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On Unification of Terms with Integer Exponents

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Abstract. We consider terms in which some patterns can be repeated n times. n is an integer variable which is part of the syntax of the terms (and hence may occur more than once in them). We show that unification of such terms is decidable and finitary, extending Chen and Hsiang's result on ρ-term unification. Finally, extending slightly the syntax yields an undecidable unification problem.

1. Introduction

In [1] H. Chen and J. Hsiang proposed a unification algorithm for what they called ω-terms, and later ρ-terms (we keep this last terminology), with intended applications to logic programming. These terms allow us to iterate terms with one hole along fixed paths. The number of iterations is part of the syntax of the terms and may include integer variables.

Example 1.1. Let the alphabet of functions symbols be f (binary), g (unary), and a (constant). Let N be an integer variable. Then, in Chen and Hsiang's formalism, $\Phi(f(\Diamond, a), N, g(a))$ is a typical ρ-term whose instances (obtained by replacing N with an actual nonnegative integer) are $g(a), f(g(a), a), f(f(g(a), a), a), \ldots$. The term $f(\Diamond, a)$ has one "hole" (the $\Diamond$) and its iteration along path 1 (which is the position of the hole) is allowed.

Such constructions can be useful in expressing infinite sets of terms which occur in logic programming (see [1] or in the Knuth–Bendix completion procedure (see [2]). In both cases the ability to express iterated terms can prevent nontermination of the deduction process. Other applications are currently under investigation, for example in unification theory (see [4]).

Constructions involving more than one occurrence of an integer variable such as

$$f(\Psi(f(\Diamond, a), N, g(a)), \Phi(f(\Diamond, a), N, f(a, a)))$$

are also allowed in [1] which shows that the ρ-terms can schematize nonregular sets of ground terms and hence they are not encompassed by the study of terms with context variables investigated in [3]. On the other hand, ρ-terms do not have the power of regular languages; there are two restrictions in these ρ-terms: nested ρ-terms are forbidden and the iterated part should not itself contain ρ-terms. Finally, in [1], the iterated parts and the terms in the holes should not contain variables.

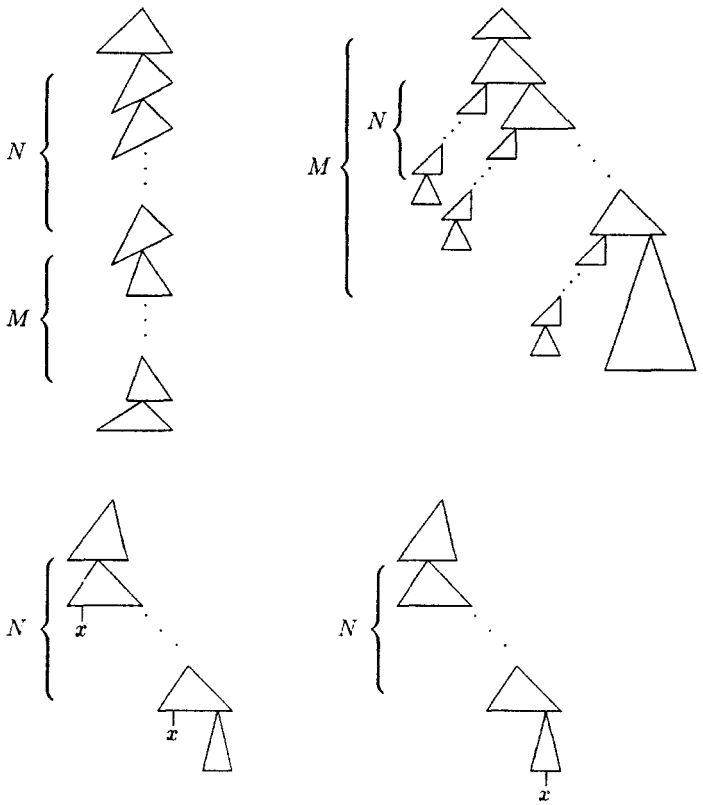


Fig. 1. Examples of terms in T which are not ρ-terms.

Example 1.2. (see also Figure 1).

$\Phi(f(\Diamond, a), N, \Phi(g(\Diamond), M, a))$ is not a ρ-term. (Nested ρ-terms are forbidden.)

$\Phi(f(\Diamond, \Phi(f(\Diamond, a), N, a)), M, g(a))$ is not a ρ-term (the iterated part itself should not contain a ρ-term).

$\Phi(f(x, \Diamond), N, g(a))$ is not a ρ-term (variables are not allowed in the iterated part).

$\Phi(f(a, \Diamond), N, g(x))$ is not a ρ-term (variables are not allowed "below" the iterated part, i.e., in the third position of the Φ construction).

In this note we give another unification algorithm where these restrictions are dropped. We keep, however, the condition that any ρ-term should not occur along an iterated path. (i.e., for example, $\Phi(\Phi(f\Diamond, a), N, g(\Diamond)), M, g(a))$ is still not allowed). This means that if the iterated part itself contains iterated parts, the two iterated paths (or if one prefers, the two positions of the hole holder) should be uncomparable with respect to the prefix ordering (as in the right-up example of Figure 1). Indeed, without this restriction, unification becomes undecidable.

2. Syntax and Interpretation of Formulae

2.1. *Terms*

Missing definitions can be found in [5]. We assume that F is a (finite or infinite) set of function symbols together with the arity function a. F is assumed to contain at least one constant (i.e., a symbol of arity 0). X is an infinite set of constants (disjoint from F) called *variables*. $\Diamond$ is a special symbol of arity 0 (the hole holder). V_N is a fixed set of symbols denoting integer variables. The set of I-terms T (also called "terms" for short) and the set of *terms with one hole* T_1 (also called "contexts") are the least sets that satisfy:

$$\begin{cases} f(\vec{s}) \in T & \Leftarrow \quad \vec{s} \in T^{a(f)}, \\ x \in T & \Leftarrow \quad x \in X, \\ \Diamond \in T_1, & \\ f(\vec{s_1}, s, \vec{s_2}) \in T_1 & \Leftarrow \quad (\vec{s_1}\, m\, s, \vec{s_2}) \in T^{n_1} \times T_1 \times T^{n_2},\ n_1 + 1 + n_2 = a(f), \\ s^N \cdot t \in T & \Leftarrow \quad s \in T_1, \quad t \in T, \quad N \in V_N, \quad s \neq \Diamond. \end{cases}$$

The construction $s^N \cdot t \in T_1 \Leftarrow s \in T_1,\ t \in T_1,\ N \in V_N$ is not allowed here. However, as we will see in Section 7, this additional construction does not increase the expressive power. We also define $Pos(t)$, the set of *positions* of a term $t \in T \cup T_1$, as follows:

$$\begin{cases} Pos(x) \stackrel{\text{def}}{=} \{\Lambda\}, \\ Pos(\Diamond) \stackrel{\text{def}}{=} \{\Lambda\}, \\ Pos(f(\vec{s})) \stackrel{\text{def}}{=} \{\Lambda\} \cup 1 \cdot Pos(s_1) \cup \cdots \cup a(f) \cdot Pos(s_{a(f)}), \\ Pos(s^N \cdot t) \stackrel{\text{def}}{=} \{\Lambda\}. \end{cases}$$

We often omit the $\cdot$ when concatenating the positions ($\cdot$ is overloaded). Positions are (partially) ordered with the prefix ordering $\geq_{\text{pref}}$. If p and q are uncomparable positions, we write $p \| q$. In the above definition, $i \cdot Pos(s_i)$ stands for the set

$\{i \cdot p | p \in Pos(s_i)\}$. If $p \in Pos(t)$, $t|_p$ is (as usual) the subterm at position p. $t(p)$ is the label (function symbol) at position p and $t[u]_p$ is the term obtained by replacing $t|_p$ with u at position p. We often use the notations $s[\Diamond]_p$, $s[\Diamond]_p^N \cdot t$ to indicate the position at which $\Diamond$ occurs in s: it follows from the definitions of T_1 and Pos that, for every term $s \in T_1$ which is not $\Diamond$ itself, there is exactly one $p \in Pos(s)$ $(p \neq \Lambda)$ such that $s|_p = \Diamond$.

Extensions of this syntax (keeping the same expressive power) are described in Section 7. In particular, contexts with multiple holes are considered.

Example 2.1. We use the same alphabet as in Example 1.1. Assuming $x, y \in X$, $N, M, Q \in V_N$, the following expressions are I-terms:

- $t_1 \equiv f(x, f(x, \Diamond))^N \cdot f(\Diamond, a)^M \cdot g(x)$ (this corresponds to I-terms with nested exponents).
- $t_2 \equiv f(f(y, \Diamond)^Q \cdot a, \Diamond)^Q \cdot y$ (this corresponds to I-terms where the iterated part itself contains integer exponents).

Of course the intended meaning of the construction $s[\Diamond]_p^N$ (which is defined precisely below) is to iterate the context $s[\ ]_p$ (i.e., the term s in which the subterm at position p has been erased) N times.

The identity of I-terms is denoted $\equiv$ (in order to avoid confusion with $=$ which is used for equations).

2.2. *Equations and Formulae*

We consider two kinds of equations: ordinary equations of the form $s = t$ where $s, t \in T$ ($=$ is assumed symmetric: there is no difference between $s = t$ and $t = s$) and linear diophantine equations over a set of variables V_N. If ψ is a conjunction of ordinary equations, then $IV(\psi)$ (*integer variables* of ψ) is the set of symbols of V_N which occur in some I-term of ψ. On the other hand, $Var(\psi)$ is the set of free variables $x \in X$ which occur in ψ.

Unification formulae are disjunctions of formulae of the form $\exists \vec{n} \cdot \varphi \wedge \psi$ where $\vec{n}$ is a finite subset of V_N, ψ is a conjunction of ordinary equations (called the *pure part* of the conjunction), and φ is a conjunction of linear diophantine equations whose variables are a subset of $\vec{n} \cup IV(\psi)$ (ϕ is called the *diophantine part* of the conjunction).

2.3. *Substitutions and Solutions*

Substitution of ordinary variables (i.e., elements of X) by I-terms are defined in the usual way: a substitution σ is defined by a finite set of pairs (variable, term) written $\{x_1 \mapsto t_1; \cdots; x_n \mapsto t_n\}$. Its application to $t \in T$ is defined by:

- $(f(s_1, \ldots, s_n)\sigma \stackrel{\text{def}}{=} f(s_1\sigma, \ldots, s_n\sigma)$.
- $x_i\{x_1 \mapsto t_1; \ldots; x_n \mapsto t_n\} \stackrel{\text{def}}{=} t_i$ and $x\{x_1 \mapsto t_1; \ldots; x_n \mapsto t_n\} \stackrel{\text{def}}{=} x$ if x is not one of the x_i's.
- $s[\Diamond]_p^N \cdot t\sigma \stackrel{\text{def}}{=} s\sigma[\Diamond]_p^N \cdot t\sigma$.

Note that this makes sense since, by induction on the structure of the terms.

$$(p \in Pos(s) \wedge s|_p \equiv \Diamond) \quad \Rightarrow \quad (p \in Pos(s\sigma) \wedge s\sigma|_p \equiv \Diamond) .$$

If σ is a mapping which assigns a nonnegative integer to each variable belonging to $IV(s = t)$, then $s\sigma$ and $t\sigma$ are defined by:

- $f(\vec{s})\sigma \stackrel{\text{def}}{=} f(\vec{s}\,\sigma)$.
- $x\sigma \stackrel{\text{def}}{=} x$ (when $x \in X$).
- $\left(s[\Diamond]_p^N \cdot u\right)\sigma \stackrel{\text{def}}{=} \underbrace{s\sigma[\cdots\, s\sigma[}_{N\sigma}\, u\sigma \underbrace{]_p \cdots]_p}_{N\sigma}$.

Similarly, we use the notation $t[\Diamond]_p^n$ where n is a (constant) nonnegative integer for the context obtained by unfolding $t[\ ]_p$ n times.

A *solution* σ to an ordinary equation $s = t$ is a pair (σ_1, σ_2) of a mapping σ_1 which assigns a nonnegative integer to each variable in $IV(s = t)$ and a substitution σ_2 which assigns an I-term to each variable in $Var(s = t)$ in such a way that $s\sigma_1\sigma_2 \equiv t\sigma_1\sigma_2$. This definition extends to any of our formulae in a straightforward way.

3. Outline of the Unification Procedure

Our goal is to transform any unification formula ψ into a formula ψ' which is equivalent (i.e., has the same set of solutions) to ψ and which is in *solved form*.

Definition 3.1. A *solved form* is either $\bot$, $\top$, or a finite disjunction of formulae of the form

$$\exists \vec{n}.\, N_1 = E_1 \wedge \cdots \wedge N_k = E_k \wedge x_1 = t_1 \wedge \cdots \wedge x_n = t_n ,$$

where $N_1, \ldots, N_k$ are integer (free) variables which occur only once in the conjunction, $E_1, \ldots, E_k$ are linear expressions over $\vec{n}$, and $x_1, \ldots, x_n$ are ordinary variables which are solved in the conjunction (i.e., that occur only once in the conjunction).

Such solved forms may also be regarded as substitutions.

The transformation of a unification formula into a solved form is described by means of *Transformation rules*, as in [5].

We split the rules into five parts. First, in Section 4, we use the classical unification rules (or slight extensions of them). Then we may assume that every unsolved ordinary equation has (at least) one member of the form $s^N \cdot t$. Such terms are called *N-terms* in the following.

The second step (again in Section 4) consists in getting rid of equations $s[\Diamond]_p^N \cdot t = u$ when p is a position of u. This step is very simple: we only "unfold" $s[\Diamond]_p^N$, after which we may again apply the unification elementary rules. Now we have only to consider equations of the form $s[t_1[\Diamond]_{q_1}^{N_1} \cdot u_1]_p = t_2[\Diamond]_{q_2}^{N_2} \cdot u_2$ (see Figure 2).

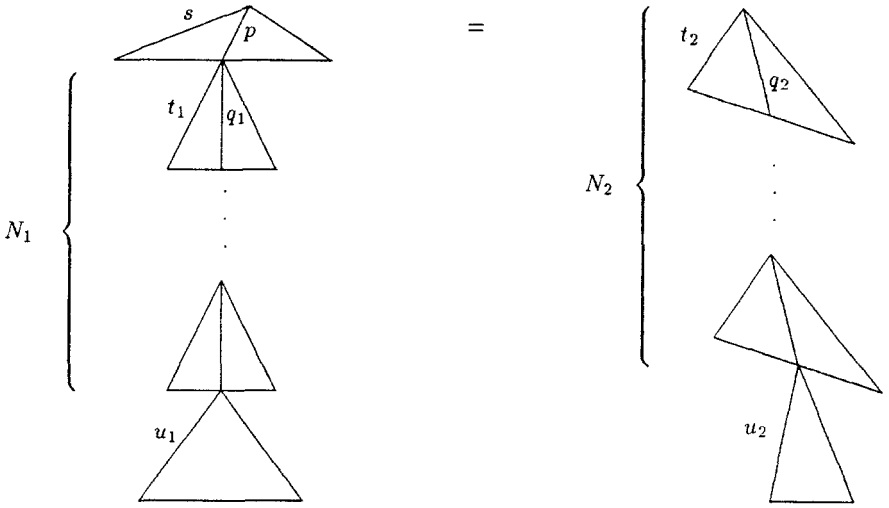


Fig. 2. The equations that remain to be considered after step 2.

The third step consists roughly in taking a "common multiple" of q_1 and q_2, reducing the general case to the case where q_1 and q_2 have the same length. We have to consider separately the cases where $N_2 \leq |q_1|$ and $N_1 \leq |q_2|$. Then, dividing N_2 by $|q_1|$ and N_1 by $|q_2|$, it is possible to replace q_1 with $q_1^{|q_2|}$ and q_2 with $q_2^{|q_1|}$ which have the same size.

The fourth part consists in again reducing the general case to the case where $q_2 \leq_{\text{pref}} p \cdot q_1 \leq_{\text{pref}} q_2 \cdot q_2$: it is sufficient to unfold once on each side. Now, if this last condition is satisfied and if $|q_1| = |q_2|$, q' must exist such that $q_2 = p \cdot q'$ and $q_1 = q' \cdot p$. (That is the main trick.)

Now, if there is no clash, we are left to solve equations of the form $s[v_1[w_1]_{q'}[\Diamond]_{q'p}^{N_1} \cdot u_1]_p = v_2[w_2]_p[\Diamond]_{pq'}^{N_2} \cdot u_2$. This situation is depicted in Figure 3. We have first to ensure the equality of subterms which are not located along the path $p \cdot (q' \cdot p)^{N_1}$; this is ensured by the equalities $s = v_2$, $v_1 = w_2$ and, $w_1 = v_2$. Then either $N_1 = N_2$, $N_1 > N_2$, or $N_1 < N_2$. In each case we simplify both sides: the path $(p \cdot q')^{\min(N_2, N_1)}$ is shared by two members of the equation.

Correctness of all the rules (i.e., the preservation of the set of solutions) is not difficult in general. Neither is termination very difficult because we never introduce new ordinary variables. Moreover, the number of integer variables which occur in the I-terms of the problem never increases. When these two numbers are constant, then the rules aim at "separating the integer variables," which is rather technical to explain formally.

4. Elementary Rules

First, we restrict our attention to solutions which assign nonnull integers to the integer variables of the problem. Indeed, we may at once nondeterministically

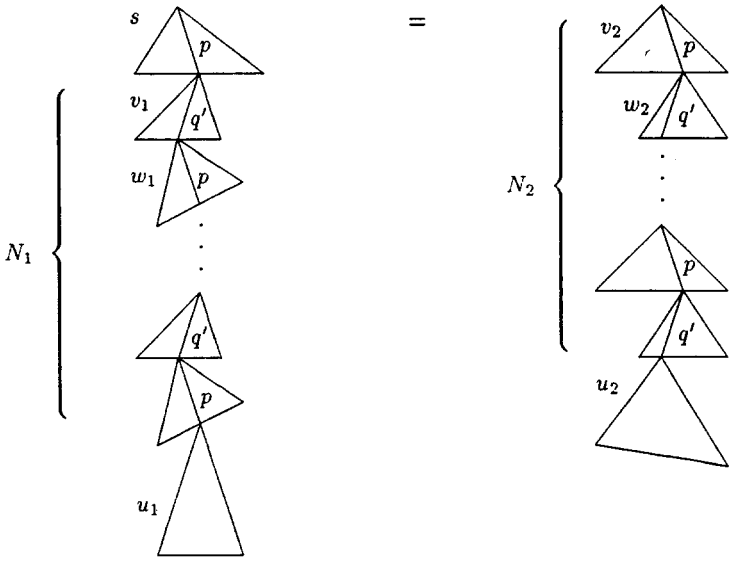


Fig. 3. The equations that remain to be solved after step 4.

choose the integer variables which are mapped to 0. Hence, from now on, we only consider such interpretations.[1]

Trivial	$s = s \to \top$	
Decompose	$f(\vec{s}) = f(\vec{t}) \to \vec{s} = \vec{t}$	
Clash	$f(\vec{s}) = g(\vec{t}) \to \bot$	if $f \neq g$
Variable Elimination	$x = s \wedge P \to x = s \wedge P\{x \mapsto s\}$	if $x \notin Var(s)$, $x \in Var(P)$, and $(s \in X \Rightarrow s \in Var(P))$
Occur Check	$x = s \to \bot$	if $s \not\equiv x$ and $x \in Var(s)$

Let R_1 be the above system of rules.

Lemma 4.1. *All rules in R_1 are correct. Moreover, R_1 terminates on any unification formula.*

Proof. Correctness is quite straightforward. Termintion is classical: it is similar to the termination proof of the usual unification rules (see [5]). □

[1] The nondeterministic choice is very inefficient and is not really necessary. We choose this presentation for the sake of simplicity (for example, we can use Occur-Checks without considering the null case separately) but this should not be followed in an implementation.

Let us call an equation *solved* in a unification formula φ if one of its member is a variable $x \in X$ which has only one occurrence in φ. (In such a case x itself is also called a *solved variable.*)

Lemma 4.2. *If φ is a unification formula which is irreducible with respect to R_1, then equations between I-terms which occur in φ are either solved or of the form $s^N \cdot t = u$ where u is not a variable.*

Proof. Every equation which is not of the form $s^N \cdot t = u$ where u is not a variable can be written as either $x = s$, where x is a variable, or $f(\vec{s}) = g(\vec{t})$. In the first case either $x = s$ is solved or else $x \in Var(s)$ (in which case, the Occur-Check rule or the Trivial rule applies) or the Variable Elimination rule applies. In the second case either Clash or Decompose applies. □

We first consider the case of equations $s = [\Diamond]_p^N \cdot u$ where no prefix q of p is such that $s|_q$ is an N-term. In such a case we may "unfold" $t[\Diamond]_p^N$ once: this will lead by decomposition along the path p to equations which have either less integer exponents or the same integer exponents and smaller terms.

Example 4.3. Consider the equation $f(\Diamond, a)^N \cdot x = f(y, f(a, \Diamond)^M \cdot z)$. We are going to unfold $f(\Diamond, a)^N$: either $N = 1$ which leads to the equation $f(x, a) = f(y, f(a, \Diamond)^M \cdot z)$ which contains less integer variables, or else N can be written $N' + 1$ (with $N' \geq 1$) and $f(f(\Diamond, a)^{N'} \cdot x, a) = f(y, f(a, \Diamond)^M \cdot z)$. This last equation can be simplified using R_1, which yields

$$f(\Diamond, a)^{N'} \cdot x = y \wedge a = f(a, \Diamond)^M \cdot z,$$

a conjunction of simpler equations.

(Unfold 1)

$$s = t[\Diamond]_p^N \cdot u \rightarrow (N = 1 \wedge s = t[u]_p) \vee (\exists M. N = M + 1 \wedge s = t[t^M \cdot u]_p).$$

If:

- There is no prefix q of p such that $s|_q$ is an N-term.
- R_1 does not apply.

Lemma 4.4. *The system R_2 obtained by adding (Unfold 1) to R_1 is correct and terminating.*

Proof. Correctness is straightforward: only termination needs a proof. We give an interpretation of formulae which are irreducible with respect to R_1.

Any unification formula can be written

$$(\exists \vec{n_1} . \varphi_1 \wedge \psi_1) \vee \cdots \vee (\exists \vec{n_k} . \varphi_k \wedge \psi_k),$$

where each φ_i is a system of linear diophantine equations and ψ_i is a pure conjunction of equations. We interpret this unification formula as the multiset

$$\{I(\psi_1), \ldots, I(\psi_k)\}$$

of interpretations of each pure part. The multisets are ordered using the multiset extension of the ordering on the components.

Now, $I(s_1 = t_1 \wedge \cdots \wedge s_n = t_n)$ is a pair $(a, \{I(s_1 = t_1), \ldots, I(s_n = t_n)\})$ which consists of the number a of unsolved (ordinary) variables in the conjunction and the multiset of interpretations of each equation. The pairs are ordered lexicographically. $I(\bot)$ is assumed to be minimal.

Before we define $I(s = t)$ we need to define some measures on the I-terms. These measures are reused in the following. Roughly, E-size(s) gives a measure of the number of *nested N-terms*. However, actually, there are two notions of nestedness: in the expression $s^N \cdot t^M \cdot u$, t^M is nested below s^N. In the expression $s[t^M \cdot u, \Diamond]^N \cdot v$, t^M is nested below s^N in another way because, for any instance n, m of N, M respectively, t will be repeated $n \times m$ times in the latter case whereas t will be repeated m times in the former case. That is why E-size(s) is a pair of natural numbers, recursively defined as follows:

- E-size (x) = E-size$(\Diamond)$ = E-size(a) = $(0, 0)$ for every variable x and every constant symbol a.
- E-size$(f(s_1, \ldots s_n))$ = max$\{$E-size$(s_i) | 1 \leq i \leq n\}$.
- E-size$(s^N \cdot t)$ = max$\{(n_1 + 1, 0), (m_1, m_2 + 1)\}$ if E-size$(s) = (n_1, n_2)$ and E-size$(t) = (m_1, m_2)$. (The maximum is considered with respect to the lexicographic extension of the ordering on natural numbers.)

$I(s = t)$ is the multiset of pairs $\{$(E-size(s), $|Pos(s)|$), (E-size(t), $|Pos(t)|$)$\}$ ordered using multiset and lexicographic extensions of the orderings.

We are going to show that the interpretation is strictly decreasing on the normal forms with respect to R_1 by application of Unfold 1).

Assume that

$$\Phi \vee (\exists \vec{n} \,.\, \varphi \wedge \psi \wedge s = t[\Diamond]_p^N \cdot u) \rightarrow$$

$$\Phi \vee (\exists \vec{n} \,.\, \phi \wedge N = 1 \wedge \psi \wedge s = t[u]_p)$$

$$\vee (\exists \vec{n}, M \,.\, \varphi \wedge N = M + 1 \wedge \psi \wedge s = t[t[\Diamond]_p^M \cdot u]_p).$$

R_1 does not introduce any disjunction. Hence, by definition of the multiset extension of an ordering, it is sufficient to prove that the normal forms (with respect to R_1) of $\psi \wedge s = t[u]_p$ and $\psi \wedge s = t[t[\Diamond]_p^M \cdot u]_p$ are both strictly smaller than $\psi \wedge s = t[\Diamond]_p^N \cdot u$ in the interpretation. If there is at least one occurrence of application of the Variable Elimination rule along the normalization by R_1, then the interpretation is decreasing because there are strictly less unsolved variables in the normal form. Similarly, if a Clash or an Occur-Check is applied during the normalization, the decrease is obvious. These situations can be excluded. Now, since only Decompose and Trivial remain, we only have to compare $s = t[\Diamond]_p^N \cdot u$ with the decomposed forms of $s = t[u]_p$ and $s = t[t[\Diamond]_p^M \cdot u]_p$, respectively. The decomposed forms of these equations are conjunctions of equations $s|_q = t|_q[u]_r$ (resp. $s|_q = t|_q[t[\Diamond]_p^M \cdot u]_r$) where q is a position of s and t and $q \cdot r = p$ and

equations of the form $s|_q = t|_q$ where $p \| q$ and q is a position of s, t. It is quite straightforward to see that E-size$(s|_q) \leq$ E-size(s). We also have:

- E-size$(t|_q[u]_r) <$ E-size$(t[\Diamond]_p^N \cdot u)$ since E-size$(t|_q[u]_r) = \max\{$E-size$(t|_q)$, E-size$(u)\}$ and both E-size(t) and E-size(u) are strictly smaller than E-size$(t[\Diamond]_p^N \cdot u)$.
- E-size$(t|_q) <$ E-size$(t[\Diamond]_p^N \cdot u)$ when $p \| q$ since E-size$(t|_q) \leq$ E-size$(t) <$ E-size$(t[\Diamond]_p^N \cdot u)$.
- $|Pos(s|_q)| < |Pos(s)|$.

Hence, we only have to consider the case of irreducible equations $s|_q = t|_q[t[\Diamond]_p^M \cdot u]_r$ such that $q \cdot r = p$. In these cases, by hypothesis on the rule (Unfold 1), the head of $s|_q$ is either a function symbol or a variable. It cannot be a variable because we assumed that there is no Variable Elimination and no Occur Check. This means that $s|_q \equiv f(s_1, \ldots, s_n)$. Then, by irreducibility of $s|_q = t|_q[t[\Diamond]_p^M \cdot u]_r$, we must have $q = p$: the equation can be written $s|_p = t[\Diamond]_p^M \cdot u$. Now the interpretation of this equation is strictly smaller than the interpretation of $s = t[\Diamond]_p^N \cdot u$.

In order to complete the proof, we only have to associate with each unification formula φ the pair consisting of the above interpretation applied on a normal form of φ with respect to R_1 and the formula itself. The pairs being ordered lexicographically and the second components being ordered by the reduction relation $\rightarrow_{R_1}^*$, we get a well-founded interpretation which is decreasing by any application of a rule. □

5. Solving Some Particular Equations

It only remains to consider equations of the form $s[t_1[\Diamond]_{q_1}^{N_1} \cdot u_1]_p = t_2[\Diamond]_{q_2}^{N_2} \cdot u_2$ where p is a prefix of q_2 (p may equal Λ, but q_1 and q_2 may not, by definition).

5.1. *Reduction to* $|q_1| = |q_2|$

The first step consists in reducing the above problem to the case $|q_1| = |q_2|$. This is done by unfolding $t_1[\Diamond]_{q_1}^{N_1}$ $|q_2|$ times and unfolding $t_2[\Diamond]_{q_2}^{N_2}$ $|q_1|$ times. Let us give an example.

Example 5.1. Consider the equation $g(f(a, g(\Diamond))^{N_1} \cdot z) = g(f(x, g(\Diamond)))^{N_2} \cdot y$. The first $\Diamond$ position $q_1 = 21$ has a length 2 and the second $\Diamond$ position $q_2 = 121$ has a length 3. We unfold the I-terms so as to have the same lengths; the above equation is equivalent to the disjunction of nine formulae. The first three formulae correspond to "small" values of N_1, N_2:

- $N_2 = 1 \wedge g(f(a, (\Diamond))^{N_1} \cdot z) = g(f(x, g(y)))$.
- $N_1 = 1 \wedge g(f(a, g(z))) = g(f(x, g(\Diamond)))^{N_2} \cdot y$.
- $N_1 = 2 \wedge g(f(a, g(f(a, g(z))))) = g(f(x, g(\Diamond)))^{N_2} \cdot y$.

The next six formulae correspond to all possible remainders of the divisions of N_1 by 3 and N_2 by 2, respectively; we gather together three iterations of $g(f(a, \Diamond))$ and two iterations of $g(f(x, g(\Diamond)))$:

- $\exists N_1', N_2' . N_2 = 2 \times N_2' \wedge N_1 = 3 \times N_1'$
  $\wedge\, g(f(a, g(f(a, g(f(a, g(\Diamond))))))^{N_1'} \cdot z) = g(f(x, g(g(f(x, g(\Diamond))))))^{N_2'} \cdot y.$
- $\exists N_1', N_2' . N_2 = 2 \times N_2' \wedge N_1 = 3 \times N_1' + 1$
  $\wedge\, g(f(a, g(f(a, g(f(a, g(\Diamond))))))^{n_1'} \cdot f(a, g(z)))$
  $= g(f(x, g(g(f(x, g(\Diamond))))))^{N_2'} \cdot y.$
- $\exists N_1', N_2' . N_2 = 2 \times N_2' \wedge N_1 = 3 \times N_1' + 2$
  $\wedge\, g(f(a, g(f(a, g(f(a, g(\Diamond))))))^{N_1'} \cdot f(a, g(f(a, g(z)))))$
  $= g(f(x, g(g(f(x, g(\Diamond))))))^{N_2'} \cdot y.$
- $\exists N_1' N_2' . N_2 = 2 \times N_2' + 1 \wedge N_1 = 3 \times N_1'$
  $\wedge\, g(f(a, g(f(a, g(f(a, g(\Diamond))))))^{N_1'} \cdot z)$
  $= g(f(x, g(g(f(x, g(\Diamond))))))^{N_2'} \cdot g(f(x, g(y))).$
- $\exists N_1', N_2' . N_2 = 2 \times N_2' + 1 \wedge N_1 = 3 \times N_1' + 1$
  $\wedge\, g(f(a, g(f(a, g(f(a, g(\Diamond))))))^{N_2'} \cdot f(a, g(z)))$
  $= g(f(x, g(g(f(x, g(\Diamond))))))^{N_2'} \cdot g(f(x, g(y))).$
- $\exists N_1', N_2' . N_2 = 2 \times N_2' + 1 \wedge N_1 = 3 \times N_1' + 2$
  $\wedge\, g(f(a, g(f(a, g(f(a, g(\Diamond))))))^{N_1'} \cdot f(a, g(f(a, g(z)))))$
  $= g(f(x, g(g(f(x, g(\Diamond))))))^{N_2'} \cdot g(f(x, g(y))).$

Now the two $\Diamond$ positions 121121 and 121212 have the same length.

(Unfold 2) $s[t_1[\Diamond]_{q_1}^{N_1} \cdot u_1]_p = t_2[\Diamond]_{q_2}^{N_2} \cdot u_2$

$$\rightarrow \bigvee_{1 \le r_1 < \alpha_2} N_1 = r_1 \wedge s[t_1[\Diamond]_{q_1}^{r_1} \cdot u_1]_p = t_2[\Diamond]_{q_2}^{N_2} \cdot u_2$$

$$\bigvee_{1 \le r_2 < \alpha_1} N_2 = r_2 \wedge s[t_1[\Diamond]_{q_1}^{N_1} \cdot u_1]_p = t_2[\Diamond]_{q_2}^{r_2} \cdot u_2$$

$$\bigvee_{\substack{0 \le r_1 < \alpha_2 \\ 0 \le r_2 < \alpha_1}} \exists M_1, M_2 . N_1 = \alpha_2 \times M_1 + r_1 \wedge N_2 = \alpha_1 \times M_2 + r_2$$

$$\wedge\, s[(t_1^{\alpha_2})^{M_1} \cdot t_1^{r_1} \cdot u_1]_p = (t_2^{\alpha_1})^{M_2} \cdot t_2^{r_2} \cdot u_2$$

If $|q_1| \neq |q_2|$, R_2 cannot be applied and p is a prefix of q_2. d is the gcd of $|q_1|$ and $|q_2|$, $\alpha_1 = |q_1|/d$, $\alpha_2 = |q_2|/d$, and $m = \alpha_1 \times \alpha_2 \times d$ is the lcm of $|q_1|$ and $|q_2|$.

Lemma 5.2. (*Unfold* 2) *is correct.*

Proof. Concerning correctness, we only have to notice that all integers $n_1, n_2 \geq 1$ (i.e., all possible assignments to N_1, N_2) have to satisfy $n_1 \in \{1, \ldots, \alpha_2 - 1\}$ or $n_2 \in \{1, \ldots, \alpha_1 - 1\}$ or else $n_1 = \alpha_2 \times m_1 + r_1$ and $n_2 = \alpha_1 \times m_2 + r_2$ for some $m_1, m_2 \geq 1$ and some $r_1 \in \{0, \ldots, \alpha_2 - 1\}$, $r_2 \in \{0, \ldots, \alpha_1 - 1\}$. □

$R_2 \cup \{(Unfold\ 2)\}$ is also terminating, but we prove this later for all rules together.

5.2. *Ensuring Some Prefix Conditions on the Positions*

The second step consists in applying rules similar to (Unfold 1) in order to eliminate uncomparable positions. More precisely, we want to ensure that $p \leq_{\text{pref}} q_2 \leq_{\text{pref}} p \cdot q_1 \leq_{\text{pref}} q_2 \cdot q_2$ in equations $s[t_1[\Diamond]_{q_1}^{N_1} \cdot u_1]_p = t_2[\Diamond]_{q_2}^{N_2} \cdot u_2$. (The first inequality is already known from (Unfold 1).)

Example 5.3. Consider the equation

$$f(a, f(f(x, \Diamond).\, y)^{N_1} \cdot z) = f(a, f(x', \Diamond))^{N_2} \cdot y'.$$

We have $p = 2$, $q_1 = 12$, and $q_2 = 22$. None of the rules which have been shown up to now can be applied. However, unfolding once on both sides, we get:

$$\begin{aligned} & N_1 = 1 \wedge f(a, f(f(x, z), y)) = f(a, f(x', \Diamond))^{N_2} \cdot y' \\ \vee\ & N_2 = 1 \wedge f(a, f(f(x, \Diamond), y)^{N_1} \cdot z) = f(a, f(x', y')) \\ \vee\ & \exists M_1, M_2 .\, N_1 = M_1 + 1 \wedge N_2 = M_2 + 1 \\ & \wedge f(a, f(f(x, f(f(x, \Diamond), y)^{M_1} \cdot z), y)) \\ & = f(a, f(x', f(a, f(x', \Diamond))^{M_2} \cdot y')). \end{aligned}$$

By decomposition the last equation reduces the exponent sizes of the members, because q_2 is not a prefix of $p \cdot q_1$; we get

$$x' = f(x, f(f(x, \Diamond), y))^{M_1} \cdot z \wedge y = f(a, f(x', \Diamond))^{M_2} \cdot y'.$$

$$\begin{aligned} \text{(Unfold 3)}\quad & s[t_1[\Diamond]_{q_1}^{N_1} \cdot u_1]_p = t_2[\Diamond]_{q_2}^{N_2} \cdot u_2 \\ & \rightarrow (N_2 = 1 \wedge t_2[u_2]_{q_2} = s[t_1^{N_1} \cdot u_1]_p) \\ & \quad \vee (N_1 = 1 \wedge s[t_1[u_1]_{q_1}]_p = t_2^{N_2} \cdot u_2) \\ & \quad \vee (N_2 = 2 \wedge t_2[t_2[u_2]_{q_2}]_{q_2} = s[t_1^{N_1} \cdot u_1]_p) \\ & \quad \vee (\exists M_1, M_2 .\, N_1 = M_1 + 1 \wedge N_2 = M_2 + 2 \\ & \qquad \wedge s[t_1[t_1^{M_1} \cdot u_1]_{q_1}]_p = t_2[t_2[t_2^{M_2} \cdot u_2]_{q_2}]_{q_2}). \end{aligned}$$

Lemma 5.4. (*Unfold* 3) *is correct.*

Proof. It is sufficient to consider that any solution either assigns N_2 to 1 or 2, or assigns N_1 to 1, or else assigns N_1 to $n_1 \geq 2$ and N_2 to $n_2 \geq 3$. □

5.3. *The Crux Decomposition Rule*

The key property is given in the following lemma. It shows that all equations that remain to be considered do have some commutation properties on their paths (actually they are of the form depicted in Figure 3).

Lemma 5.5. *If* $s[t_1[\Diamond]_{q_1}^{N_1} \cdot u_1]_p = t_2[\Diamond]_{q_2}^{N_2} \cdot u_2$ *is irreducible by* (*Unfold* 1), (*Unfold* 2), *and* (*Unfold* 3), *then there is a* q' *such that* $q_2 = p \cdot q'$ *and* $q_1 = q' \cdot p$.

Proof. By irreducibility with respect to (Unfold i) ($i = 1, 2, 3$), we know that

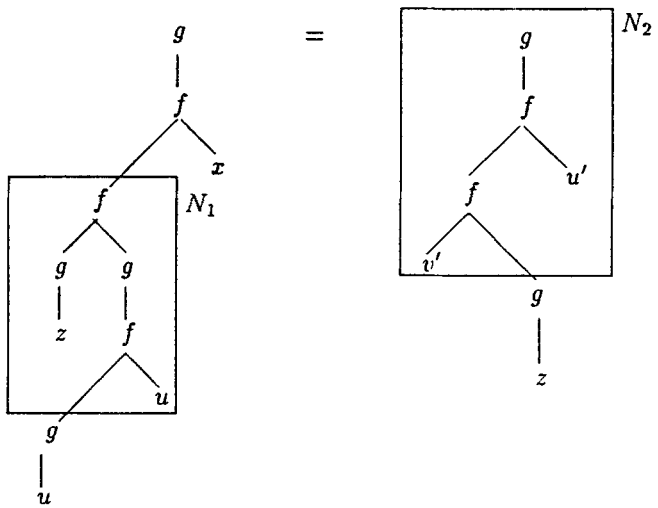
$$p \leq_{\text{pref}} q_2 \leq_{\text{pref}} p \cdot q_1 \leq_{\text{pref}} q_2 \cdot q_2.$$

We also know that $|q_1| = |q_2|$. Let $q_2 = p \cdot q'$ and $p \cdot q_1 = q_2 \cdot q''$. Then $p \cdot q_1 = p \cdot q' \cdot q''$, hence $q_1 = q' \cdot q''$. From $q_2 = p \cdot q'$, $q_1 = q' \cdot q''$, and $|q_1| = |q_2|$, we derive that $|p| = |q''|$.

Now $p \cdot q_1$ is a prefix of $q_2 \cdot q_2$, Hence, for some r, $q_2 \cdot q_2 = p \cdot q_1 \cdot r$. Which means that $p \cdot q' = q'' \cdot r$. Now using $|p| = |q''|$ we derive $p = q''$. It follows that $q_2 = p \cdot q'$ and $q_1 = q' \cdot p$. □

In order to have an intuition of the next decomposition rule, look at Figure 3: in such equations, we must have $s = v_2 = w_1$ and $v_1 = w_2$. Then, rearranging the parentheses, we lift up the N_1 exponent and guess whether $N_1 \geq N_2$ or $N_2 \geq N_1$. Assume, for example, $N_1 \geq N_2$, then we can remove $t_2[\Diamond]_{q_2}^{N_2}$ on both sides.

Example 5.6. Consider the equation $g(f(f(g(z), g(f(\Diamond, u)))^{N_1} \cdot g(u), x)) = g(f(f(v', \Diamond), u'))^{N_2} \cdot g(z)$ which can be depicted as



We have $s \equiv g(f(a, x))$, $v_1 \equiv f(g(z), a)$, $w_1 \equiv g(f(a, u))$, $v_2 \equiv g(f(a, u'))$, $w_2 \equiv f(v', a)$, $p = 11$, and $q' = 2$. The equation can be decomposed into the following problems (the a can be replaced by any term, we only replace the $\Diamond$ in

order to keep terms in T):

$$
\begin{aligned}
& g(f(a, x)) = g(f(a, u')) && \text{(i.e., } s = v_2\text{)} \\
\wedge\ & g(f(a, x)) = g(f(a, u)) && \text{(i.e., } s = w_1\text{)} \\
\wedge\ & f(g(z), a) = f(v', a) && \text{(i.e., } v_1 = w_2\text{)} \\
\wedge\ & ((N_1 = N_2 \wedge g(f(g(u), u)) = g(z)) \\
& \quad \vee (\exists M_1 . N_1 = N_2 + M_1 \wedge g(f(f(g(z), g(f(\Diamond, u)))^{M_1} \cdot g(u), x)) = g(z) \\
& \quad \vee (\exists M_2 . N_2 = N_1 + M_2 \wedge g(f(g(u), u)) = g(f(f(f(v', \Diamond), u'))^{M_2} \cdot g(z))).
\end{aligned}
$$

More generally, we get the following rule (where the righthand side has actually to be put in disjunctive normal form in order to get a unification formula):

(Decompose 2)

$$
\begin{aligned}
& s[(v_1[w_1[\Diamond]_p]_{q'})^{N_1} \cdot u_1]_p = (v_2[w_2[\Diamond]_{q'}]_p)^{N_2} \cdot u_2 \rightarrow s[a]_p = w_1[a]_p \\
\wedge\ & s[a]_p = v_2[a]_p \wedge v_1[a]_{q'} = w_2[a]_{q'} \\
\wedge\ & ((N_1 = N_2 \wedge s[u_1]_p = u_2) \\
& \quad \vee (\exists M_1 . N_1 = N_2 + M_1 \wedge s[(v_1[w_1[\Diamond]_p]_{q'})^{M_1} \cdot u_1]_p = u_2) \\
& \quad \vee (\exists M_2 . N_2 = N_1 + M_2 \wedge w_1[u_1]_p = (v_2[w_2[\Diamond]_{q'}]_p)^{M_2} \cdot u_2)).
\end{aligned}
$$

Lemma 5.7. *The rule (Decompose 2) is correct (a being any constant).*

Proof. This is again quite easy: the first three equations are obtained by considering the top part of both sides for an arbitrary assignment (remember that N_1 and N_2 must be assigned to integers larger or equal to 1). The next three disjunctions are obtained by guessing how N_1 and N_2 compare. Assume, for example, that N_1 is assigned to a strictly larger integer than N_2. Writing N_1 as $N_2 + M_1$, and using the equations $s = v_2$ and $v_1 = w_2$, the equation becomes

$$v_2[(w_2[v_2[\Diamond]_p]_{q'})^{N_2} \cdot (v_1[w_1[\Diamond]_p]_{q'})^{M_1} \cdot u_1]_p = (v_2[w_2[\Diamond]_{q'}]_p)^{N_2} \cdot u_2,$$

which becomes, after reorganizing the terms and simplifying by $(v_2[w_2[\Diamond]_{q'}]_p)^{N_2}$ on both sides,

$$v_2[(v_1[w_1[\Diamond]_p]_{q'})^{M_1} \cdot u_1]_p = u_2.$$

Replacing again v_2 with s, we get the corresponding equation. □

6. Unification of Terms with Integer Exponents

Now we have the whole set of rules. Let us first prove termination. Then solving the diophantine equations part leads to solved forms.

6.1. *Termination*

Lemma 6.1. *The system R_3 which consists of applying R_2 as long as possible and (Unfold 2), (Unfold 3), and (Decompose 2) on irreducible problems with respect to R_2 is correct and terminating.*

Proof. Correctness has been shown in Lemmas 4.4, 5.2, 5.4, and 5.7.

Regarding termination, we use the same interpretation as in Lemma 4.4, except for the definition of $I(s = t)$ which is modified as follows:

$$I(s = t) = (\{\text{E-size}(s),\ \text{E-size}(t)\},\ \{|Pos(s)|,\ |Pos(t)|\},\ \text{status}(s = t)),$$

where status$(s = t)$ is equal to 1 if (Unfold 2) can be applied on $s = t$. Otherwise, status $(s = t) = 0$.

As in the proof of Lemma 4.4, we only have to prove that, for each rule, each equation in the pure part of a normal form with respect to R_1 of the right-hand side is strictly smaller than the left-hand side.

In Table 1 we summarize, for each rule, and each equation in a normal form with respect to R_1 of the right-hand side of this rule, how it compares with the left-hand side of the rule. We assume here (as in the proof of Lemma 4.4) that there is no occurrence of a Variable Elimination rule along the simplification. (Because, otherwise, the interpretation is obviously decreasing.) A pair $\{<, =\}$ on the line $s = t$ for the interpretation i means, for example, that $s = t$ has one member for which i is strictly decreasing and one member for which i is constant. In some places we use the notation $u|_q$ without any further explanation. In such cases q is assumed to be a position of u, such that the displayed equation is irreducible with respect to (Decompose 1). Finally, most of the results reported in the table are quite easy. How look the irreducible forms of equations in the right-hand side of (Unfold 1) has been investigated in the proof of Lemma 4.4. We only give further explanation for the five results which are marked (1)–(5) in the table:

(1) It should be noted that, in these situations, the number of occurrences of integer variables may increase. However, E-size is still strictly decreasing. Indeed,

$$\text{E-size}(s[t_1[\Diamond]_{q_1}^{r_1} \cdot u_1]_p) = \max\{\text{E-size}(s),\ \text{E-size}(t_1),\ \text{E-size}(u_1)\},$$

since, by definition, for all terms u, E-size$(u) = \max\{[\text{E-size}(u|_q)|q \in Pos(u)\}$ and $p \cdot q_1^{r_1} \in Pos(s[t_1[\Diamond]_{q_1}^{r_1} \cdot u_1]_p)$. Moreover, as already noticed in the proof of Lemma 4.4, for every term $t^N \cdot u$, E-size(t) and E-size(u) are both strictly smaller than E-size$(t^N \cdot u)$.

(2) As above, E-size$(t_1^{\alpha_2}) =$ E-size(t_1) and E-size$(t_1^{r_1} \cdot u_1) = \max\{\text{E-size}(t_1),$ E-size$(u_1)\}$. Now, assume that E-size$(t_1) = (n_1, n_2)$ and E-size$(u_1) = (m_1, m_2)$. There are two situations: either $N_1 \geq M_1$ or $N_1 < M_1$. In the first case E-size$(t_1^{r_1} \cdot u_1) \leq (n_1, \max\{m_1, m_2\})$, hence

$$\text{E-size}((t_1^{\alpha_2})^{M_1} \cdot t_1^{r_1} \cdot u_1) = (n_1 + 1,\ 0) = \text{E-size}(t_1^{N_1} \cdot u_1).$$

In the second case E-size$(t_1^{r_1} \cdot u_1) = (m_1, m_2)$, hence

$$\text{E-size}((t_1^{\alpha_2})^{M_1} \cdot t_1^{r_1} \cdot u_1) = (m_1,\ m_2 + 1) = \text{E-size}(t_1^{N_1} \cdot u_1).$$

Table 1. Influence of the rules on the interpretation

		E-size	Size	Status
(Unfold 1)	$s\|_q = t[u]_p\|_q$	$\{\leq, <\}$		
	$s\|_q = t\|_q$	$\{\leq, <\}$		
	$s\|_p = t[\Diamond]_p^M \cdot u$	$\{\leq, =\}$	$\{<, =\}$	
(Unfold 2)	$s[t_1[\Diamond]_{q_1}^{r_1} \cdot u_1]_p = t_2[\Diamond]_{q_2}^{N_2} \cdot u_2$	$\{<, =\}$ (1)		
	$s[t_1[\Diamond]_{q_1}^{N_1} \cdot u_1]_p\|_q = (t_2[\Diamond]_{q_2}^{r_2} \cdot u_2)\|q$	$\{\leq, <\}$ (1)		
	$s[(t_1^{\alpha_2})^{M_1} \cdot t_1^{r_1} \cdot u_1]_p = (t_2^{\alpha_1})^{M_2} \cdot t_2^{r_2} \cdot u_2$	$\{=, =\}$ (2)	$\{=, =\}$ (3)	$<$ (4)
(Unfold 3)	$t_2[u_2]_{q2}\|_q = s[t_1^{N_1} \cdot u_1]_p\|q$	$\{<, \leq\}$		
	$s[t_1[u_1]_{q_1}]_p = t_2^{N_2} \cdot u_2$	$\{<, =\}$		
	$t_2[t_2[u_2]_{q2}]_{q2}\|_q = s[t_1^{N_1} \cdot u_1]_p\|_q$	$\{<, \leq\}$		
	$s[t_1[t_1^{M_1} \cdot u_1]_{q1}]_p\|_q = t_2[t_2[t_2^{M_2} \cdot u_2]_{q2}]_{q2}\|_q$	$\{<, \leq\}$ (5)		
(Decompose 2)	$s\|_q = w_1\|_q$	$\{<, <\}$		
	$s\|q = v_2\|_q$	$\{<, <\}$		
	$v_1\|_q = w_2\|_q$	$\{<, <\}$		
	$s[u_1]_p\|_q = u_2\|_q$	$\{<, <\}$		
	$s[(v_1[w_1[\Diamond]_p\|_{q'})^{M_1} \cdot u_1]_p\|_q = u_2\|_q$	$\{\leq, <\}$		
	$w_1[u_1]_p\|_q = ((v_2[w_2[\Diamond]_{q'}]_p)^{M_2} \cdot u_2)\|_q$	$\{<, \leq\}$		

(3) By definition of Pos, $|Pos(s[(t_1^{\alpha_2})^{M_1} \cdot t_1^{r_1} \cdot u_1]_p)| = |Pos(s)| = |Pos(s[t_1^{N_1} \cdot u_1]_p)|$.

(4) By definition, the equation on which (Unfold 2) is applied has a status 1. We have to check that (Unfold 2) cannot be applied again on the equation we consider. In which case the status is 0 and hence strictly decreasing. Note first that p is fixed (because it must be a prefix of q_2). Then note that $q_1^{\alpha_2}$ has the same length as $q_2^{\alpha_1}$, which prevents the application of (Unfold 2).

(5) Several cases have to be considered:

- If $q \| p \cdot q_1$, then

$$\text{E-size}(s[t_1[t_1^{M_1} \cdot u_1]_{q_1}]_p|_q) = \text{E-size}(s[t_1]_p|_q) < \text{E-size}(s[t_1[\Diamond]_{q_1}^{N_1} \cdot u_1]_p)$$

  and the result follows.
- If $q \| q_2 \cdot q_2$, we get a similar result for the other member. (Note that, in any case, E-size is never increasing.)
- Now assume that q is a prefix of both $p \cdot q_1$ and $q_2 \cdot q_2$. By irreducibility with respect to (Decompose 1), we must have $q = p \cdot q_1$ or $q = q_2 \cdot q_2$. Which means that either $p \cdot q_1$ is a prefix of $q_2 \cdot q_2$ or else $q_2 \cdot q_2$ is a prefix of $p \cdot q_1$. However, both situations are prevented by the condition

on the application of (Unfold 3). (See the rule.) Hence, this last case cannot occur.

As in the proof of Lemma 4.4, these decreasing properties complete the termination proof. □

6.2. *Irreducible Formulae*

Lemma 6.2. *If a formula is irreducible with respect to R_3, then it is a disjunction of conjunctions of the form*

$$\exists \vec{n} \,.\, \varphi \wedge x_1 = t_1 \wedge \cdots \wedge x_n = t_n \,,$$

where φ is a conjunction of linear diophantine equations, $\vec{n}$ is a set of integer variables, and $x_1, \ldots, x_n$ are ordinary variables which are solved in the conjunction.

Proof. We only have to check that each unification formula which is not in the form of the lemma can be reduced using one of the rules in R_3. As already noticed at the beginning of Section 5, every irreducible unsolved equation (with respect to R_2) must be of the form $s[t_1[\Diamond]_{q_1}^{N_1} \cdot u_1]_p = t_2[\Diamond]_{q_2}^{N_2}$ with $p \leq_{\text{pref}} q_2$. By irreducibility with respect to (Unfold 2), q_1 and q_2 must have the same size. By irreducibility with respect to (Unfold 3), we know that $q_2 \leq_{\text{pref}} p \cdot q_1 \leq_{\text{pref}} q_2 \cdot q_2$, and, in this case, by Lemma 5.5, we can apply (Decompose 2). □

6.3. *The Last Step*

Now we solve the linear diophantine equations parts. This means that the diophantine parts are replaced with finite disjunctions of systems of the form

$$\exists \vec{n} \,.\, N_1 = E_1 \wedge \cdots \wedge N_k = E_k,$$

where $E_1, \ldots, E_k$ are linear expressions and $N_1, \ldots, N_k$ are integer variables occurring only once in the problem.

The values of $N_1, \ldots, N_k$ can be replaced in the corresponding pure part, leading to equivalent solved forms. (t^E is expanded according to the rule $t^{N+M} \to t^N \cdot t^M$; then replacing integer variables with linear expressions yields a unification formula.)

Theorem 6.3. *Unification of I-terms is decidable and finitary: there is a correct and terminating algorithm which computes a finite set of solved forms for $s = t$.*

6.4. *Minimality*

It should be noticed that the solved forms of our unification algorithms do not contain redundancies: if $d_1 \vee d_2$ is a solved form of an equation $s = t$, then d_1 and d_2 do not share any solution. This can be checked on each rule: each disjunction introduced by a rule splits the solutions in disjoint sets. However, we cannot claim

that the solved form are *minimal*, because there are solved forms $d_1 \vee d_2$, where d_1, d_2 are conjunctions of equations, that can be "folded" into a single conjunction of equations. For example, consider the equation $f(x) = f(f(\Diamond))^N \cdot f(a)$. By (Unfold 1) and further reductions, we get

$$(N = 1 \wedge x = f(f(a))) \vee \exists N' . N = N' + 1 \wedge x = f(f(f(\Diamond))^{N'} \cdot f(a)).$$

Which is obviously not minimal since $x = f(f(\Diamond))^N \cdot a$ is equivalent to the original problem.

7. Extensions of the Syntax

We may allow a number of new syntactic constructions while keeping the result of Theorem 6.3. Let T_n be the set of terms with n "holes" ($T' = T_0$) defined as the least sets which satisfy

$$\begin{cases} x \in T_0 & \Leftarrow & x \in X \quad \text{or} \quad x \in F, \quad a(x) = 0, \\ f(s_1, \ldots, s_n) \in T_{m_1 + \cdots + m_n} & \Leftarrow & s_1 \in T_{m_1}, \ \ldots, \ s_n \in T_{m_n}, \quad 1 \leq n, \\ s^{N_1 \cdots N_k}_{p_1 \cdots p_k} \in T_k & \Leftarrow & s \in T_0, \ 1 \leq k, \ p_1, \ \ldots, \ p_k \in Pos(s) - \{\Lambda\}, \\ & & \forall i \neq j \, . \, p_i \| p_j, \ N_1, \ \ldots, \ N_k \in V_N, \\ s \cdot t \in T_{n+k-1} & \Leftarrow & s \in T_n, \quad 1 \leq n, \quad t \in T_k. \end{cases}$$

Positions are defined as before; in particular, $Pos(s \cdot t) = \{\Lambda\}$. We do not use the special symbol $\Diamond$ in this syntax for the reason shown in the following example.

Example 7.1. Let $s \equiv (f(a, b)^{N_1, N_2}_{1,2} \cdot c) \cdot d$. Assume that N_1 and N_2 are both assigned to 2. Then, according to the semantics defined below, $s\sigma \equiv f(f(c, b), f(a, d))$. Now, using the special place holder $\Diamond$, what should $(f(\Diamond, \Diamond)^{2,2} \cdot c) \cdot d$ be? There are several possibilities for unfolding $f(\Diamond, \Diamond)^{2,2}_{1,2}$, but all of them lead to terms with more than two occurrences of $\Diamond$. (This could be $f(f(\Diamond, \Diamond), f(\Diamond, \Diamond))$ or $f(f(\Diamond, \Diamond), f(f(\Diamond, \Diamond), ; \Diamond))$.) Hence, we would have to decide which occurrences of $\Diamond$ have to be replaced with c and which have to be replaced with d. This leads to another interpretation of the terms which is more complicated. Even if we do, the expressive power is not increased. In order to be able to express other sets of terms we have either to allow s to be $u \cdot v$ (which is investigated in the next section) or, e.g., unfold simultaneously with respect to all holes (as in [6]).

Now, given an assignment to the integer variables of $s \in T_m$, $s\sigma$ is defined by

$$x\sigma \stackrel{\text{def}}{=} x \qquad \text{if} \quad x \in X \quad \text{or} \quad x \in F,$$

$$f(s_1, \ldots, s_n)\sigma \stackrel{\text{def}}{=} f(s_1\sigma, \ldots, s_n\sigma),$$

$$s^{N_1, \ldots, N_k}_{p_1, \ldots, p_k}\sigma \stackrel{\text{def}}{=} s^{N_1, \ldots, N_{k-1}}_{p_1, \ldots, p_{k-1}}\sigma \underbrace{[s\sigma[\cdots[s\sigma[\Diamond]_{p_k}]\cdots]_{p_k})]_{p_k}}_{N_k\sigma - 1},$$

$$(s \cdot t)\sigma \stackrel{\text{def}}{=} s\sigma[t\sigma]_p \qquad \text{if } p \text{ is the smallest position of } \Diamond \text{ in } s\sigma.$$

Assignments to 0 are excluded (because this would greatly complicate the definitions). Positions are compared lexicographically in the above definition.

The above additional constructions actually do not increase the expressive power of terms with exponents. That is what is shown in the next proposition. $t \in T_n$ is said to be *equivalent* to $t' \in T_n$ if $IV(t) = IV(t')$ and, for every assignment to these integer variables, $t\sigma \equiv t'\sigma$. A term $s \in T$ is *weakly equivalent* to a term $t \in T_0$ if $IV(s) = IV(t)$ and:

- For every assignment σ of nonnull integers to these variables, there is an assignment θ such that $s\theta \equiv t\sigma$.
- For every assignment θ to these variables, there is an assignment σ of nonnull integers such that $s\theta \equiv t\sigma$.

Proposition 7.2. *For every $t' \in T_0$ there is a weakly equivalent $t \in T$.*

Sketch of the Proof. We show the proposition by structural induction on t':

- If $t' \equiv f(\vec{s'})$ with $\vec{s'} \in T_0^{a(f)}$, Then it is sufficient to choose $t \equiv f(\vec{s})$ where each term $s_i \in \vec{s}$ satisfies, for all σ, $s_i\sigma \equiv s_i'\sigma$. ($\vec{s}$ exists by the induction hypothesis.)
- If $t' \in X$, then let $t \equiv t'$.
- If we are not in one of the two above cases, then $t' \equiv s \cdot u$ with $s \in T_1$ and $u \in T_0$. Note that $\cdot$ is "associative" in the following sense: when $s \in T_n$, $t \in T_m$, $u \in T_k$ with $n, m \leq 1$, then $(s \cdot t) \cdot u$ and $s \cdot (t \cdot u)$ are equivalent. Hence, we may assume that t' has the form $(s \cdot u_1) \cdot \cdots \cdot u_k$ where $u_1, \ldots, u_k \in T_0$ and $s \in T_{k+1}$ cannot be decomposed as $s_1 \cdot s_2$. There are still two possible cases:

  —$s \equiv f(s_1, \ldots, s_n)$. In this case t' is equivalent to a term $f(v_1, \ldots, v_n)$, where $v_i \equiv s_i \cdot u_{m_i} \cdot \cdots \cdot u_{m_{i+1}-1}$ and $m_1 = 1$, $m_{n+1} = k + 1$. Therefore, this case reduces to the case $t' \equiv f(\vec{s'})$ which has already been considered.

  —$s \equiv (s_0)^{N_1, \ldots, N_m}_{p_1, \ldots, p_m}$ where $s_0 \equiv f(s_1, \ldots, s_n)$. Then we have necessarily $k = m$. If $m = 1$, then we are done: let $v_0 \in T$ be equivalent to s_0 and let $v_i \in T$ be equivalent to u_1. Then $v_0[\Diamond]^{N_1}_{p_1} \cdot v_1$ is equivalent to $(s_0)^{N_1}_{p_1} \cdot u_1$. Assume that $p_1, \ldots, p_m$ are listed in increasing order. This is possible since $s^{N,M}_{p,q}$ is equivalent to $s^{M,N}_{q,p}$. We claim that $(s^{N_1, \ldots, N_k}_{p_1, \ldots, p_k} \cdot u_1) \cdot \cdots \cdot u_k$ is weakly equivalent to the term $(s[s^{M_1}_{p_1} \cdot u_1]_{p_1}) \ldots [s^{M_k}_{p_k} \cdot u_k]_{p_k}$ in which every occurrence of N_i has been replaced with M_i. For, consider an assignment σ to the integer variables of the terms. We have

$$(s^{N_1, \ldots, N_k}_{p_1, \ldots, p_k} \cdot u_1)\sigma \equiv s^{N_2, \ldots, N_k}_{p_2, \ldots, p_k} \sigma \underbrace{[s\sigma[\cdots [s\sigma[u_1]_{p_1}] \cdots]_{p_1}]}_{N_1\sigma - 1} p_1,$$

since $p <_{\text{lex}} q$ implies $p^n <_{\text{lex}} q^n$ for all positive integers n. Moreover, since $p_1, \ldots, p_k$ are disjoint positions, $p_1^{N_1\sigma}, \ldots, p_k^{N_k\sigma}$ are also disjoint positions.

Hence

$$(s^{N_1,\ldots,N_k}_{p_1,\ldots,p_k} \cdot u_1 \cdot \cdots \cdot u_k)\sigma \equiv (s^{N_2,\ldots,N_k}_{p_2,\ldots,p_k} \cdot u_2 \cdot \cdots \cdot u_k)\sigma \underbrace{[s\sigma[\cdots[s\sigma[u_1]_{p_1}]\cdots]_{p_1}]}_{N_1\sigma-1} p_1,$$

which shows the desired equivalence, by induction on k: it is sufficient to choose $M_i\theta = N_i\sigma - 1$, whatever substitution is considered first. □

8. An Unsound Generalization

Now let T_U be defined by

$$\begin{cases} f(\vec{s}) \in T_U & \Leftarrow & \vec{s} \in T_U^{a(f)}, \\ x \in T_U & \Leftarrow & x \in X, \\ \Diamond \in T_1, & & \\ f(\vec{s_1}, s, \vec{s_2}) \in T_1 & \Leftarrow & (\vec{s_1}, s, \vec{s_2}) \in T_U^{n_1} \times T_1 \times T_U^{n_2},\ n_1 + n_2 + 1 = a(f), \\ s \cdot t \in T_1 & \Leftarrow & s, t \in T_1 \\ s^N \in T_1 & \Leftarrow & s \in T_1, \quad N \in V_N, \\ s \cdot t \in T_U & \Leftarrow & s \in T_1, \quad t \in T_U, \end{cases}$$

and *Pos* is defined as previously. This definition is close to the definition of T. The main difference is that we now allow a construction such as $(s^N \cdot s')^M$ which enables us to encode the multiplication of integer variables and hence diophantine equations.

Proposition 8.1. *Unification of terms in T_U is undecidable.*

Proof. Encoding diophantine equations is straightforward. We only need one unary symbol f and one constant symbol a in the signature. $code(e)$ is defined on every monomial by:

- $code(1) \stackrel{\text{def}}{=} f(\Diamond)$.
- $code(N) \stackrel{\text{def}}{=} f(\Diamond)^N$ for a single variable N.
- $code(e \times N) \stackrel{\text{def}}{=} code(e)^N$ for every nonempty product of variables e and every variable N.

This coding extends to every integer polynomial, setting $code(e + e') \stackrel{\text{def}}{=} code(e) \cdot code(e')$. Then the integer polynomial equation $P = Q$ has a solution iff $code(P) \cdot a = code(Q) \cdot a$ has a solution. Indeed, f can be seen as the successor function: $code(P)$ can be seen as $f(\Diamond)^P$. (Although this is not allowed by the above syntax.)

For example, $x^2 + 2 \times x \times y + 1$ is encoded by $(f(\Diamond)^x)^x \cdot (f(\Diamond)^x)^y \cdot (f(\Diamond)^x)^y \cdot f(a)$. □

9. Further Remarks

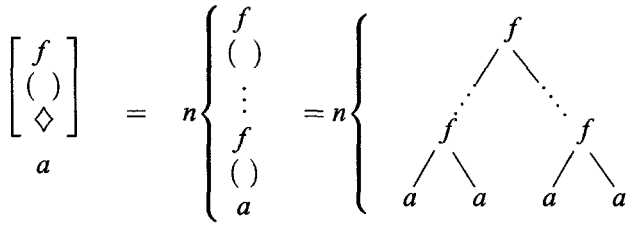
9.1. *Complexity Issues*

We do not know the exact complexity of the problem of unifying two I-terms. However, as in the previous section, it is easy to see that if we restrict the alphabet to a single unary function symbol and a constant, then the problem of unifying two I-terms is equivalent to solving an arbitrary linear diophantine equation. Hence, for arbitrary alphabets, the problem is at least as hard as solving systems of linear diophantine equations,[2] which may have an exponential number of minimal solutions. In the same way, checking the existance of a solution is NP-hard.

On the other hand, if we only consider one disjunct when applying a rule (i.e., choosing "carefully" which disjunct will lead to a solution), and if we use an appropriate representation of terms (which prevents the exponential behavior of variable elimination), we conjecture that our algorithms decide the existence of a solution in NP-time. This would mean that the unifiability of I-terms is NP-complete.

9.2. *Related Work*

Since the first version of this paper, we were informed of the work of Salzer, which has been done independently from ours [6]. He uses a quite different formalism, but he shows essentially the same results as ours, except that his formalism is slightly more powerful since he can express, for example, the set of all complete finite binary trees with internal nodes labeled with f. Indeed, his syntax allows for multiple holes in the terms, in which case, the semantics is different from what we considered in Section 7. In order to express it shortly, you can assume that the holes are shared, leading to a DAG on which we apply the semantics defined in Section 2. For example, the instances of $f(\diamondsuit, \diamondsuit)^N \cdot a$ are all complete binary trees:



We believe that there is no important difference if we use DAGs instead of terms. Hence, the technique of our paper should apply to this interpretation as well.

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[2]Note that this has nothing to do with the linear diophantine equations which are solved in our unification procedure. Indeed, the diophantine equations that we have to consider in our procedure are not arbitrary linear diophantine systems, and their solution might be simpler to solve than the general case.

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