2(\tfrac{1}{2}(P_1 - P_1)) > V_2(P_1) = 1,$$

$\frac{1}{2}(P_1 - P_1)$ is a (VP)-polygon. We now consider $V_L(Q)$, where

$$Q = Q_1 + \dots + Q_r + J_1 + \dots + J_s \;.$$

We note that we change Q to $-Q$ if we replace P_1 by $-P_1$ (and hence Q_1 by $-Q_1$), since the remaining P_j and the I_k are centrally symmetric.

Now $V_L(Q)$ can be expressed as a sum of three terms (actually mixed volumes), which are, respectively, quadratic, linear or constant in Q_1. We look at these three terms in turn.

The first, quadratic, term arises from those cells of a dissection of Q which come from faces of P containing a translate of P_1. We can consider varying the dissection, by replacing Q_1 by $\lambda Q_1 - \mu Q_1$ ($\lambda, \mu \geqslant 0$); the case $\lambda = \frac{1}{2} = \mu$ corresponds to the zonotope case, discussed before, but the dissections can be made to vary continuously. So, all the terms contributing to the quadratic term can be put into one to one correspondence with the terms of $V_{L^\perp}(Q' = \Phi_{L^\perp} P)$ which are constant in $Q'_1 = \Phi_{L^\perp} P_1$, and by the same token as above, these volumes are numerically equal. The symmetrical argument equates the constant term in $V_L(Q)$ with the quadratic term in $V_{L^\perp}(Q')$.

There remains the linear term. Since the area of P_1 does not enter into the earlier argument, we can now equate the terms linear in Q_1 in $V_L(\bar{Q})$ and $V_{L^\perp}(\bar{Q}')$, where in $\bar{Q}$ we replace Q_1 by $\frac{1}{2}(Q_1 - Q_1)$, and similarly for $\bar{Q}'$. But by this linearity, and the symmetry between Q_1 and $-Q_1$, we finally equate the original linear terms, and hence show that $V_L(Q) = V_{L^\perp}(Q')$, which proves Theorem 4.

We now wish to propose:

Conjecture 1. *The condition of Theorem 4 is necessary as well as sufficient for a polytope P to be a (VP)-polytope.*

As we have already remarked, a (VP)-polytope P must satisfy $\Pi P = DP$. The first step in proving Conjecture 1 would be to establish

Conjecture 2. *If a polytope P satisfies $\Pi P = DP$, then P is a cartesian product of polygons and line segments.*

By considering certain special projections, the fact that no more that one polygonal component in the cartesian product P can be non-centrally symmetric if P is to be a (VP)-polytope would follow from:

Conjecture 3. *Let P_1, P_2 be non-centrally symmetric polygons in $\mathbb{E}^2$. Then there is a rotation Ψ of $\mathbb{E}^2$ such that $V_2(P_1 + \Psi P_2) \neq V_2(P_1 - \Psi P_2)$.*

Rolf Schneider (private communication) has pointed out that this conjecture can fail for pairs of non-centrally symmetric planar convex sets.

So far, we have said nothing about arbitrary convex bodies. Theorem 2 generalizes in the obvious way, to describe the (VP)-*discs*, as we shall call the planar convex bodies which have property (VP). But while cartesian products of centrally symmetric (VP)-discs and unit line segments have property (VP), it is far from obvious that this characterizes centrally symmetric convex bodies satisfying (VP).

A further generalization is possible. Suppose that K is a convex body in $\mathbb{E}^d$, such that there exist numbers $\lambda_0, \ldots, \lambda_d$, with $V_{L^\perp}(\Phi_{L^\perp} K) = \lambda_r V_L(\Phi_L K)$ whenever L is an r-dimensional linear subspace. (Thus $\lambda_r \lambda_{d-r} = 1$ for $r = 0, \ldots, d$.) If K is a centrally symmetric polytope, and if $\mu^{d-2} = \lambda_1$, then we see that $\Pi(\mu K) = D(\mu K)$, and by Theorem 3 (the equivalence of conditions (a) and (b)), we see that μK is a (VP)-polytope; in other words, we get nothing essentially new. But clearly any ball has this more general property, although it is nowhere near a product of discs and line segments. This generalization may well be very difficult to investigate, and we propose no solution here.

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