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Probability Theory and Related Fields

Existence and uniqueness of semimartingale reflecting Brownian motions in an orthant*

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Summary. This work is concerned with the existence and uniqueness of a class of semimartingale reflecting Brownian motions which live in the non-negative orthant of $\mathbb{R}^d$. Loosely speaking, such a process has a semimartingale decomposition such that in the interior of the orthant the process behaves like a Brownian motion with a constant drift and covariance matrix, and at each of the $(d-1)$-dimensional faces that form the boundary of the orthant, the bounded variation part of the process increases in a given direction (constant for any particular face) so as to confine the process to the orthant. For historical reasons, this "pushing" at the boundary is called instantaneous reflection. In 1988, Reiman and Williams proved that a necessary condition for the existence of such a semimartingale reflecting Brownian motion (SRBM) is that the reflection matrix formed by the directions of reflection be completely-$\mathcal{S}$. In this work we prove that condition is sufficient for the existence of an SRBM and that the SRBM is unique in law. It follows from the uniqueness that an SRBM defines a strong Markov process. Our results have potential application to the study of diffusions arising as approximations to *multi-class* queueing networks.

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1 Introduction

This work is concerned with the existence and uniqueness of a class of semimartingale reflecting Brownian motions which live in the non-negative orthant of $\mathbb{R}^d$. For a precise description of these processes, let $S = \{x \in \mathbb{R}^d : x_i \geqq 0, i = 1, \ldots, d\}$, θ be

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a vector in $\mathbb{R}^d$, Γ be a $d \times d$ non-degenerate covariance matrix (symmetric and positive definite), and R be a $d \times d$ matrix. A triple $(\Omega, \mathscr{F}, \{\mathscr{F}_t\})$ will be called a *filtered space* if Ω is a set, $\mathscr{F}$ is a σ-field of subsets of Ω, and $\{\mathscr{F}_t\} \equiv \{\mathscr{F}_t, t \geqq 0\}$ is an increasing family of sub-σ-fields of $\mathscr{F}$, i.e., a filtration. If, in addition, P is a probability measure on $(\Omega, \mathscr{F})$, then $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P)$ is called a filtered probability space.

Definition 1.1 For $x \in \mathbf{S}$, a *semimartingale reflecting Brownian motion* (abbreviated as SRBM) *associated with the data* $(\mathbf{S}, \theta, \Gamma, R)$ *that starts from* x is a continuous, $\{\mathscr{F}_t\}$-adapted, d-dimensional process Z defined on some filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P_x)$ such that under P_x,

$$Z(t) = X(t) + RY(t) \in \mathbf{S} \quad \text{for all } t \geqq 0 \,, \tag{1.1}$$

where

(i) X is a d-dimensional Brownian motion with drift vector θ and covariance matrix Γ such that $\{X(t) - \theta t, \mathscr{F}_t, t \geqq 0\}$ is a martingale and $X(0) = x$ P_x-a.s.,

(ii) Y is an $\{\mathscr{F}_t\}$-adapted, d-dimensional process such that P_x-a.s. for each $i \in \{1, \ldots, d\}$, the i^{th} component Y_i of Y satisfies

(a) $Y_i(0) = 0$,

(b) Y_i is continuous and non-decreasing,

(c) Y_i can increase only when Z is on the face $F_i \equiv \{x \in \mathbf{S} : x_i = 0\}$, i.e., $\int_0^t 1_{\mathbf{S} \setminus F_i}(Z(s))\, dY_i(s) = 0$ for all $t \geqq 0$.

An SRBM associated with the data $(\mathbf{S}, \theta, \Gamma, R)$ is a continuous, $\{\mathscr{F}_t\}$-adapted, d-dimensional process Z together with a family of probability measures $\{P_x, x \in \mathbf{S}\}$ defined on some filtered space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\})$ such that for each $x \in \mathbf{S}$, under P_x, (1.1) and (i)–(ii) above hold; that is, on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P_x)$, Z is an SRBM associated with $(\mathbf{S}, \theta, \Gamma, R)$ that starts from x.

Remarks. 1. Here we have made a slight modification to the definition given in [23] in that Y need only be P_x-a.s. continuous here. This does not affect the applicability of the results in [23]. Our reason for allowing this flexibility is that we may be dealing with uncompleted probability spaces and to require all paths of Y to be continuous seems unnecessarily restrictive. Note that for a fixed x, one could always complete the filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P_x)$ and then modify X, Y on a P_x-null set to make them continuous everywhere. See Remark 2 below for another alternative.

2. The notion of an SRBM with a family of probability measures (one for each possible starting point) is formulated because we want this to generate a strong Markov process. A priori, it may seem unnecessarily restrictive to require the family of probability measures to all be defined on the same filtered space and to use the same Z for each x. However, whenever there is an SRBM starting from x for each $x \in \mathbf{S}$, then there is an SRBM where X, Y and Z are defined on the same filtered space, are continuous everywhere, and are defined independently of the $\{P_x, x \in \mathbf{S}\}$. This can be achieved by considering the measures induced on the (Z, Y)-path space by SRBM's starting from x, where x is allowed to run over all points in $\mathbf{S}$ (cf. Theorem 1.3). The definition of an SRBM with a particular starting point, as opposed to a family of starting points, is made largely to facilitate as sharp a statement as possible of the uniqueness result.

3. In the language of stochastic differential equations, the triple (X, Y, Z) (or just Z) can be thought of as a "weak" solution of the stochastic equation (1.1) and

conditions (i)–(ii), in the sense that one is free to choose the filtered probability space and processes (X, Y, Z) that realize these properties.
4. For brevity, in the sequel we say that X is a (θ, Γ)-Brownian motion if X is an $\mathbb{R}^d$-valued Brownian motion with drift vector θ and covariance matrix Γ, and Z is sometimes called an SRBM when the accompanying measures $\{P_x, x \in \mathbf{S}\}$ are clear from the context.

Heuristically, the behavior of an SRBM may be described as follows. Under P_x, Z behaves like a Brownian motion in the interior of the orthant and it is confined to the orthant by instantaneous "reflection" (or "pushing") at the boundary, where the direction of reflection on the i^{th} face F_i is given by the i^{th} column of the *reflection matrix R*.

In [23], Reiman and Williams showed that a *necessary* condition for the existence of an SRBM is that the matrix R be completely-$\mathscr{S}$, as defined below.

Definition 1.2 A *principal submatrix* of the $d \times d$ matrix R is any square matrix obtained from R by deleting all rows and columns of R with indices in some (possibly empty) subset of $\{1, \ldots, d\}$. The matrix R is *completely-$\mathscr{S}$* if and only if for each principal submatrix $\tilde{R}$ of R there is $\tilde{x} \geqq 0$ such that $\tilde{R}\tilde{x} > 0$.

Remarks. 1. Matrices that are completely-$\mathscr{S}$ are known in the operations research literature as strictly semimonotone or completely-$\mathscr{Q}$ matrices (see [4]).
2. The completely-$\mathscr{S}$ property is invariant under transpose, i.e., R is completely-$\mathscr{S}$ if and only if its transpose R' is completely-$\mathscr{S}$ (see [23, Lemma 3, p. 91]).

Some sufficient (but not necessary and sufficient) conditions for existence of an SRBM have been given previously by Harrison and Reiman [13] and more recently by Dupuis and Ishii [8]. These are based on showing the existence of a Lipschitz map from continuous paths in $\mathbb{R}^d$ (starting in S) to continuous paths in S, such that when applied to the paths of a Brownian motion X, the map yields an SRBM Z that is adapted to X. However, the conditions on R required in [8, 13] are stronger than the completely-$\mathscr{S}$ condition. Recently, Mandelbaum and Van der Heyden [21] and Bernard and El Kharroubi [2] have shown that R being completely-$\mathscr{S}$ is sufficient for the existence of a path-to-path mapping that when applied to any continuous path X starting in S yields continuous paths Y and Z satisfying (1.1) and (a)–(c) above. However, due to an inherent non-uniqueness of this mapping [2, 20], these authors were unable to establish that Y and Z are adapted to X. Consequently, when X is regarded as a Brownian motion, one cannot deduce from their results the important property that $\{X(t) - \theta t, t \geqq 0\}$ is a martingale with respect to a filtration to which Y is also adapted. The results described above were all aimed at proving existence and uniqueness of a "strong" solution of the stochastic equation (1.1) and conditions (i)–(ii). In this paper, we focus on obtaining a "weak" solution.

Define $\mathbf{C} = \{(z, y): [0, \infty) \to \mathbf{S} \times \mathbf{S}, z \text{ and } y \text{ are continuous functions}\}$, $\mathscr{M} = \sigma\{(z, y)(s):\ 0 \leqq s < \infty,\ (z, y) \in \mathbf{C}\}$, $\mathscr{M}_t = \sigma\{(z, y)(s):\ 0 \leqq s \leqq t,\ (z, y) \in \mathbf{C}\}$, for all $t \geqq 0$. Our main result is the following.

Theorem 1.3 *Assume that R is completely-$\mathscr{S}$. Fix $x \in \mathbf{S}$. There exists an* SRBM *associated with* $(\mathbf{S}, \theta, \Gamma, R)$ *that starts from x. Let Z with probability measure P_x defined on some filtered space be such an* SRBM *and let Y denote its "pushing" process as described in Definition* 1.1(ii). *Let Q_x denote the probability measure* *induced on* $(\mathbf{C}, \mathscr{M})$ *by* (Z, Y):

$$(1.2) \qquad Q_x(A) = P_x((Z, Y) \in A) \quad \textit{for all } A \in \mathscr{M} .$$

Then Q_x *is unique and hence the law of any* SRBM, *together with its associated pushing process, for the data* $(\mathbf{S}, \theta, \Gamma, R)$ *and starting point* x *is unique.*

The canonical process $z(\cdot)$ *together with the family of probability measures* $\{Q_x, x \in \mathbf{S}\}$ *defines an* SRBM *on* $(\mathbf{C}, \mathscr{M}, \{\mathscr{M}_t\})$, *where for the semimartingale decomposition* (1.1) *one can take* $Y(\cdot) = y(\cdot)$ *and* $X(\cdot) = z(\cdot) - Ry(\cdot)$. *The family* $\{Q_x, x \in \mathbf{S}\}$ *is Feller continuous and together with the canonical process* $z(\cdot)$ *defines a strong Markov process.*

Terminology. The family of probability measures $\{Q_x, x \in \mathbf{S}\}$ is Feller continuous if for each $x \in \mathbf{S}$ and sequence $\{x_n\} \subset \mathbf{S}$ that converges to x we have $\{Q_{x_n}\}$ converges weakly to Q_x.

Remark. Note that when the measures $\{Q_x, x \in \mathbf{S}\}$ are restricted to the canonical $z(\cdot)$ path space: $\{z : [0, \infty) \to \mathbf{S}, z \text{ is continuous}\}$ with the restriction of the σ-field $\mathscr{M}$, we have that $z(\cdot)$ together with these restricted measures is a Feller continuous strong Markov process there.

Combining the above theorem with the results of Reiman and Williams [23] we have the following.

Corollary 1.4 *There exists an* SRBM *with data* $(\mathbf{S}, \theta, \Gamma, R)$ *if and only if* R *is completely-*$\mathscr{S}$. *In this case, the* SRBM *is unique in law and defines a Feller continuous strong Markov process.*

As a guide to the reader, we now summarize the organization of this paper. Our proof of Theorem 1.3 is by induction on the dimension d. In Sect. 2, after some notations and terminology are defined, we develop a preliminary result and prove Theorem 1.3 for the case $d = 1$. Assuming that Theorem 1.3 holds on $\mathbb{R}^k_+$ for all $k \leq d - 1$ and some $d \geq 2$, and that R is a completely-$\mathscr{S}$ matrix, in Sect. 3 we construct semimartingale reflecting Brownian motions in $\mathbf{K}$-troughs of the form $\{x \in \mathbb{R}^d : x_i \geq 0 \text{ for all } i \in \mathbf{K}\}$ where $\mathbf{K} \subset \{1, \ldots, d\}$ and $|\mathbf{K}| \leq d - 1$. Here the directions of reflection are given by the columns of R that are indexed by $\mathbf{K}$. For each starting point in such a trough, the law of the semimartingale reflecting Brownian motion together with its pushing process is shown to be unique. The results of this section not only provide a key building block for the construction of an SRBM in the orthant, but are also essential to the proof of uniqueness given in Sect. 6. Now observe that there are d "trough" processes with $|\mathbf{K}| = d - 1$, such that the i^{th} process has freedom in the i^{th} coordinate direction for $1 \leq i \leq d$. In Sect. 4 we "patch" together the measures on $(\mathbf{C}, \mathscr{M})$ associated with these trough processes and their pushing processes to obtain an SRBM in the orthant *with absorption at the origin.* In Sect. 5, an SRBM in the orthant is obtained as a weak limit of an approximating family of processes $\{Z^\delta, \delta \in \mathbf{S} \setminus \{0\}\}$ as δ tends to the origin (0) through a particular sequence, where Z^δ behaves like an SRBM with absorption at the origin until the first time the latter hits the origin, at which time Z^δ instantaneously jumps to the point δ and continues from there as if it had started there. The proof that Z^δ and its associated pushing process Y^δ converge weakly as a pair to an SRBM and its associated pushing process uses results of Kurtz and Protter [18]. In Sect. 6, uniqueness in law of an SRBM for each starting point in the orthant is proved. Our proof uses an argument similar to that of Bass and Pardoux [1, Sect. 5] or Kwon and Williams [19, Sect. 3], in conjunction with some crucial estimates that are particular to this problem (see Sect. 6.2). Essential to the proof are a Girsanov transformation to remove the drift (see Sect. 6.1), the scaling property of Lemma 6.5, the compactness of the operator Q established in Lemma 6.6, and the ergodic property established in Lemma 6.7. Both Lemmas 6.6 and 6.7 depend on the particular estimates in Sect. 6.2. Since the pushing process (Y) associated with an SRBM Z can be almost surely recovered as a functional of Z (see Lemma 2.1), uniqueness in law for the SRBM implies uniqueness in law for the pair (Z, Y). The strong Markov property follows from the uniqueness in law and the Feller continuity is also established.

SRBM's of the type constructed by Harrison and Reiman [13] arise naturally as diffusion approximations to single-class open queueing networks, under conditions of heavy traffic [22]. It has been hypothesized [12, 14, 15] that SRBM's with more general reflection matrices than those in [13] arise as approximations to *multi-class* open queueing networks. A general heavy traffic limit theorem justifying this has not been proved to date, in part because of the previous lack of a sufficiently general existence and uniqueness theorem for SRBM's. The results of this paper on existence and uniqueness provide a solid mathematical foundation for SRBM's and potentially could be used in a proof of heavy traffic limit theorems for multi-class open queueing networks, and for the further analysis of SRBM's. In fact, Dai and Kurtz [7] have recently used our results in establishing a characterization of the stationary distributions for SRBM's in terms of a *basic adjoint relationship*. This relationship is the starting point for a numerical algorithm proposed by Dai and Harrison [6] for the computation of stationary distributions of SRBM's. For a summary of other recent results on reflecting Brownian motions and their connections with multi-class queueing networks, see [15, 16].

2 Preliminaries and the induction hypothesis

2.1 Notations and terminology

The following notations and terminology are used throughout this paper. The set of natural numbers $\{1, 2, \ldots\}$ is denoted by $\mathbb{N}$. The set of real numbers is denoted by $\mathbb{R}$ and for $d \geqq 1$, $\mathbb{R}^d$ denotes d-dimensional Euclidean space. We endow $\mathbb{R}$, $\mathbb{R}^d$ with their Borel σ-fields. The set of non-negative real numbers will be denoted by $\mathbb{R}_+$ and the positive orthant in $\mathbb{R}^d$ will be denoted by $\mathbb{R}^d_+ \equiv \{x \in \mathbb{R}^d : x_i \geqq 0$ for $i = 1, \ldots, d\}$. The Euclidean distance between two points x, $y \in \mathbb{R}^d$ will be denoted by $d(x, y)$. Similarly, $d(x, A)$ will denote the Euclidean distance between a point x and a set A in $\mathbb{R}^d$, and $d(A, B)$ will denote the Euclidean distance between two sets A and B in $\mathbb{R}^d$. The open ball in $\mathbb{R}^d$ with center x and radius $r > 0$ will be denoted by $B(x, r)$. A vector $v \in \mathbb{R}^d$ will be treated as a column vector with components v_i for $i = 1, \ldots, d$. We write $v > 0$ ($\geqq 0$) if and only if each component of v is positive (non-negative). The Euclidean length of v will be denoted by $|v|$. Throughout this paper we let $\mathbf{J} = \{1, \ldots, d\}$ and for any set $\mathbf{K} \subset \mathbf{J}$, we let $|\mathbf{K}|$ denote the cardinality (size) of $\mathbf{K}$ and $v_{\mathbf{K}}$ denote the vector whose components are those of v with indices in $\mathbf{K}$. We let $v_{|\mathbf{K}}$ denote the vector obtained from v by deleting the components with indices in $\mathbf{K}$. If $\mathbf{K} = \{i\}$, we abuse notation by writing v_i and $v_{|i}$ instead of $v_{\{i\}}$ and $v_{|\{i\}}$, respectively. We use similar notation for any d-dimensional process. If H is a $d \times d$ matrix, then $H_{\mathbf{K}}$ denotes the matrix whose elements come from those in H with row and column indices in $\mathbf{K}$, and $H_{|\mathbf{K}}$ denotes the matrix obtained by deleting the rows and columns with indices in $\mathbf{K}$. If $\mathbf{K} = \{i\}$, we write H_i and $H_{|i}$ in place of $H_{\{i\}}$ and $H_{|\{i\}}$, respectively. We let H' denote the transpose of H and write $H > 0 (\geqq 0)$ if and only if each entry of H is positive (non-negative). We let I_d denote the $d \times d$ identity matrix. For each $i \in \{1, \ldots, d\}$, we let R_{ii} denote the i^{th} diagonal element of R and n_i denote the inward unit normal to the i^{th} face F_i of $\mathbf{S}$. Let $\mathbf{S}^\circ$ denote the interior of $\mathbf{S}$ and F_i° denote the relative interior of F_i. As a convention we will assume that stochastic processes evaluated at time $t = \infty$ are at some isolated cemetery point. The (i, j) component of the mutual variation process associated with a multi-dimensional continuous semimartingale X will be denoted by $\langle X_i, X_j \rangle$. For a one-dimensional continuous semimartingale X, its quadratic variation process will be denoted by $[X]$.

2.2 Preliminary lemma

The following lemma will be used several times throughout this work.

Lemma 2.1 *Suppose Z defined on a filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P_x)$ is an* SRBM *associated with* $(\mathbf{S}, \theta, \Gamma, R)$ *that starts from $x \in \mathbf{S}$, and X, Y have the properties described in Definition* 1.1. *Then* P_x-a.s.,

(2.1)
$$\int_0^\infty 1_{\partial S}(Z(s))ds = 0 ,$$

(2.2)
$$X(t) = Z(0) + \int_0^t 1_{S^\circ}(Z(s))dZ(s) \quad \textit{for all } t \geqq 0 ,$$

(2.3)
$$Y_i(t) = R_{ii}^{-1} \int_0^t 1_{F_i^\circ}(Z(s))d(n_i \cdot Z)(s) \quad \textit{for all } t \geqq 0 .$$

Remark. The integrals in (2.2)–(2.3) can be defined as right continuous, P_x-a.s. continuous adapted processes on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P_x)$ (cf. [26, p. 97]), or as continuous adapted processes on the completion of $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P_x)$.

Proof. To prove (2.1), it suffices to show that for each i, P_x-a.s.,

$$\int_0^\infty 1_{\{0\}}(n_i \cdot Z(s))ds = 0 ,$$

where n_i denotes the inward unit normal to F_i. Now, $n_i \cdot Z$ is a continuous one-dimensional semimartingale and so by [24, VI.1], $n_i \cdot Z$ has a local time $\{L_t^y; t \geqq 0, y \in \mathbb{R}\}$ that is P_x-a.s. continuous in t and right continuous with finite left limits in y, and such that P_x-a.s. for all $t \geqq 0$,

$$\int_0^t 1_{\{0\}}(n_i \cdot Z(s))d[n_i \cdot Z]_s = \int_{\mathbb{R}} L_t^y \, 1_{\{y=0\}} \, dy = 0 .$$

Since $[n_i \cdot Z]_s = [n_i \cdot X]_s = \Gamma_{ii} s$ where $\Gamma_{ii} > 0$, (2.1) follows.

It follows from (2.1) and the fact that $\{X(t) - \theta t, t \geqq 0\}$ is a Brownian motion with P_x-a.s. continuous paths that P_x-a.s. for all $t \geqq 0$,

$$X(t) = X(0) + \int_0^t 1_{S^\circ}(Z(s))dX(s)$$

$$= Z(0) + \int_0^t 1_{S^\circ}(Z(s))dZ(s) ,$$

where the last equality follows from the fact that P_x-a.s., Y can increase only when Z is on ∂S.

It follows from [23] that P_x-a.s., Y_i only charges the set $\{s \geqq 0: Z(s) \in F_i^\circ\}$, i.e., P_x-a.s. for all $t \geqq 0$,

$$Y_i(t) = \int_0^t 1_{F_i^\circ}(Z(s))dY_i(s) .$$

Since X and the $Y_j, j \neq i$, as integrators do not charge the set $\{s \geqq 0: Z(s) \in F_i^\circ\}$, we can replace dY_i by $R_{ii}^{-1} d(n_i \cdot Z)$ in the above to obtain (2.3). □

2.3 Induction hypothesis

Theorem 2.2 *Theorem* 1.3 *holds for* $d = 1$.

Proof. Suppose $d = 1, \theta \in \mathbb{R}$ and $\Gamma \in \mathbb{R}_+ \setminus \{0\}$. Then $R = \alpha, \alpha \in \mathbb{R}$, is completely-$\mathscr{S}$ if and only if $\alpha > 0$. Assuming this, fix $x \in \mathbb{R}_+$ and let X be a continuous one-dimensional process defined on a probability space $(\Omega, \mathscr{F}, P_x)$, such that X is a one-dimensional Brownian motion with drift θ, variance parameter Γ, and $X(0) = x$ P_x-a.s. Let $\mathscr{F}_t \equiv \sigma\{X(s), 0 \leqq s \leqq t\}$. Then $\{X(t) - \theta t, \mathscr{F}_t, t \geqq 0\}$ is a P_x-martingale. Define

$$Y(t) = \alpha^{-1}\left(- \min_{0 \leqq s \leqq t} X(s)\right)^+ \quad \text{for all } t \geqq 0 , \tag{2.4}$$

and

$$Z = X + \alpha Y ,$$

where $w^+ = w \vee 0$. Then Z is an SRBM on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P_x)$ associated with $(\mathbb{R}_+, \theta, \Gamma, R)$ that starts from x.

The fact that pathwise uniqueness is known to hold when $d = 1$ (cf. [9]) for Eq. (1.1) with the attendant properties (i)–(ii) of Definition 1.1, ensures that for any SRBM Z associated with $(\mathbb{R}_+, \theta, \Gamma, \alpha)$ that starts from x, its Y process must be the functional of its Brownian motion X exhibited in formula (2.4). This implies that the law of the pair (Z, Y) is uniquely determined. Hence, the measure Q_x as defined in Theorem 1.3 is unique.

For Z, Y as defined in the first paragraph of this proof, $\{Z(t) - \alpha Y(t) - \theta t, t \geqq 0\}$ is a $(0, \Gamma)$-Brownian motion P_x-martingale relative to the filtration generated by (Z, Y), and so it follows that $\{z(t) - \alpha y(t) - \theta t, t \geqq 0\}$ is a (continuous) $(0, \Gamma)$-Brownian motion Q_x-martingale with respect to the $\{\mathscr{M}_t\}$-filtration. The desired properties of the canonical process $y(\cdot)$ under Q_x are inherited from those of Y under P_x. The Feller continuity of the measures $\{Q_x, x \in \mathbb{R}_+\}$ and the strong

Markov property of $z(\cdot)$ with these measures follow from the corresponding properties for Brownian motion and the fact that Y is a continuous additive functional of X. □

We now proceed to prove Theorem 1.3 by induction. By Theorem 2.2, Theorem 1.3 is true for $d = 1$. So we now fix $d \geqq 2$ and make the following induction hypothesis.

Induction hypothesis. *Theorem* 1.3 *holds for all dimensions less than or equal to* $d - 1$.

We henceforth take θ, Γ, and R to be as in the hypotheses of Theorem 1.3. In particular, R is assumed to be completely-$\mathscr{S}$.

3 SRBM in a trough

In this section we prove existence and uniqueness in law of SRBM's in state spaces that we call $\mathbf{K}$-troughs, $\mathbf{K} \subset \mathbf{J}, \mathbf{K} \neq \mathbf{J}$. These results for $|\mathbf{K}| = d - 1$ are used for the proof of existence, and the results for all $\mathbf{K}: |\mathbf{K}| \leqq d - 1$ are used for the proof of uniqueness, of an SRBM in $\mathbf{S}$.

3.1 *Definition of an* SRBM *in a trough*

Definition 3.1 Let $\mathbf{K} \subset \mathbf{J}$ with size $k \equiv |\mathbf{K}| \in \{1, \ldots, d-1\}$. Let $\mathbf{S}^{\mathbf{K}} = \{x \in \mathbb{R}^d : x_i \geqq 0 \text{ for all } i \in \mathbf{K}\}$ and let $R^{\mathbf{K}}$ be the $d \times k$ matrix obtained from R by deleting those columns of R with indices in $\mathbf{J} \setminus \mathbf{K}$. Order the elements of $\mathbf{K}$ in increasing order and let $i: \{1, \ldots, k\} \to \mathbf{K}$ be such that $i_j \equiv i(j)$ is the j^{th} element of $\mathbf{K}$. For $x \in \mathbf{S}^{\mathbf{K}}$, a *semimartingale reflecting Brownian motion* (SRBM) *associated with* $(\mathbf{S}^{\mathbf{K}}, \theta, \Gamma, R^{\mathbf{K}})$ *that starts from* x is a continuous, $\{\mathscr{F}_t\}$-adapted, d-dimensional process Z defined on some filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P_x)$ such that under P_x,

$$(3.1) \qquad Z(t) = X(t) + R^{\mathbf{K}} Y(t) \in \mathbf{S}^{\mathbf{K}} \quad \text{for all } t \geqq 0 ,$$

where

(i) X is a d-dimensional Brownian motion with drift vector θ and covariance matrix Γ such that $\{X(t) - \theta t, \mathscr{F}_t, t \geqq 0\}$ is a martingale and $X(0) = x$ P_x-a.s.,

(ii) Y is an $\{\mathscr{F}_t\}$-adapted, $\mathbb{R}_+^k$-valued process such that P_x-a.s. for each $j \in \{1, \ldots, k\}$, the j^{th} component Y_j of Y satisfies

(a) $Y_j(0) = 0$,
(b) Y_j is continuous and non-decreasing,
(c) Y_j can increase only when Z is on the j^{th} face $F_{i_j}^{\mathbf{K}}$ of $\mathbf{S}^{\mathbf{K}}$: $F_{i_j}^{\mathbf{K}} \equiv \{x \in \mathbf{S}^{\mathbf{K}} : x_{i_j} = 0\}$, i.e., $\int_0^t 1_{\mathbf{S}^k \setminus F_{i_j}^k}(Z(s)) dY_j(s) = 0$ for all $t \geqq 0$.

An SRBM associated with $(\mathbf{S}^{\mathbf{K}}, \theta, \Gamma, R^{\mathbf{K}})$ is a continuous, $\{\mathscr{F}_t\}$-adapted, d-dimensional process Z together with a family of probability measures $\{P_x, x \in \mathbf{S}^{\mathbf{K}}\}$ defined on some filtered space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\})$ such that for each $x \in \mathbf{S}^{\mathbf{K}}$, on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P_x)$, Z is an SRBM associated with $(\mathbf{S}^{\mathbf{K}}, \theta, \Gamma, R^{\mathbf{K}})$ that starts from x.

For the remainder of this section, let $k, \mathbf{K}, \mathbf{S}^{\mathbf{K}}$ be as in Definition 3.1. Let $\mathbf{C}^{\mathbf{K}} = \{(z,y): [0,\infty) \to \mathbf{S}^{\mathbf{K}} \times \mathbb{R}_+^k, z \text{ and } y \text{ are continuous functions}\}$, $\mathcal{M}^{\mathbf{K}} = \sigma\{(z,y)(s): 0 \leqq s < \infty, (z,y) \in \mathbf{C}^{\mathbf{K}}\}$, and $\mathcal{M}_t^K = \sigma\{(z,y)(s): 0 \leqq s \leqq t, (z,y) \in \mathbf{C}^{\mathbf{K}}\}$ for all $t \geqq 0$.

3.2 Existence of an SRBM *in a trough*

The following proposition can be proved using standard martingale arguments for characterizing Brownian motions; accordingly its proof is omitted.

Proposition 3.2 *Suppose a continuous process $\hat{X}$, together with a family of probability measures $\{\hat{P}_{\hat{x}}, \hat{x} \in \mathbb{R}_+^k\}$, is defined on some filtered space $(\hat{\Omega}, \hat{\mathcal{F}}, \{\hat{\mathcal{F}}_t\})$ such that for each $\hat{x} \in \mathbb{R}_+^k$, under $\hat{P}_{\hat{x}}$, $\hat{X}$ is a k-dimensional Brownian motion with covariance matrix $\Gamma_{\mathbf{K}}$ and drift vector $\hat{\theta} \equiv \theta_{\mathbf{K}}$, such that $\{\hat{X}(t) - \hat{\theta}t, \hat{\mathcal{F}}_t, t \geqq 0\}$ is a martingale and $\hat{X}(0) = \hat{x}$ $\hat{P}_x$-a.s.*

Let $\Lambda = (\Gamma_{ij})_{i \in \mathbf{K}, j \notin \mathbf{K}}$. Then there is an invertible $(d-k) \times (d-k)$ matrix H such that $HH' = \Gamma_{|\mathbf{K}} - \Lambda' \Gamma_{\mathbf{K}}^{-1} \Lambda$. Let a continuous process $\tilde{B}$, together with a family of probability measures $\{\tilde{P}_{\tilde{x}}, \tilde{x} \in \mathbb{R}^{d-k}\}$, be defined on a measurable space $(\tilde{\Omega}, \tilde{\mathcal{F}})$ which is distinct from $(\hat{\Omega}, \hat{\mathcal{F}})$, such that for each $\tilde{x} \in \mathbb{R}^{d-k}$, under $\tilde{P}_{\tilde{x}}$, $\tilde{B}$ is a $(d-k)$-dimensional Brownian motion with covariance matrix I_{d-k} and zero drift, such that $\tilde{B}(0) = \tilde{x}$ $\tilde{P}_{\tilde{x}}$-a.s.

Let $\Omega = \hat{\Omega} \times \tilde{\Omega}$, $\mathcal{F} = \hat{\mathcal{F}} \times \tilde{\mathcal{F}}$, and $\mathcal{F}_t = \hat{\mathcal{F}}_t \times \tilde{\mathcal{F}}_t$ for all $t \geqq 0$, where $\tilde{\mathcal{F}}_t = \sigma\{\tilde{B}(s): 0 \leqq s \leqq t\}$. Define a d-dimensional process X on $(\Omega, \mathcal{F})$ by

$$X_{\mathbf{K}}(t, (\hat{\omega}, \tilde{\omega})) = \hat{X}(t, \hat{\omega}),$$

$$X_{|\mathbf{K}}(t, (\hat{\omega}, \tilde{\omega})) = (\Lambda' \Gamma_{\mathbf{K}}^{-1}(\hat{X}(t, \hat{\omega}) - \hat{\theta}t) + H\tilde{B}(t, \tilde{\omega})) + \theta_{|\mathbf{K}} t,$$

for all $t \geqq 0$ and $(\hat{\omega}, \tilde{\omega}) \in \hat{\Omega} \times \tilde{\Omega}$. For each $x \in \mathbf{S}^{\mathbf{K}}$, define $P_x = \hat{P}_{\hat{x}} \times \tilde{P}_{\tilde{x}}$, where $\hat{x} = x_{\mathbf{K}}$ and

$$\tilde{x} = H^{-1}(x_{|\mathbf{K}} - \Lambda' \Gamma_{\mathbf{K}}^{-1} \hat{x}). \tag{3.2}$$

Then under P_x, the continuous process X is a d-dimensional Brownian motion with covariance matrix Γ and drift vector θ, such that $\{X(t) - \theta t, \mathcal{F}_t, t \geqq 0\}$ is a martingale and $X(0) = x$ P_x-a.s.

Theorem 3.3 *There exists an* SRBM *associated with* $(\mathbf{S}^{\mathbf{K}}, \theta, \Gamma, R^{\mathbf{K}})$.

Proof. Let $\hat{\mathbf{S}} = \mathbb{R}_+^k$, $\hat{\mathbf{C}} = \{(\hat{z}, \hat{y}): [0,\infty) \to \hat{\mathbf{S}} \times \hat{\mathbf{S}}, \hat{z} \text{ and } \hat{y} \text{ are continuous functions}\}$, $\hat{\mathcal{M}} = \sigma\{(\hat{z}, \hat{y})(s): 0 \leqq s < \infty, (\hat{z}, \hat{y}) \in \hat{\mathbf{C}}\}$, and $\hat{\mathcal{M}}_t = \sigma\{(\hat{z}, \hat{y})(s): 0 \leqq s \leqq t, (\hat{z}, \hat{y}) \in \hat{\mathbf{C}}\}$ for each $t \geqq 0$. Let $\theta_{\mathbf{K}}, \Gamma_{\mathbf{K}}, R_{\mathbf{K}}$ be defined from θ, Γ, R in the manner described in Sect. 2.1. By the induction hypothesis, Theorem 1.3 holds for dimension k and so there exists a unique family of probability measures $\{\hat{P}_{\hat{x}}, \hat{x} \in \hat{\mathbf{S}}\}$ defined on $(\hat{\mathbf{C}}, \hat{\mathcal{M}}, \{\hat{\mathcal{M}}_t\})$ such that the canonical process $\hat{z}(\cdot)$ together with these probability measures defines an SRBM associated with $(\hat{\mathbf{S}}, \theta_{\mathbf{K}}, \Gamma_{\mathbf{K}}, R_{\mathbf{K}})$, where the pushing process and Brownian motion in the SRBM decomposition of $\hat{z}(\cdot)$ can be taken to be given by $\hat{y}(\cdot)$ and $\hat{X}(\cdot) \equiv \hat{z}(\cdot) - R_K \hat{y}(\cdot)$, respectively.

By applying Proposition 3.2 with $\hat{\Omega} = \hat{\mathbf{C}}$ we see that we can extend $\hat{X}$ to X so that if we define $Y(t, (\hat{\omega}, \tilde{\omega})) = \hat{y}(t, \hat{\omega})$, where $\hat{\omega} = (\hat{z}, \hat{y})$, and $Z = X + R^{\mathbf{K}} Y$, then Z together with $\{P_x, x \in \mathbf{S}^{\mathbf{K}}\}$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ is an SRBM associated with $(\mathbf{S}^{\mathbf{K}}, \theta, \Gamma, R^{\mathbf{K}})$. □

Remark. For the construction of an SRBM in $\mathbf{S}^{\mathbf{K}}$, we really only need that $R_{\mathbf{K}}$ is completely-$\mathscr{S}$. However, there is no loss of generality in assuming that $R^{\mathbf{K}}$ such that $R^{\mathbf{K}}_{\mathbf{K}}$ is completely-$\mathscr{S}$ is derived from an R that is completely-$\mathscr{S}$. For if $R^{\mathbf{K}}$ is such that $R^{\mathbf{K}}_{\mathbf{K}}$ is completely-$\mathscr{S}$, one can always add $d-k$ columns of the vector $(1, 1, \ldots, 1)'$ to $R^{\mathbf{K}}$ to obtain an R that is completely-$\mathscr{S}$.

3.3 Uniqueness of an SRBM *in a trough*

Theorem 3.4 *Fix $x \in \mathbf{S}^{\mathbf{K}}$ and let Z defined on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P_x)$ be an* SRBM *associated with $(\mathbf{S}^{\mathbf{K}}, \theta, \Gamma, R^{\mathbf{K}})$ that starts from x. Let Y be the k-dimensional pushing process associated with Z, as in Definition* 3.1. *Then the law Q_x induced on $(\mathbf{C}^{\mathbf{K}}, \mathscr{M}^{\mathbf{K}})$ by (Z, Y) under P_x is unique, i.e., the joint law of an* SRBM *and its attendant pushing process for the data $(\mathbf{S}^{\mathbf{K}}, \theta, \Gamma, R^{\mathbf{K}})$ and starting point x is unique.*

Proof. We first prove the result for the case where $|\mathbf{K}| = d-1$. Indeed, for this we may and do assume that $\mathbf{K} = \{1, \ldots, d-1\}$. Let $\hat{\mathbf{S}} = \mathbb{R}^{d-1}_+$, $\hat{\theta} = \theta_{|d}$, $\hat{\Gamma} = \Gamma_{|d}$ and $\hat{R} = R_{|d}$. Now $Z_{|d} = X_{|d} + \hat{R}Y$ on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P_x)$ is an SRBM associated with $(\hat{\mathbf{S}}, \hat{\theta}, \hat{\Gamma}, \hat{R})$ that starts from $\hat{x} = x_{|d}$. By the induction hypothesis, Theorem 1.3 holds for dimension $d-1$, and so the law of $(Z_{|d}, Y)$ under P_x is unique. To characterize (Z, Y) under P_x we need to know the joint law of $(Z_{|d}, Y, X_d)$ under P_x. If we could show for Λ and H as in Proposition 3.2 with $\mathbf{K} = \mathbf{J}\backslash\{d\}$ that under P_x, $B \equiv \{X_d(t) - \theta_d t - \Lambda'\hat{\Gamma}^{-1}(X_{|d}(t) - \hat{\theta}t), t \geqq 0\}$ is independent of $Z_{|d}$, then since B is a driftless Brownian motion with variance parameter HH' under P_x and Y is P_x-a.s. a functional of $Z_{|d}$ (see Lemma 2.1), it would follow that the joint law of $(Z_{|d}, Y, X_d)$ under P_x is unique. However, we do not see how to show the desired independence because although we can show B is independent of $X_{|d}$, this is not sufficient because we do not know that $X_{|d}$ generates the same σ-field as $Z_{|d}$. Instead, we observe that since Y is P_x-a.s. a functional of $Z_{|d}$, it suffices to show that Z under P_x is unique in law. For this we approximate Z by processes Z^m in which the Brownian motion part and the Y-bounded variation part of Z^m_d are supported on disjoint stochastic intervals, as follows. Let v^i denote the i^{th} column of R for $i = 1, \ldots, d$, and let $\{K_m, m \geqq 1\}$ be a sequence of compact sets in $\hat{\mathbf{S}}$ such that $K_m \subset \hat{\mathbf{S}}^\circ$ (the interior of $\hat{\mathbf{S}}$), $K_m \subset K_{m+1}$, $\bigcup_m K_m = \hat{\mathbf{S}}^\circ$, and $1/m \leqq d(K_m, \partial\hat{\mathbf{S}}) \leqq 2/m$. For each m, define a sequence of stopping times as follows:

$$\sigma_0 = 0\,,$$
$$\tau_0 = \inf\{t \geqq \sigma_0 \colon Z_{|d}(t) \in \partial\hat{\mathbf{S}}\}\,,$$
$$\vdots$$
$$\sigma_n = \inf\{t \geqq \tau_{n-1} \colon Z_{|d}(t) \in K_m\}\,,$$
$$\tau_n = \inf\{t \geqq \sigma_n \colon Z_{|d}(t) \in \partial\hat{\mathbf{S}}\}\,,$$

and let

$$Z^m(\cdot) = \begin{bmatrix} Z_{|d}(\cdot) \\ x_d + \sum_{n \geqq 0} (X_d(\cdot \wedge \tau_n) - X_d(\cdot \wedge \sigma_n)) + \sum_{i=1}^{d-1} v^i_d Y_i(\cdot) \end{bmatrix}. \tag{3.3}$$

By considering the probability measure induced on $(\mathbf{C}^{\mathbf{K}}, \mathscr{M}^{\mathbf{K}})$ by (Z, Y) if necessary, we may assume that P_x has regular conditional probability distributions relative to the filtration $\{\mathscr{F}_t\}$.

We will show by induction on n that the law of $Z^m(\cdot \wedge \sigma_n)$ under P_x is unique. Clearly this is true for $n = 0$, since $\sigma_0 = 0$ and $P_x(Z^m(0) = x) = 1$. For the induction step, we suppose that the claim is true for some $n \geqq 0$ and we shall then prove that it is true for $n + 1$. As an intermediate step, we first verify that the law of $(Z^m(\cdot \wedge \tau_n), \tau_n)$ under P_x is unique. Note that since σ_n is determined by $Z_{|d}(\cdot \wedge \sigma_n)$, the joint law of $(Z^m(\cdot \wedge \sigma_n), \sigma_n)$ under P_x is unique. Now, P_x-a.s. Y does not increase on $[\sigma_n, \tau_n)$, and P_x-a.s. on $\{\sigma_n < \infty\}$, by the martingale Brownian motion property of $X, X(\cdot + \sigma_n) - X(\sigma_n)$ under $P_x(\cdot|\mathscr{F}_{\sigma_n})$ is a (θ, Γ)-Brownian motion starting from the origin. It follows from the definition of Z^m and τ_n that the law of $(Z^m(\cdot \wedge \tau_n), \tau_n)$ under P_x is unique.

Now we turn to proving that the law of $Z^m(\cdot \wedge \sigma_{n+1})$ under P_x is unique. Using the martingale Brownian motion property of $X_{|d}$ and functional representations for $X_{|d} - X_{|d}(0)$ and Y (cf. Lemma 2.1), we see that P_x-a.s. on $\{\tau_n < \infty\}$, $Z_{|d}(\cdot + \tau_n)$ under $P_x(\cdot|\mathscr{F}_{\tau_n})$ has the law of an SRBM associated with $(\hat{\mathbf{S}}, \hat{\theta}, \hat{\Gamma}, \hat{R})$ that starts from $Z_{|d}(\tau_n)$, which is unique and Feller continuous in the starting point, by the induction assumption that Theorem 1.3 holds in dimensions less than d. Since on $\{\tau_n < \infty\}$, $Y(\cdot + \tau_n) - Y(\tau_n)$ and $\sigma_{n+1} - \tau_n$ can be expressed P_x-a.s. as functionals of $Z_{|d}(\cdot + \tau_n)$, it follows using conditioning and the already established fact that the law of $(Z^m(\cdot \wedge \tau_n), \tau_n)$ under P_x is unique, that the law of $Z^m(\cdot \wedge \sigma_{n+1})$ under P_x is unique. This completes the induction step. Since $\sigma_n \to \infty$ as $n \to \infty$, it follows that the law of Z^m under P_x is unique.

It now remains to show that for each $T \in \mathbb{R}_+$, $\sup_{t\in[0,T]} |Z^m(t) - Z(t)| \to 0$ in $L^2 \equiv L^2(\Omega, \mathscr{F}, P_x)$ as $m \to \infty$. By the definition of Z^m,

$$\sup_{t\in[0,T]} |Z^m(t) - Z(t)| \leqq \sup_{t\in[0,T]} \left| \sum_{n\geqq 0} \int_{t\wedge\tau_n}^{t\wedge\sigma_{n+1}} dX_d(s) \right| .$$

The sum of stochastic integrals in the right member above defines a P_x-a.s. continuous L^2-martingale and so by Doob's inequality and the L^2-isometry for stochastic integrals we have

$$\text{(3.4)} \qquad \overline{\lim_{m\to\infty}}\, E\left[\left(\sup_{t\in[0,T]} |Z^m(t) - Z(t)|\right)^2\right] \leqq \overline{\lim_{m\to\infty}}\, 4\Gamma_{dd}\, E^{P_x}\left[\int_0^T \sum_{n\geqq 0} 1_{[\tau_n,\sigma_{n+1}]}(s)ds\right]$$

$$= 4\Gamma_{dd} E^{P_x}\left[\int_0^T 1_{\partial\hat{\mathbf{S}}}(Z_{|d}(s))ds\right] = 0 ,$$

where the last equality follows from the fact that P_x-a.s., $Z_{|d}$ spends zero Lebesgue time on the boundary of $\partial\hat{\mathbf{S}}$ (see Lemma 2.1).

By minor modification of the above proof, one can show that if τ is any stopping time adapted to the filtration generated by the continuous d-dimensional process Z and $(Z(\cdot \wedge \tau), Y(\cdot \wedge \tau), X(\cdot \wedge \tau))$ satisfy (3.1) and (i)–(ii) of Definition 3.1 with $t \wedge \tau$ in place of t and "stopped Brownian motion" in place of "Brownian motion" there, then $(Z(\cdot \wedge \tau), Y(\cdot \wedge \tau))$ is unique in law. We shall refer to this by saying that a stopped SRBM and its associated Y process form a pair that is unique in law.

We now turn to the case where $1 \leqq |\mathbf{K}| < d - 1$. Note that since $Z_{\mathbf{K}} = X_{\mathbf{K}} + R_{\mathbf{K}} Y$ under P_x is an SRBM in $\mathbb{R}^k_+$, by Lemma 2.1, Y is P_x-a.s. a functional of $Z_{\mathbf{K}}$, and hence it suffices to show that Z is unique in law. For a proof by contradiction, suppose that there are $k, \mathbf{K}, x$, such that $1 \leqq k < d - 1$, $|\mathbf{K}| = k, x \in \mathbf{S}^{\mathbf{K}}$, and there are different SRBM's Z^i defined on $(\Omega^i, \mathscr{F}^i,$

$\{\mathscr{F}_t^i\}, P^i)$, $i = 1, 2$, associated with $(\mathbf{S}^{\mathbf{K}}, \theta, \Gamma, R^{\mathbf{K}})$ and starting from x. For $i = 1, 2$ and each positive integer n, let $\tau_n^i = \inf\{t \geqq 0 : |Z_{|\mathbf{K}}^i - x_{|\mathbf{K}}| \geqq n\}$. Let $\Omega^{\mathbf{K}} = \{z : [0, \infty) \to \mathbf{S}^{\mathbf{K}}, z \text{ is continuous}\}$, $\mathscr{F}_t^{\mathbf{K}} = \sigma\{z(s) : 0 \leqq s \leqq t\}$ for each $t \geqq 0$, and $\tau_n = \inf\{t \geqq 0 : |z_{|\mathbf{K}}(t) - x_{|\mathbf{K}}| \geqq n, z \in \Omega^{\mathbf{K}}\}$. Since Z^1 is not equivalent in law to Z^2, there must be $n \geqq 1$ and $A \in \mathscr{F}_{\tau_n}^{\mathbf{K}}$ such that

$$P^1(Z^1(\cdot \wedge \tau_n^1) \in A) \neq P^2(Z^2(\cdot \wedge \tau_n^2) \in A) . \tag{3.5}$$

Let $c = |x_{|\mathbf{K}}|$, and define $a \in \mathbb{R}_+^d$ such that $a_{\mathbf{K}} = 0$ and $a_j = c + 2n$ for all $j \in \mathbf{J} \backslash \mathbf{K}$. Note that for $i = 1, 2$, P^i-a.s., $Z^i(\cdot \wedge \tau_n^i) + a \in \mathbf{S}$. From (3.5), we have

$$P^1(Z^1(\cdot \wedge \tau_n^1) + a \in A + a) \neq P^2(Z^2(\cdot \wedge \tau_n^2) + a \in A + a) , \tag{3.6}$$

where $A + a = \{z(\cdot) + a : z \in A\}$. Now let $\mathbf{L} \subset \{1, 2, \ldots, d\}$ such that $|\mathbf{L}| = d - 1$ and $\mathbf{K} \subset \mathbf{L}$. Then for $i = 1, 2$, under P^i, $Z^i(\cdot \wedge \tau_n^i) + a$ is an SRBM associated with $(\mathbf{S}^{\mathbf{L}}, \theta, \Gamma, R^{\mathbf{L}})$ starting from $x + a$ and stopped at τ_n^i. It follows from the first part of the proof of this theorem that the law of $Z^1(\cdot \wedge \tau_n^1) + a$ under P^1 is equal to that of $Z^2(\cdot \wedge \tau_n^2) + a$ under P^2. But this contradicts (3.6) and so the result is proved.

The same comments that applied regarding uniqueness in law for stopped SRBM's when $|\mathbf{K}| = d - 1$ also apply when $|\mathbf{K}| < d - 1$. □

Corollary 3.5 *Let Q_x be as described in Theorem* 3.4. *Then*

$$Q_x(A) = (\hat{P}_{x_{\mathbf{K}}} \times \tilde{P}_{x_{|\mathbf{K}}})((Z, Y) \in A) \quad \text{for all } A \in \mathscr{M}^{\mathbf{K}} , \tag{3.7}$$

where $\hat{P}_{x_{\mathbf{K}}}$, $\tilde{P}_{x_{|\mathbf{K}}}$, Z, Y are as defined in the proofs of Theorem 3.3 *and Proposition* 3.2. *Furthermore, the family $\{Q_x, x \in \mathbf{S}^{\mathbf{K}}\}$ is Feller continuous and the canonical process $z(\cdot)$ on $(\mathbf{C}^{\mathbf{K}}, \mathscr{M}^{\mathbf{K}})$ together with this family has the strong Markov property.*

Proof. Equation (3.7) follows immediately from the uniqueness proved in Theorem 3.4 and the construction contained in the proofs of Theorem 3.3 and Proposition 3.2. The Feller continuity of $\{Q_x, x \in \mathbf{S}^{\mathbf{K}}\}$ follows from this, the Feller continuity of $\{\hat{P}_{\hat{x}}, \hat{x} \in \mathbb{R}_+^k\}$, the Feller continuity that can be assumed to hold for $\{\tilde{P}_{\tilde{x}}, \tilde{x} \in \mathbb{R}_+^{d-k}\}$, and the explicit form of Z.

For the proof of the strong Markov property, fix $x \in \mathbf{S}^{\mathbf{K}}$ and note that the canonical process $z(\cdot)$ on $\mathbf{C}^{\mathbf{K}}$ has an SRBM decomposition with respect to the filtration $\{\mathscr{M}_t^{\mathbf{K}}\}$ under Q_x, where the pushing process is given by the canonical process $y(\cdot)$. Let τ be a stopping time relative to the filtration generated by $z(\cdot)$ and let $\{Q_x^{\omega}(\cdot), \omega \in \mathbf{C}^{\mathbf{K}}\}$ be a regular conditional probability distribution (r.c.p.d.) for $Q_x(\cdot \,|\, \mathscr{M}_\tau^{\mathbf{K}})$. It follows from the SRBM decomposition of $z(\cdot)$ and the martingale property of the Brownian motion in this decomposition, that Q_x-a.s. on $\{\tau < \infty\}$, $z(\tau + \cdot)$ under Q_x^{ω} is an SRBM starting from $z(\tau)$, and so by the uniqueness established in Theorem 3.4, it has the law of the canonical process $z(\cdot)$ under $Q_{z(\tau)}$. The strong Markov property follows from this and the Feller continuity established above. □

4 SRBM in an orthant with absorption at the origin

4.1 Definition of an SRBM *with absorption at the origin*

Definition 4.1 For $x \in \mathbf{S}$, a *semimartingale reflecting Brownian motion* (SRBM) *associated with* $(\mathbf{S}, \theta, \Gamma, R)$ *that starts from x and is absorbed at the origin*, is a continuous, $\{\mathscr{F}_t\}$-adapted, d-dimensional process Z defined on some filtered probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P_x)$ such that for $\tau = \inf\{t \geqq 0 : Z(t) = 0\}$, under P_x,

$$(4.1) \qquad Z(t) = \begin{cases} X(t) + RY(t) \in \mathbf{S} & \text{for all } t \leqq \tau , \\ 0 & \text{for all } t \geqq \tau , \end{cases}$$

where

(i) X is an $\{\mathscr{F}_t\}$-adapted, d-dimensional, P_x-a.s. continuous process such that $B \equiv \{X(t \wedge \tau) - \theta(t \wedge \tau), \mathscr{F}_t, t \geqq 0\}$ is a martingale with mutual variation process: $\langle B_i, B_j \rangle_t = \Gamma_{ij}(t \wedge \tau)$ for all $t \geqq 0$, and $X(0) = x$ P_x-a.s.,

(ii) Y is an $\{\mathscr{F}_t\}$-adapted, d-dimensional process such that P_x-a.s. for each $i \in \{1, \ldots, d\}$, the i^{th} component Y_i of Y satisfies

(a) $Y_i(0) = 0$,

(b) Y_i is continuous and non-decreasing,

(c) Y_i can increase only when Z is on the i^{th} face F_i of $\mathbf{S}$, i.e., $\int_0^t 1_{\mathbf{S}\setminus F_i}(Z(s))\,dY_i(s) = 0$ for all $t \geqq 0$,

(d) $Y_i(t) = Y_i(\tau)$ for all $t \geqq \tau$.

An SRBM associated with $(\mathbf{S}, \theta, \Gamma, R)$ and with absorption at the origin is a continuous, $\{\mathscr{F}_t\}$-adapted, d-dimensional process Z together with a family of probability measures $\{P_x, x \in \mathbf{S}\}$ defined on some filtered space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\})$ such that for each $x \in \mathbf{S}$, on $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}, P_x)$, Z is an SRBM associated with $(\mathbf{S}, \theta, \Gamma, R)$ that starts from x and is absorbed at the origin.

4.2 *Existence of an* SRBM *with absorption at the origin*

In this subsection, we construct an SRBM for the data $(\mathbf{S}, \theta, \Gamma, R)$ with absorption at the origin. We achieve this by patching together measures on $(\mathbf{C}, \mathscr{M})$ induced by SRBM's (and their attendant pushing processes) for various troughs.

For each $i \in \mathbf{J}$, let $\mathbf{S}^i = \{x \in \mathbb{R}^d : x_j \geqq 0 \text{ for all } j \in \mathbf{J}\setminus\{i\}\}$, $\mathbf{C}^i = \mathbf{C}^{\mathbf{J}\setminus\{i\}}$, $\mathscr{M}^i = \mathscr{M}^{\mathbf{J}\setminus\{i\}}$, $F^i_j = \{x \in \mathbf{S}^i : x_j = 0\}$ for all $j \in \mathbf{J}$, R^i denote the matrix obtained by deleting the i^{th} column from R. Let $\{Q^i_x, x \in \mathbf{S}^i\}$ denote the family of probability measures on $(\mathbf{C}^i, \mathscr{M}^i)$ defined in Corollary 3.5 with $\mathbf{K} = \mathbf{J}\setminus\{i\}$ there. Let (z^i, y^i) denote the canonical pair of processes on the space $(\mathbf{C}^i, \mathscr{M}^i)$. Let Y^i denote the d-dimensional process whose i^{th} component is identically zero and whose other components are given by those of y^i: $Y^i_j = y^i_j$ for $j < i$ and $Y^i_j = y^i_{j-1}$ for $j > i$. Define $Z^i = z^i$ and $T^i = \inf\{t \geqq 0 : Z^i_i(t) = 0\}$. For $x \in \mathbf{S}$, consider the probability measure $\hat{P}^i_x$ induced on $(\mathbf{C}, \mathscr{M})$ by (Z^i, Y^i, T^i) and Q^i_x via

$$\hat{P}^i_x(A) = Q^i_x((Z^i(\cdot \wedge T^i), Y^i(\cdot \wedge T^i)) \in A) \quad \text{for all } A \in \mathscr{M} .$$

It follows from the Feller continuity of the $\{Q^i_x, x \in \mathbf{S}^i\}$ that the mapping $x \to \hat{P}^i_x(A)$, $x \in \mathbf{S}$, is Borel measurable for each $A \in \mathscr{M}$.

For $x \in \mathbf{S}$ fixed, define

$$r_1 = \max\{j : d(x, F_j) = \max_{i \in \mathbf{J}} d(x, F_i)\} .$$

Let (z, y) denote the pair of canonical processes on $(\mathbf{C}, \mathscr{M})$. On this space, define $P^1_x = \hat{P}^{r_1}_x$, $\tau_0 = 0$, and $\tau_1 = \inf\{t \geqq 0 : z_{r_1}(t) = 0\}$. On $\mathscr{M}_{\tau_1}$, (z, y) under P^1_x has the law of our desired SRBM and its associated Y-process, up until the time τ_1. At τ_1, we need to switch to a new trough to determine the law of our desired processes beyond that time. Indeed, we shall define a sequence $\{(r_n, \tau_n, P_x^n)\}_{n=1}^{\infty}$ such that $\mathbf{S}^{r_n}$ is the trough that $z(\cdot)$ is moving in between the stopping times τ_{n-1} and τ_n and P_x^n describes the law of our desired SRBM together with its associated Y-process up until the time τ_n, where τ_n is the time of switching from the n^{th} to the $(n+1)^{\text{st}}$ trough. For the precise definition of these quantities, suppose $\{(r_i, \tau_i, P_x^i)\}_{i=1}^{n}$ have been defined for some $n \geqq 1$, such that properties (i)–(ii) below hold.

(i) $x^n \equiv \{z(t\wedge\tau_n) - Ry(t\wedge\tau_n) - \theta(t\wedge\tau_n), \mathscr{M}_{t\wedge\tau_n}, t \geqq 0\}$ is a continuous martingale under P_x^n and has mutual variation process: $\langle x_i^n, x_j^n\rangle_t = \Gamma_{ij}(t\wedge\tau_n)$, $t \geqq 0$, and consequently x^n is a $(0, \Gamma)$-Brownian motion starting from x and stopped at τ_n.

(ii) P_x^n-a.s. for $i = 1, \ldots, d$, $y_i(0) = 0$, y_i is non-decreasing, and y_i can increase only when z is on F_i.

We paraphrase this by saying that under P_x^n, $z(\cdot\wedge\tau_n)$ is an SRBM associated with $(\mathbf{S}, \theta, \Gamma, R)$, starting from x and stopped at the time τ_n.

Now, $r_{n+1}, \tau_{n+1}, P_x^{n+1}$ are defined as follows. Let

$$r_{n+1} = 1_{\{\tau_n<\infty\}} \max\{j : d(z(\tau_n), F_j) = \max_{i\in\mathbf{J}} d(z(\tau_n), F_i)\}\,, \tag{4.2}$$

and

$$\tau_{n+1} = \begin{cases} \inf\{t \geqq \tau_n : z_{r_{n+1}}(t) \in F_{r_{n+1}}\} & \text{on } \{\tau_n < \infty\} \\ \infty & \text{on } \{\tau_n = \infty\}\,. \end{cases} \tag{4.3}$$

Since $x \to \hat{P}_x^i(A)$ is Borel measurable for each $A \in \mathscr{M}$, it follows from [26, Theorem 6.1.2] that on $(\mathbf{C}, \mathscr{M})$ there exists a unique probability measure P_x^{n+1} such that $P_x^{n+1} = P_x^n$ on $\mathscr{M}_{\tau_n}$ and on $\{\tau_n < \infty\}$ there is an r.c.p.d. $\tilde{P}_{z,y}^{n+1}$ of $P_x^{n+1}(\cdot\,|\mathscr{M}_{\tau_n})$ such that for P_x^{n+1}-a.e. $(z, y) \in \{\tau_n < \infty\}$,

$$\tilde{P}_{z,y}^{n+1}((\tilde{z}(\cdot+\tau_n), \tilde{y}(\cdot+\tau_n) - \tilde{y}(\tau_n)) \in A) = \hat{P}_{z(\tau_n)}^{r_{n+1}}(A) \quad \text{for all } A \in \mathscr{M}\,, \tag{4.4}$$

where $\tau_n = \tau_n(z, y)$, $r_{n+1} = r_{n+1}(z, y)$, and $(\tilde{z}, \tilde{y})$ denotes a generic element of $\mathbf{C}$, to be distinguished from the particular element (z, y). It follows from the construction of P_x^{n+1} and the properties of the $\{Q_w^i : w \in \mathbf{S}^i, i \in \mathbf{J}\}$ that (i)–(ii) above hold with P_x^{n+1} and τ_{n+1} in place of P_x^n and τ_n, respectively.

For later reference we record the following which is a consequence of the results of Bernard and El Kharroubi [2, Lemma 1]. Since under P_x^n, $z(\cdot\wedge\tau_n)$ is an SRBM stopped at τ_n, there is a constant $c > 1$ that depends only on R, d, $T > 0$, such that for each n, P_x^n-a.s. for any interval $[t_1, t_2]$ in $[0, T]$,

$$\begin{aligned} &\operatorname{Osc}(z^n, [t_1, t_2]) \leqq c \operatorname{Osc}(x^n, [t_1, t_2]) \\ &\text{and } \operatorname{Osc}(y^n, [t_1, t_2]) \leqq c \operatorname{Osc}(x^n, [t_1, t_2])\,, \end{aligned} \tag{4.5}$$

where $z^n \equiv z(\cdot\wedge\tau_n)$, $y^n \equiv y(\cdot\wedge\tau_n)$, and for a continuous function f defined on $[0, T]$,

$$\operatorname{Osc}(f, [t_1, t_2]) = \sup\{|f(t) - f(s)| : t_1 \leqq s \leqq t \leqq t_2\}\,. \tag{4.6}$$

We wish to extend the consistent sequence $\{P_x^n\}_{n=1}^{\infty}$ to a probability measure Q_x° on $(\mathbf{C}, \mathscr{M})$ such that $Q_x^\circ = P_x^n$ on $\mathscr{M}_{\tau_n}$ for all n. Intuitively, this amounts to showing that an SRBM in the orthant defined up to the time τ_n for each n, can be extended continuously to an SRBM defined on the time interval $[0, \tau] \cap [0, \infty)$ where $\tau = \lim_n \tau_n$. The problem is that we do not know a priori that such an SRBM has a well defined limit as $t \uparrow \tau$ on $\{\tau < \infty\}$. To deal with this, we first extend the space of paths on which our measures P_x^n are defined, and define an extension of the P_x^n as $n \to \infty$ there. It will turn out that we can project this limit probability measure back down onto the space of continuous paths in $\mathbf{S} \times \mathbf{S}$, so that it gives a probability measure Q_x° as described above.

For the description of the extended space, let Δ be a point isolated from $\mathbf{S} \times \mathbf{S}$. Let $(\mathbf{S} \times \mathbf{S})^\Delta = (\mathbf{S} \times \mathbf{S}) \cup \{\Delta\}$, $F_i^\Delta = F_i \cup \{\Delta\}$, $F_0^\Delta = \{\Delta\}$. Define $\Omega^\Delta = \{(z, y)\colon [0, \infty) \to (\mathbf{S} \times \mathbf{S})^\Delta$ such that (z, y) is right continuous on $[0, \infty)$ with finite left limits on $(0, \zeta)$ and $(z, y)(t) = \Delta$ for all $t \geqq \zeta(z)$ where $\zeta = \inf\{t \geqq 0 : (z, y)(t) = \Delta\}\}$. Let $\mathcal{M}^\Delta = \sigma\{(z, y)(s) : 0 \leqq s < \infty, (z, y) \in \Omega^\Delta\}$. Define the probability measure $P_x^{n,\Delta}$ on $(\Omega^\Delta, \mathcal{M}^\Delta)$ so that $P_x^{n,\Delta} = P_x^n$ on $\mathbf{C}$ and $P_x^{n,\Delta}(\Omega^\Delta \backslash \mathbf{C}) = 0$. On $(\Omega^\Delta, \mathcal{M}^\Delta)$, define $\tau_0^\Delta = 0$ and define $(\tau_n^\Delta, r_n^\Delta)$ for $n \geqq 1$ inductively such that

$$r_n^\Delta = 1_{\{\tau_{n-1}^\Delta < \infty, z(\tau_{n-1}^\Delta) \in S\}} \max\left\{j : d(z(\tau_{n-1}^\Delta), F_j) = \max_{i \in J} d(z(\tau_{n-1}^\Delta), F_i)\right\},$$

$$\tau_n^\Delta = \begin{cases} \inf\{t \geqq \tau_{n-1}^\Delta : z(t) \in F_{r_n^\Delta}^\Delta\} & \text{on } \{\tau_{n-1}^\Delta < \infty\}, \\ \infty & \text{on } \{\tau_{n-1}^\Delta = \infty\}. \end{cases}$$

Note that if $\tau_n^\Delta = \zeta$ for some $n \geqq 1$, then $\tau_{n+j}^\Delta = \tau_n^\Delta$ for all $j \geqq 0$. Also note that for $(z, y) \in \mathbf{C}$, $r_n(z, y) = r_n^\Delta(z, y)$ and $\tau_n(z, y) = \tau_n^\Delta(z, y)$.

By an extension of the Ionescu–Tulcea theorem (cf. Sharpe [25, Theorem 62.5, p. 290]), there is a probability measure P_x^Δ on $(\Omega^\Delta, \mathcal{M}^\Delta)$ such that $P_x^\Delta = P_x^{n,\Delta}$ on $\mathcal{M}_{\tau_n^\Delta}^\Delta$ for all n. Now, we prove that P_x^Δ can be used to induce a probability measure on $(\mathbf{C}, \mathcal{M})$ that agrees with P_x^n on $\mathcal{M}_{\tau_n}$ for each n. Define $\tau^\Delta = \lim_{n \to \infty} \tau_n^\Delta$, $A = \{\tau_n^\Delta < \tau^\Delta < \infty$ for all $n\}$. By construction,

$$P_x^\Delta(z \text{ is continuous on } [0, \tau^\Delta)) = 1 . \tag{4.7}$$

We want to prove that

$$P_x^\Delta\left(A, \lim_{t \uparrow\uparrow \tau^\Delta} z(t) = 0\right) = P_x^\Delta(A) . \tag{4.8}$$

For this it suffices to show that

$$P_x^\Delta\left(A, \overline{\lim_{t \uparrow\uparrow \tau^\Delta}} |z(t)| > 0\right) = 0 . \tag{4.9}$$

Observe that

$$\begin{aligned} &\left\{A, \overline{\lim_{t \uparrow\uparrow \tau^\Delta}} |z(t)| > 0\right\} \\ &= \bigcup_{m \in \mathbb{N}} \left[\left\{A, \underline{\lim_{t \uparrow\uparrow \tau^\Delta}} |z(t)| > 1/m\right\} \cup \left\{A, \underline{\lim_{t \uparrow\uparrow \tau^\Delta}} |z(t)| \leqq 1/m, \overline{\lim_{t \uparrow\uparrow \tau^\Delta}} |z(t)| \geqq 2/m\right\}\right]. \end{aligned} \tag{4.10}$$

Fix $m \in \mathbb{N}$ and consider $\{A, \underline{\lim}_{t \uparrow\uparrow \tau^\Delta} |z(t)| > 1/m\}$ (the other type of set in (4.10) can be considered in a similar manner and we leave its treatment to the reader).

Note that on $\{\tau_n^\Delta < \zeta\}$, $F_{r_{n+1}^\Delta}$ denotes a face whose distance from $z(\tau_n^\Delta) \in F_{r_n^\Delta}$ is maximal, and so $d(F_{r_{n+1}^\Delta}, z(\tau_n^\Delta)) \geqq 1/(dm)$ when $|z(\tau_n^\Delta)| > 1/m$. Hence, on $\{|z(\tau_n^\Delta)| > 1/m, \tau_{n+1}^\Delta < \zeta\}$, $|z(\tau_{n+1}^\Delta) - z(\tau_n^\Delta)| \geqq 1/(dm)$. By construction, on $\{\tau_n^\Delta < \zeta\}$, under $P_x^\Delta(\cdot|\mathscr{M}_{\tau_n^\Delta}^\Delta)$, the process $z((\cdot + \tau_n^\Delta) \wedge \tau_{n+1}^\Delta)$ is a stopped SRBM associated with $(\mathbf{S}, \theta, \Gamma, R)$ that starts from $z(\tau_n^\Delta)$. Thus, by the same results of Bernard and El Kharroubi [2] as used to obtain (4.5), a.s. under $P_x^\Delta(\cdot|\mathscr{M}_{\tau_n^\Delta}^\Delta)$ the oscillation of $z((\cdot + \tau_n^\Delta) \wedge \tau_{n+1}^\Delta)$ on any finite subinterval of [0, 1], is bounded by a constant times the oscillation of a (θ, Γ)-Brownian motion on that time interval, where the constant depends only on R, d. It follows that there is $s \in (0, 1)$ and $\delta > 0$ such that P_x^Δ-a.s. on $\{\tau_n^\Delta < \zeta\}$,

$$P_x^\Delta(\mathrm{Osc}(z((\cdot + \tau_n^\Delta) \wedge \tau_{n+1}^\Delta), [0, s]) \geqq 1/dm | \mathscr{M}_{\tau_n^\Delta}^\Delta) < 1 - \delta ,$$

and hence

$$P_x^\Delta(\tau_{n+1}^\Delta - \tau_n^\Delta > s | \mathscr{M}_{\tau_n^\Delta}^\Delta) \geqq \delta > 0 .$$

Putting the above together, we obtain P_x^Δ-a.s.,

$$\text{(4.11)} \qquad \sum_{n=1}^{\infty} P_x^\Delta(\tau_n^\Delta < \zeta, |z(\tau_n^\Delta)| > 1/m, \tau_{n+1}^\Delta - \tau_n^\Delta > s | \mathscr{M}_{\tau_n^\Delta}^\Delta)$$

$$\geqq \delta \sum_{n=1}^{\infty} 1_{\{\tau_n^\Delta < \zeta; |z(\tau_n^\Delta)| > 1/m\}} .$$

Hence, up to a P_x^Δ-null set, we have

$$\left\{A, \varlimsup_{t \uparrow\uparrow \tau^\Delta} |z(t)| > 1/m\right\} \subset \left\{\sum_{n=1}^{\infty} 1_{\{\tau_n^\Delta < \zeta; |z(\tau_n^\Delta)| > 1/m\}} = \infty\right\}$$

$$\subset \left\{\sum_{n=1}^{\infty} P_x^\Delta(\tau_n^\Delta < \zeta, |z(\tau_n^\Delta)| > 1/m, \tau_{n+1}^\Delta - \tau_n^\Delta > s | \mathscr{M}_{\tau_n^\Delta}^\Delta) = \infty\right\}.$$

Thus, by an extension of the Borel–Cantelli lemma [11, Corollary 2.3], we have up to a P_x^Δ-null set,

$$\left\{A, \varlimsup_{t \uparrow\uparrow \tau^\Delta} |z(t)| > 1/m\right\} \subset \{\tau_n^\Delta < \zeta, |z(\tau_n^\Delta)| > 1/m, \tau_{n+1}^\Delta - \tau_n^\Delta > s \text{ i.o.}\} .$$

But $\{\tau_{n+1}^\Delta - \tau_n^\Delta > s \text{ i.o.}\} \subset \{\tau^\Delta \equiv \lim_{n \to \infty} \tau_n^\Delta = +\infty\}$. Since A does not meet the last set, it follows that $P_x^\Delta(A, \varlimsup_{t \uparrow\uparrow \tau^\Delta} |z(t)| > 1/m) = 0$.

Now, by (4.7)–(4.8) and by considering the other possibilities: $\tau_n^\Delta = \tau^\Delta$ for some n, and $\tau^\Delta = \infty$, we conclude that P_x^Δ-a.s., $z(\cdot)$ is continuous on $[0, \tau^\Delta)$ and $\lim_{t \uparrow\uparrow \tau^\Delta} z(t) = 0$ on $\{\tau^\Delta < \infty\}$. Moreover, by using the definition of P_x^Δ and the oscillation estimate (4.5) we can show that P_x^Δ-a.s., $y(\cdot)$ is a continuous, non-decreasing $\mathbb{R}^d$-valued process on $[0, \tau^\Delta)$ and $\lim_{t \uparrow\uparrow \tau^\Delta} y(t) < \infty$ on $\{\tau^\Delta < \infty\}$. Thus, on defining

$$\tilde{z}(t) = \begin{cases} z(t) & \text{for } t < \tau^\Delta \\ 0 & \text{for } t \geqq \tau^\Delta, \end{cases}$$

and

$$\tilde{y}(t) = \begin{cases} y(t) & \text{for } t < \tau^{\Delta} \\ \lim_{t \uparrow\uparrow \tau^{\Delta}} y(t) & \text{for } t \geqq \tau^{\Delta}, \end{cases}$$

we have that $P_x^{\Delta}((\tilde{z}, \tilde{y})$ is continuous on $[0, \infty)) = 1$ and $\tilde{x}(t) \equiv \tilde{z}(t \wedge \tau^{\Delta}) - R\tilde{y}(t \wedge \tau^{\Delta}) - \theta(t \wedge \tau^{\Delta})$ defines a P_x^{Δ}-a.s. continuous $\{\mathcal{M}^{\Delta}_{t \wedge \tau^{\Delta}}\}$-adapted martingale with mutual variation process: $\langle \tilde{x}_i, \tilde{x}_j \rangle_t = \Gamma_{ij}(t \wedge \tau^{\Delta})$. For each $x \in \mathbf{S}$, define Q_x° on $(\mathbf{C}, \mathcal{M})$ by

$$Q_x^{\circ}(B) = P_x^{\Delta}((\tilde{z}, \tilde{y}) \in B) \quad \text{for all } B \in \mathcal{M} .$$

Then, we have the following.

Theorem 4.2 *The collection* $\{Q_x^{\circ}, x \in \mathbf{S}\}$ *is a family of probability measures on* $(\mathbf{C}, \mathcal{M})$ *such that for each* $x \in \mathbf{S}$, $Q_x^{\circ} = P_x^n$ *on* $\mathcal{M}_{\tau_n}$ *for each n, and* $Q_x^{\circ}(z(t) = 0, y(t) = y(\tau)$ *for all* $t \geqq \tau) = 1$ *where* $\tau = \inf\{t \geqq 0: z(t) = 0\}$. *Furthermore, the canonical process* $z(\cdot)$ *together with the probability measures* $\{Q_x^{\circ}, x \in \mathbf{S}\}$ *defines an* SRBM *with absorption at the origin on* $(\mathbf{C}, \mathcal{M}, \{\mathcal{M}_t\})$ *and the attendant pushing process can be taken to be the canonical process* $y(\cdot)$.

4.3 Uniqueness of an SRBM *with absorption at the origin*

Theorem 4.3 *Fix* $x \in \mathbf{S}$. *Let Z defined on* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P_x)$ *be an* SRBM *associated with* $(\mathbf{S}, \theta, \Gamma, R)$ *that starts from x and is absorbed at the origin. Let Y be the associated pushing process. Then the law* P_x° *induced on* $(\mathbf{C}, \mathcal{M})$ *by the pair* (Z, Y) *under* P_x *is unique, i.e., the law of an* SRBM *and its attendant pushing process for the data* $(\mathbf{S}, \theta, \Gamma, R)$ *and starting point x with absorption at the origin is unique.*

Proof. Let $\{\tau_n\}, \{r_n\}$ be defined on $\mathbf{C}$ as in Sect. 4.2 and let τ be defined as in Theorem 4.2. Since the canonical processes z and y are continuous and P_x°-a.s., $(z(t), y(t)) = (z(\tau), y(\tau)) = (0, y(\tau))$ for all $t \geqq \tau$, it suffices to show that P_x° is unique on $\mathcal{M}_{\tau_n}$ for each n. In the following we shall use (z, y) and $(\tilde{z}, \tilde{y})$ to denote generic elements of $\mathbf{C}$.

Clearly P_x° is unique on $\mathcal{M}_{\tau_0}$. For an induction proof, suppose that P_x° is unique on $\mathcal{M}_{\tau_n}$ for some $n \geqq 0$. Let $\{P_{z,y}^n : (z, y) \in \mathbf{C}\}$ be an r.c.p.d. of $P_x^{\circ}(\cdot | \mathcal{M}_{\tau_n})$. For each $(z, y) \in \{\tau_n < \infty\}$, define $\tilde{P}_{z,y}^n$ on $(\mathbf{C}^{r_{n+1}(z,y)}, \mathcal{M}^{r_{n+1}(z,y)})$ by

$$\tilde{P}_{z,y}^n(A) = P_{z,y}^n((\tilde{z}((\cdot + \tau_n) \wedge \tau_{n+1}), \tilde{y}((\cdot + \tau_n) \wedge \tau_{n+1}) - \tilde{y}(\tau_n)) \in A)$$

for all $A \in \mathcal{M}^{r_{n+1}(z,y)}$, where $\tau_n = \tau_n(z, y)$ and $\tau_{n+1} = \tau_{n+1}(\tilde{z}, \tilde{y})$. It follows from the properties of an SRBM with absorption at the origin and the uniqueness for stopped SRBM's established in the proof of Theorem 3.4, that for each $i \in \mathbf{J}$ and P_x°-a.e. $(z, y) \in \{\tau_n < \infty, r_{n+1} = i\}$, $\tilde{P}_{z,y}^n = \hat{P}_{z(\tau_n)}^i$, where the latter is defined in Sect. 4.2. Combining this with the uniqueness of P_x° on $\mathcal{M}_{\tau_n}$, it follows that P_x° is unique on $\mathcal{M}_{\tau_{n+1}}$. This completes the induction step. □

Remark. By combining Theorems 4.2 and 4.3, we see that the unique law P_x° defined in Theorem 4.3 is equal to Q_x° defined in Sect. 4.2.

4.4 Tightness and the strong Markov property

Theorem 4.4 *Let K denote a compact subset of* $\mathbf{S}$. *Then the family* $\{Q_x^\circ, x \in K\}$ *of probability measures on* $(\mathbf{C}, \mathscr{M})$ *is tight.*

Proof. Since $z(\cdot)$ is an SRBM with absorption at the origin under each Q_w°, $w \in \mathbf{S}$, it follows from Bernard and El Kharroubi [2, Lemma 1] that the oscillation estimates (4.5)–(4.6) hold Q_w°-a.s. for each $w \in \mathbf{S}$, with $z, y, x \equiv z - Ry$, in place of z^n, y^n, x^n, respectively, where the constant c depends only on R, d, T. By combining this with the tightness for Brownian motions starting from points lying in a compact subset of $\mathbb{R}^d$, the desired tightness follows. □

Lemma 4.5 *For each bounded continuous function* $h: \mathbf{S} \times \mathbf{S} \to \mathbb{R}$ *and* $t \geqq 0$,

$$x \to E_x[h((z, y)(t))]$$

is a Borel measurable function on $\mathbf{S}$, *where* E_x *denotes expectation under* Q_x°.

Proof. For $\{\tau_n\}$, τ, and P_x^n, as defined in Sect. 4.2,

$$E_x[h((z, y)(t))] = E_x[h((z, y)(t \wedge \tau))]$$

$$= \lim_n E_x^n[h((z, y)(t \wedge \tau_n))] ,$$

where E_x^n denotes expectation under P_x^n. By the construction in Sect. 4.2, the expectation $E_x^n[h((z, y)(t \wedge \tau_n))]$ is Borel measurable in x. □

In the following, $C_b(\mathbf{S})$ denotes the set of real-valued, bounded continuous functions defined on $\mathbf{S}$.

Corollary 4.6 *Let* $f \in C_b(\mathbf{S})$, *T be an* $\{\mathscr{M}_t\}$*-stopping time, and* $t \geqq 0$. *Then*

$$E_x[1_{\{T<\infty\}} f(z(T+t)) | \mathscr{M}_T] = 1_{\{T<\infty\}} E_{z(T)}[f(z(t))] , \tag{4.13}$$

where E_x *denotes expectation under* Q_x°. *Thus,* $z(\cdot)$ *together with* $\{Q_x^\circ, x \in \mathbf{S}\}$ *defines a strong Markov process.*

Proof. Now, Q_x°-a.s. on $\{T < \infty\}$, $z(\cdot + T)$ under an r.c.p.d. of $Q_x^\circ(\cdot | \mathscr{M}_T)$ is an SRBM starting from $z(T)$ with absorption at the origin. The uniqueness in law established in Theorem 4.3 together with the measurability established in Lemma 4.5 then yield the desired conclusion. □

4.5 A scaling property

Theorem 4.7 *Suppose* $\theta = 0$. *Then for each* $r > 0$ *and* $x \in \mathbf{S}$,

$$Q_x^\circ(A) = Q_{rx}^\circ(r^{-1}(z, y)(r^2 \cdot) \in A) \quad \text{for each } A \in \mathscr{M} . \tag{4.14}$$

Proof. It follows readily from Theorem 4.2, the definition (4.1) of an SRBM with absorption at the origin, the scaling properties of Brownian motion with zero drift, and the identity: $\tau(z) = r^2 \tau(r^{-1}(z(r^2 \cdot)))$, that under Q_{rx}°, $r^{-1} z(r^2 \cdot)$ is an SRBM starting from x with absorption at the origin and with attendant pushing process $r^{-1} y(r^2 \cdot)$. Then by the uniqueness established in Theorem 4.3, $r^{-1}(z(r^2 \cdot), y(r^2 \cdot))$ under Q_{rx}° has the law of $(z(\cdot), y(\cdot))$ under Q_x°. □

5 SRBM in an orthant-Existence

5.1 An approximating family

The measures $\{Q_x^\circ, x \in \mathbf{S}\}$ will now be used to define an approximation to an SRBM associated with $(\mathbf{S}, \theta, \Gamma, R)$. For this, let $\mathbf{D}$ denote the space of functions $(z, y):[0, \infty) \to \mathbf{S} \times \mathbf{S}$ that are right continuous on $[0, \infty)$ and have finite left limits on $(0, \infty)$. We endow $\mathbf{D}$ with the Skorokhod topology (cf. [10, Sect. 3.5]). The Borel σ-field $\mathcal{M}^{\mathbf{D}}$ associated with the space $\mathbf{D}$ is the same as the σ-field generated by the coordinate maps, i.e., $\mathcal{M}^{\mathbf{D}} = \sigma\{(z, y)(s): 0 \leqq s < \infty, (z, y) \in \mathbf{D}\}$. The restriction of $\mathcal{M}^{\mathbf{D}}$ to $\mathbf{C}$ is $\mathcal{M}$, and so for each $x \in \mathbf{S}$, Q_x° may be thought of as a probability measure on $(\mathbf{D}, \mathcal{M}^{\mathbf{D}})$, concentrated on $\mathbf{C}$. Let $\mathcal{M}_t^{\mathbf{D}} = \sigma\{(z, y)(s): 0 \leqq s \leqq t, (z, y) \in \mathbf{D}\}$ for each $t \geqq 0$.

Since R is completely-$\mathcal{S}$, there is $\lambda > 0$ in $\mathbb{R}^d$ such that $R\lambda > 0$. For $\varepsilon \in (0, 1)$, let $\delta = \varepsilon R\lambda$. For each $x \in \mathbf{S}$ an ε-approximate process will be defined that starts from x and behaves like an SRBM with absorption at the origin prior to the hitting time of the origin, but rather than being absorbed at the origin, the process instantaneously jumps to δ, and then continues from there as if it had started there. The probability measure induced on $(\mathbf{D}, \mathcal{M}^{\mathbf{D}})$ by this process and its attendant pushing process will be denoted by Q_x^δ. A more precise description follows.

For each $k \in \mathbb{N}$, let $(\mathbf{C}^k, \mathcal{M}^k)$ be a distinct copy of $(\mathbf{C}, \mathcal{M})$ and let (Z^k, Y^k) be the canonical pair of processes there: $(Z^k, Y^k)(t, (z^k, y^k)) = (z^k, y^k)(t)$ for all $t \geqq 0$ and $(z^k, y^k) \in \mathbf{C}^k$. Let $\Omega = \prod_{k=1}^\infty \mathbf{C}^k$, $\mathcal{G} = \prod_{k=1}^\infty \mathcal{M}^k$, and let $P_x^\delta = \prod_{k=1}^\infty Q_{x^k}^\circ$, where $x^k = x$ when $k = 1$ and $x^k = \delta$ for $k \geqq 2$. On $(\Omega, \mathcal{G})$, define the pair of processes (Z^δ, Y^δ) as follows. For each $t \geqq 0$ and $\omega = ((z^1, y^1), (z^2, y^2), \ldots) \in \Omega$, define

$$
(5.1) \qquad Z^\delta(t, \omega) = \begin{cases} Z^1(t, (z^1, y^1)) & \text{for } 0 \leqq t < \tau_1^\delta , \\ \vdots & \vdots \\ Z^k(t - \tau_{k-1}^\delta, (z^k, y^k)) & \text{for } \tau_{k-1}^\delta \leqq t < \tau_k^\delta, k \geqq 2 , \\ \vdots & \vdots \\ \Delta & \text{for } \tau_\infty^\delta \leqq t , \end{cases}
$$

$$
(5.2) \qquad Y^\delta(t) = \sum_{k=1}^\infty 1_{\{\tau_{k-1}^\delta \leqq t\}} Y^k((t - \tau_{k-1}^\delta) \wedge \sigma_k^\delta) + \sum_{k=1}^\infty 1_{\{\tau_k^\delta \leqq t\}} \varepsilon\lambda ,
$$

where $\tau_0^\delta = 0$ and for $k \geqq 1$,

$$
\tau_k^\delta(\omega) = \begin{cases} \inf(t \geqq \tau_{k-1}^\delta : Z^k(t - \tau_{k-1}^\delta, (z^k, y^k)) = 0\} & \text{on } \{\tau_{k-1}^\delta < \infty\} , \\ +\infty & \text{on } \{\tau_{k-1}^\delta = \infty\} , \end{cases}
$$

$$
\tau_\infty^\delta = \lim_{k \to \infty} \tau_k^\delta ,
$$

$$
\sigma_k^\delta(z^k, y^k) = \inf\{t \geqq 0 : Z^k(t, (z^k, y^k)) = 0\} ,
$$

and Δ is a point not in $\mathbf{S}$ that is regarded as an isolated point of $\mathbf{S} \cup \{\Delta\}$. Note that on $\{\tau_{k-1}^\delta < \infty\}$, $(\tau_k^\delta - \tau_{k-1}^\delta)(\omega) = \sigma_k^\delta(z^k, y^k)$.

We first observe that $P_x^\delta(\tau_\infty^\delta < \infty) = 0$. This follows from a Borel–Cantelli argument and the fact that $\{\tau_k^\delta - \tau_{k-1}^\delta, k \geqq 2\}$ is a sequence of independent identically distributed random variables that are P_x^δ-a.s. non-zero. Thus, P_x^δ-a.s., (Z^δ, Y^δ) has paths in $\mathbf{D}$.

By Theorem 4.2, for each $k \geqq 1$, Z^k on $(\mathbf{C}^k, \mathcal{M}^k, \{\mathcal{M}_t^k\}, Q_{x^k}^\circ)$ has the following semimartingale decomposition:

$$Z^k(t, (z^k, y^k)) = \begin{cases} x^k + X^k(t, (z^k, y^k)) + RY^k(t, (z^k, y^k)) \in \mathbf{S} & \text{for } t \leqq \sigma_k^\delta \\ 0 & \text{for } t \geqq \sigma_k^\delta , \end{cases}$$

where $B^k \equiv \{X^k(t \wedge \sigma_k^\delta) - \theta(t \wedge \sigma_k^\delta), \mathcal{M}_t^k, t \geqq 0\}$ is a Q_x°-a.s. continuous martingale starting from the origin with mutual variation process: $\langle B_i^k, B_j^k \rangle_t = \Gamma_{ij}(t \wedge \sigma_k^\delta)$ for all $t \geqq 0$, and for each $j \in \mathbf{J}$, Q_x°-a.s., Y_j^k is a continuous, non-decreasing process such that $Y_j^k(0) = 0$ and Y_j^k can increase only when $Z_j^k = 0$. Setting $X^\delta = Z^\delta - RY^\delta$, from the construction and the observation that P_x^δ-a.s.,

$$\sum_{k=1}^{\infty} 1_{\{\tau_k^\delta \leqq t\}} Z^{k-1}(\tau_k^\delta - \tau_{k-1}^\delta, (z^{k-1}, y^{k-1})) = 0 ,$$

we see that P_x^δ-a.s.,

$$X^\delta(t) = x + \sum_{k=1}^{\infty} 1_{\{\tau_{k-1}^\delta \leqq t\}} X^k((t - \tau_{k-1}^\delta) \wedge \sigma_k^\delta) \quad \text{for all } t \geqq 0 . \tag{5.3}$$

Let $\mathcal{G}_t = \sigma\{(Z^\delta, Y^\delta)(s) : 0 \leqq s \leqq t\}$ for each $t \geqq 0$. Using the above it can be verified that $\{X^\delta(t) - \theta t, \mathcal{G}_t, t \geqq 0\}$ as a P_x^δ-a.s. continuous martingale starting from x such that $\langle X_i^\delta, X_j^\delta \rangle_t = \Gamma_{ij} t$ for all $t \geqq 0$. It follows that X^δ is a (θ, Γ)-Brownian motion. Let $\hat{Y}^\delta(t) = Y^\delta(t) - \Sigma_{k=1}^\infty 1_{\{\tau_k^\delta \leqq t\}} \varepsilon\lambda$. Then $\hat{Y}^\delta$ and Y^δ are $\{\mathcal{G}_t\}$-adapted processes such that P_x^δ-a.s. $\hat{Y}^\delta(0) = Y^\delta(0) = 0$ and for each $j \in \mathbf{J}$, $\hat{Y}_j^\delta$ is continuous, non-decreasing and can only increase when $Z_j^\delta = 0$.

The pair of processes (Z^δ, Y^δ), along with the probability measure P_x^δ on $(\Omega, \mathcal{G})$, induce a probability measure Q_x^δ on $(\mathbf{D}, \mathcal{M}^{\mathbf{D}})$ via $Q_x^\delta(A) = P_x^\delta((Z^\delta, Y^\delta) \in A)$ for all $A \in \mathcal{M}^{\mathbf{D}}$. We denote by (Z, Y) the canonical mapping $(Z, Y)(t, (z, y)) = (z, y)(t)$ on $(\mathbf{D}, \mathcal{M}^{\mathbf{D}})$, and let $\{\tau_k, k \geqq 1\}$ be defined by

$$\tau_1 = \inf\{t > 0 : Z(t-) = 0\} ,$$

$$\tau_k = \inf\{t > \tau_{k-1} : Z(t-) = 0\} , \quad \text{for } k \geqq 2 .$$

Then Q_x^δ-a.s., τ_k is the time of the k^{th} jump of Z. Since the process Z^k with measures $\{Q_x^\circ, x \in \mathbf{S}\}$ has the strong Markov property for each $k \geqq 1$, it follows by construction that Z with the measures $\{Q_x^\delta, x \in \mathbf{S}\}$ has the strong Markov property. This plays an essential role in the following.

5.2 *Weak convergence to an* SRBM *in an orthant*

Theorem 5.1 *Let $\{\varepsilon_n\}_{n=1}^\infty$ be a sequence in* $(0, 1)$ *that converges to zero as $n \to \infty$. For each n, let $\delta_n = \varepsilon_n R\lambda$. Then for each $x \in \mathbf{S}$, the family $\{Q_x^{\delta_n}, n \geqq 1\}$ is tight.*

Proof. Note that it is equivalent to show that the sequence of pairs of processes with attendant probability measures: $\{(Z^{\delta_n}, Y^{\delta_n}); P_x^{\delta_n})\}$ forms a tight sequence.

By the completely-$\mathcal{S}$ property of R' (inherited from that of R), there is $v > 0$ in $\mathbb{R}^d$ such that $\eta \equiv R'v > 0$. Accordingly, for any $0 \leqq t_1 < t_2 < \infty$,

$$\begin{aligned} \eta'(Y^\delta(t_2) - Y^\delta(t_1)) &= v'R(Y^\delta(t_2) - Y^\delta(t_1)) \\ &= v'(Z^\delta(t_2) - Z^\delta(t_1)) - v'(X^\delta(t_2) - X^\delta(t_1)) . \end{aligned} \tag{5.4}$$

Now, since X^{δ_n} is a (θ, Γ)-Brownian motion starting from x under each $P_x^{\delta_n}$, $\{(X^{\delta_n}, P_x^{\delta_n})\}$ forms a trivially tight sequence. It follows from this, (5.4), and the fact that Y^{δ_n} is non-decreasing $P_x^{\delta_n}$-a.s., that it suffices to show that $\{(Z^{\delta_n}; P_x^{\delta_n})\}$ forms a tight sequence, or equivalently that $\{(Z; Q_x^{\delta_n})\}$ is tight.

To verify the tightness conditions of [10, Chap. 3, Theorem 7.2(a), Proposition 8.3], since the jumps of Z^{δ_n} are all from the origin to δ_n and $|\delta_n| \to 0$ as $n \to \infty$, and Z together with the $Q_x^{\delta_n}$ has the strong Markov property, it suffices to prove (a) and (b) below. For each $\beta > 0$, let $\tau(\beta) = \inf\{t \geqq 0 : |Z(t) - Z(0)| \geqq \beta\}$.

(a) For each $x \in \mathbf{S}$, $\gamma > 0$ and $t > 0$, there is $M > 0$:

$$\sup_n Q_x^{\delta_n}\left(\sup_{0 \leqq s \leqq t} |Z(s)| \geqq M\right) < \gamma . \tag{5.5}$$

(b) For each $\gamma > 0$ and $\beta > 0$ there is $t > 0$:

$$\overline{\lim_{n \to \infty}} \sup_{x \in \mathbf{S}} Q_x^{\delta_n}(\tau(\beta) \leqq t) < \gamma . \tag{5.6}$$

The compact containment condition (5.5) follows from the facts that Z under $Q_x^{\delta_n}$ has no jumps outside of $\{w \in \mathbf{S} : |w| \leqq \delta_n\}$, Z has the strong Markov property under $\{Q_x^{\delta_n}, x \in \mathbf{S}\}$, for $x \neq 0$, $Q_x^{\delta_n}$ agrees with Q_x° on $\mathscr{M}_{\sigma-}^{\mathbf{D}}$, where $\sigma = \inf\{s > 0 : Z(s-) = 0\}$, and by the proof of Theorem 4.4, the $\{Q_x^\circ\}$ satisfy a compact containment condition of the form of (5.5) with Q_x° in place of $Q_x^{\delta_n}$ that is uniform for x in a compact set.

To prove (5.6), fix $\beta > 0$. Without loss of generality, we may assume $|\delta_n| < 3\beta/8$. Then using the strong Markov property again, for $\sigma(3\beta/8) \equiv \inf\{t \geqq 0 : |Z(t)| \geqq 3\beta/8\}$ we have

$$\begin{aligned}
\sup_x Q_x^{\delta_n}(\tau(\beta) \leqq t) &\leqq \sup_x Q_x^{\delta_n}\left(\sigma(3\beta/8) \leqq t,\, Q_{Z(\sigma(\frac{3\beta}{8}))}^{\delta_n}(\tau(\beta/4) \leqq t)\right) \\
&\leqq \sup_{|x| \geqq \frac{3\beta}{8}} Q_x^{\delta_n}(\tau(\beta/4) \leqq t) \\
&= \sup_{|x| \geqq \frac{3\beta}{8}} Q_x^\circ(\tau(\beta/4) \leqq t) .
\end{aligned} \tag{5.7}$$

By the tightness of the measures $\{Q_x^\circ, x \in \mathbf{S}\}$ established in the proof of Theorem 4.4, it follows that the above can be made arbitrarily small by choosing t sufficiently small. Hence (5.6) holds. □

Theorem 5.2 *Let $x \in \mathbf{S}$ and Q_x be a weak limit point of the sequence $\{Q_x^{\delta_n}\}$ defined in Theorem* 5.1. *Then the following hold.*

(i) $Q_x(\mathbf{C}) = 1$.

(ii) *Under the restriction of Q_x to $(\mathbf{C}, \mathscr{M})$, the canonical process $z(\cdot)$ is an* SRBM *associated with $(\mathbf{S}, \theta, \Gamma, R)$ that starts from x, with attendant pushing process given by the canonical process $y(\cdot)$.*

Proof. Recall that $(Z, Y)(\cdot,(z, y)) = (z, y)(\cdot)$. It follows from the weak convergence and the properties previously established for $X^\delta = Z^\delta - RY^\delta$ under P_x^δ, that under

Q_x, $X \equiv Z - RY$ is an almost surely continuous (θ, Γ)-Brownian motion that starts from x, and $\{X(t) - \theta t, \mathcal{M}_t^{\mathrm{P}}, t \geqq 0\}$ is a martingale. From the fact that Z^{δ_n} only has jumps from the origin of size $|\delta_n|$ under $P_x^{\delta_n}$, it follows that Z has almost surely continuous paths under Q_x. It also follows from the weak convergence and the corresponding properties for Y^{δ_n} under $P_x^{\delta_n}$ that Q_x-a.s., $Y(0) = 0$ and Y is non-decreasing. The almost sure continuity of the paths of Y under Q_x then follows from that for X and Z, in combination with (5.4) with the δ's removed. To see the remaining property of Y under Q_x, namely (ii)(c) of Definition 1.1, note that since all components of Z are non-negative and Y is almost surely non-decreasing, it is enough to show that

$$\int_0^{\cdot} Z(s) \cdot dY(s) = 0 \quad Q_x - \text{a.s.}$$

But it follows from Theorem 2.2 of Kurtz and Protter [18] that the above integral process under Q_x is a weak limit point of the sequence

$$\left\{\left(\int_0^{\cdot} Z^{\delta_n}(s-) \cdot dY^{\delta_n}(s) \colon P_x^{\delta_n}\right)\right\}. \tag{5.8}$$

(The condition (C2.2(i)) in [18] can be verified using (5.4) and stopping X^{δ_n} and Y^{δ_n} at times at which Z^{δ_n} gets a certain distance from the origin, and using the compact containment condition (5.5) to show that these stopping times have a uniform lower bound with high probability.) Now all of the integral processes in (5.8) are zero almost surely, since $P_x^{\delta_n}$-a.s. $Y_j^{\delta_n}$ only increases at times t when $Z_j^{\delta_n}(t) = 0$ or when Z^{δ_n} jumps to δ_n, in which case $Z^{\delta_n}(t-) = 0$. □

Corollary 5.3 *Henceforth let $\{Q_x, x \in \mathbf{S}\}$ denote the restriction to $(\mathbf{C}, \mathcal{M})$ of the family of measures defined in Theorem 5.2. The canonical process $z(\cdot)$ on $(\mathbf{C}, \mathcal{M}, \{\mathcal{M}_t\})$ together with these measures $\{Q_x, x \in \mathbf{S}\}$ defines an* SRBM *associated with $(\mathbf{S}, \theta, \Gamma, R)$, and the other canonical process $y(\cdot)$ gives the associated pushing process.*

5.3 Tightness

Theorem 5.4 *For each $x \in \mathbf{S}$, let P_x denote a probability measure induced on $(\mathbf{C}, \mathcal{M})$ by an* SRBM *and its associated pushing process for the data $(\mathbf{S}, \theta, \Gamma, R)$ and starting point x. Fix $x_0 \in \mathbf{S}$ and suppose $\{x_n\}_{n=1}^{\infty}$ is a sequence in $\mathbf{S}$ such that $x_n \to x_0$ as $n \to \infty$. Then the sequence $\{P_{x_n}, n = 1, 2, \ldots\}$ of probability measures on $(\mathbf{C}, \mathcal{M})$ is tight, and any weak limit point of this sequence together with the canonical processes $z(\cdot)$ on $(\mathbf{C}, \mathcal{M}, \{\mathcal{M}_t\})$ defines an* SRBM *associated with $(\mathbf{S}, \theta, \Gamma, R)$ that starts from x, and the attendant pushing process is given by the other canonical process $y(\cdot)$.*

Proof. The tightness can be proved in a similar manner to Theorem 4.4, using the oscillation estimate for an SRBM (cf. (4.5)) given in Bernard and El Kharroubi [2], and the tightness for (θ, Γ)-Brownian motions with starting points lying in a compact set. The identification of any weak limit of the $\{P_{x_n}\}$ as the law of an SRBM and an associated pushing process, can be justified in a similar manner to the last part of the proof of Theorem 5.2. □

6 SRBM in an orthant – Uniqueness

In this section we will prove the uniqueness part of Theorem 1.3. Unless stated otherwise, throughout this section (z, y) will denote the canonical pair of processes on $(\mathbf{C}, \mathcal{M})$.

6.1 A Girsanov transformation

We first show that it suffices to consider the case where $\theta = 0$.

Lemma 6.1 *Fix $x \in \mathbf{S}$ and $\theta \in \mathbb{R}^d$. Let P_x^θ be a probability measure induced on $(\mathbf{C}, \mathcal{M})$ by an* SRBM *and its associated pushing process for the data $(\mathbf{S}, \theta, \Gamma, R)$ and starting point x. The measure P_x^θ is unique if and only if it is unique for $\theta = 0$.*

Proof. Let $(Z, Y)(z, y) = (z, y)$ for all $(z, y) \in \mathbf{C}$. Now under $P_x^\theta, X = Z - RY$ is a (θ, Γ)-Brownian motion and $\{X(t) - \theta t, \mathcal{M}_t, t \geqq 0\}$ is a continuous martingale. Hence by a Girsanov transformation, there is a unique probability measure P_x^0 on $(\mathbf{C}, \mathcal{M})$ such that

$$(6.1) \qquad \frac{dP_x^0}{dP_x^\theta} = \exp\left(- \theta(X(t) - X(0)) + \frac{t}{2}|\theta|^2 \right) \quad \text{on } \mathcal{M}_t \text{ for all } t \geqq 0\,,$$

and under P_x^0, X is a $(0, \Gamma)$-Brownian motion $\{\mathcal{M}_t\}$-martingale starting from x, and the properties of Y under P_x^θ are retained under P_x^0. It follows that Z on $(\mathbf{C}, \mathcal{M}, \{\mathcal{M}_t\}, P_x^0)$ is an SRBM starting from x with pushing process Y, for the data $(\mathbf{S}, 0, \Gamma, R)$. If the joint law of such an SRBM and its pushing process is unique, it follows that P_x^0 is unique and hence by inversion of the Girsanov transformation and the fact that $X = Z - RY$, P_x^θ is unique. This proves the "if" part of the lemma, the only if part follows in a similar manner. □

Remark. We observe that by Lemma 2.1, the pushing process Y is almost surely a functional of its associated SRBM Z. Combining this with Lemma 6.1 above, we see that to prove the uniqueness in Theorem 1.3 it suffices to prove for each $x \in \mathbf{S}$ that the law of an SRBM associated with the data $(\mathbf{S}, 0, \Gamma, R)$ and starting point x is unique.

6.2 Some crucial estimates

Throughout this subsection we assume that $\theta = 0$. Recall from Sect. 4 that the probability measures $\{Q_x^\circ, x \in \mathbf{S}\}$ defined on $(\mathbf{C}, \mathcal{M})$ are the laws of SRBM's and their associated pushing processes for the data $(\mathbf{S}, 0, \Gamma, R)$ with absorption at the origin.

Lemma 6.2 *Fix $x_0 \in \mathbf{S} \setminus \{0\}$. For each $r \geqq 0$, define $\zeta_r \equiv \inf\{t \geqq 0 : |z(t) - x_0| \geqq r\}$ and $\mathcal{I}_r = \{x \in \mathbf{S} : d(x, \partial \mathbf{S}) > r/8d\}$. There are constants $\kappa > 0, \gamma \in (0, \frac{1}{2}]$ and $\beta \in (0, \frac{1}{2})$, such that for each r satisfying $0 < r \leqq \gamma$ and $x \in \mathbf{S}$ satisfying $|x - x_0| \leqq \beta r$, we have*

$$(6.2) \qquad Q_x^\circ(z(\zeta_r) \in \mathcal{I}_r) \geqq \kappa\,.$$

Remark. The constants κ, γ and β in the above may depend on x_0, but not on r.

Proof. Let $F_0 = \mathbf{S}$. Consider sets $\mathbf{K}_0 \subset \{0, 1, \ldots, d\}$ such that $0 \in \mathbf{K}_0$ and $1 \leqq |\mathbf{K}_0| < d+1$. Then for $x_0 \in \mathbf{S}\backslash\{0\}$, there is a unique set $\mathbf{K}_0$ such that $x_0 \in (\bigcap_{i\in\mathbf{K}_0} F_i)\backslash(\bigcup_{j\in\mathbf{K}_0^c} F_j)$, where $\mathbf{K}_0^c = \{0, 1, \ldots, d\}\backslash\mathbf{K}_0$. We prove by induction on $|\mathbf{K}_0|$ that the result of the lemma holds for all $x_0 \in (\bigcap_{i\in\mathbf{K}_0} F_i)\backslash(\bigcup_{j\in\mathbf{K}_0^c} F_j)$. First consider $|\mathbf{K}_0| = 1$. Then $x_0 \in \mathbf{S}\backslash(\bigcup_{j=1}^d F_j) = \mathbf{S}^\circ$. For $\gamma = \frac{1}{2}d(x_0, \partial\mathbf{S}) \wedge \frac{1}{2}$ and $0 < r \leqq \gamma$, $\partial B(x_0, r) \subset \mathscr{I}_r$ and for all x satisfying $|x - x_0| < r$, $z(\cdot)$ under Q_x° behaves like a (θ, Γ)-Brownian motion up to the time ζ_r, and so it follows that $Q_x^\circ(\zeta_r < \infty) = 1$ and $Q_x^\circ(z(\zeta_r) \in \mathscr{I}_r) = 1$. The desired result then holds with any $\beta \in (0, \frac{1}{2})$ and $\kappa = 1$.

Now, for the induction step, assume that the result holds for all $\mathbf{K}_0$ satisfying $|\mathbf{K}_0| \leqq k$ for some $k \in \{1, \ldots, d-1\}$. Then consider a $\mathbf{K}_0$ satisfying $|\mathbf{K}_0| = k+1$. Let $\mathbf{K} = \mathbf{K}_0\backslash\{0\}$. Fix $x_0 \in (\bigcap_{i\in\mathbf{K}_0} F_i)\backslash(\bigcup_{j\in\mathbf{K}_0^c} F_j) = (\bigcap_{i\in\mathbf{K}} F_i)\backslash(\bigcup_{j\in\mathbf{K}^c} F_j)$, where $\mathbf{K}^c = \mathbf{J}\backslash\mathbf{K}$. For $F \equiv \bigcup_{j\in\mathbf{K}^c} F_j$, we have $d(x_0, F) > 0$. Let $\tau = \inf\{t \geqq 0 : z(t) \in F\}$. Then by the proof of Theorem 3.4 and Corollary 3.5, we have for $x \in \mathbf{S}$ that $z(\cdot \wedge \tau)$ under Q_x° is equivalent in law to $Z(\cdot \wedge \hat{\tau})$ under P_x, where $\hat{\tau} = \inf\{t \geqq 0 : Z(t) \in F\}$ and Z and P_x denote the process and associated probability measure, respectively, constructed in the proof of Theorem 3.3 for the trough data $(\mathbf{S}^\mathbf{K}, \theta, \Gamma, R^\mathbf{K})$. Let $\gamma = \frac{1}{2}d(x_0, F) \wedge \frac{1}{2}$ and $\varepsilon = \gamma/(4c+4)$, where the constant $c > 0$ will be determined later. First note that $Z_\mathbf{K} = X_\mathbf{K} + R_\mathbf{K} Y$ under P_x is an SRBM associated with $(\mathbb{R}_+^k, 0, \Gamma_\mathbf{K}, R_\mathbf{K})$, and so by the oscillation estimates of Bernard and El Kharroubi [2, Lemma 1], there is a constant $c_1 > 1$ that does not depend on x such that P_x-a.s.

$$|Y(1)| \leqq c_1 \max_{0\leqq s\leqq 1} |X_\mathbf{K}(s) - X_\mathbf{K}(0)| \,. \tag{6.3}$$

Recall the construction of Z and P_x, and the definitions of $\tilde{B}$, H, Λ, from Theorem 3.3 and Proposition 3.2. By the independence of $\tilde{B}$ from $(X_\mathbf{K}, Y)$, and the fact that both $\tilde{B}(\cdot) - \tilde{B}(0)$ and $X_\mathbf{K}(\cdot) - X_\mathbf{K}(0)$ are Brownian motions starting from the origin, it follows that for each $\delta \in (0, \frac{1}{2})$ there exists $t_\delta \in (0, 1)$ such that for all $x \in \mathbf{S}$ we have

$$P_x\left(\max_{0\leqq s\leqq t_\delta} |\tilde{B}(s) - \tilde{B}(0)| \leqq \varepsilon/2,\ \max_{0\leqq s\leqq t_\delta} |X_\mathbf{K}(s) - X_\mathbf{K}(0)| \leqq \varepsilon/2c_1\right) \geqq 1 - \delta \,. \tag{6.4}$$

Let a_1 denote the operator norm of $\Lambda' \Gamma_\mathbf{K}^{-1}$ as a linear operator from $\mathbb{R}^k$ into $\mathbb{R}^{d-k}$ and similarly let a_2 and a_3 denote the operator norms of the matrices H and $R^\mathbf{K}$ respectively. Then,

$$|Z_{|\mathbf{K}}(t) - Z_{|\mathbf{K}}(0)| \leqq a_1 |X_\mathbf{K}(t) - X_\mathbf{K}(0)| + a_2 |\tilde{B}(t) - \tilde{B}(0)| + a_3 |Y(t)| \,.$$

Hence, if each of the magnitudes in the right member above is less than or equal to $\varepsilon/2$ and $Z(0) = x$ where $|x - x_0| \leqq \varepsilon/2$, then for $c \equiv (a_1 + a_2 + a_3)$ we have

$$\begin{aligned} |\mathbf{Z}_{|\mathbf{K}}(t) - x_{0|\mathbf{K}}| &\leqq |Z_{|\mathbf{K}}(t) - x_{|\mathbf{K}}| + |x_{|\mathbf{K}} - x_{0|\mathbf{K}}| \\ &\leqq \frac{c\varepsilon}{2} + \frac{\varepsilon}{2} \\ &\leqq \gamma/8 < \gamma/4 \,. \end{aligned}$$

It follows from (6.4), (6.3), and the facts that Y is P_x-a.s. non-decreasing, $c_1 > 1$, and $t_\delta \in (0, 1)$, that for $x \in \mathbf{S}$ satisfying $|x - x_0| \leqq \varepsilon/2$ we have

$$(6.5) \qquad P_x\left(\max_{0 \leqq s \leqq t_\delta} |Z_{|\mathbf{K}} - x_{0|\mathbf{K}}| < \gamma/4\right) \geqq 1 - \delta .$$

Now, since $R'_{\mathbf{K}}$ is completely-$\mathscr{S}$, there is $v \in \mathbb{R}^k$ such that $v > 0, |v| = 1$ and $R'_{\mathbf{K}} v > 0$. Thus, $v \cdot Z_{\mathbf{K}} = v \cdot X_{\mathbf{K}} + v' R_{\mathbf{K}} Y$ where P_x-a.s., $v' R_{\mathbf{K}} Y \geqq 0$ and so

$$(6.6) \qquad v \cdot Z_{\mathbf{K}} \geqq v \cdot X_{\mathbf{K}} .$$

Note that under P_x, $v \cdot (X_{\mathbf{K}} - x_{\mathbf{K}})$ is a one-dimensional Brownian motion that almost surely starts from the origin. Hence, for fixed $\delta \in (0, \frac{1}{2})$ and the associated $t_\delta \in (0, 1)$, there exists $\alpha \in (0, \gamma/4)$ not depending on x such that

$$(6.7) \qquad P_x\left(\max_{0 \leqq s \leqq t_\delta} (v \cdot (X_{\mathbf{K}}(s) - x_{\mathbf{K}})) \geqq \alpha\right) \geqq 2\delta .$$

Note that $v > 0$ and $x_{\mathbf{K}} - x_{0\mathbf{K}} = x_{\mathbf{K}} \geqq 0$, since $x_0 \in \bigcap_{i \in \mathbf{K}} F_i$. By combining this with (6.6) and (6.7), we obtain

$$(6.8) \qquad P_x\left(\max_{0 \leqq s \leqq t_\delta} (v \cdot (Z_{\mathbf{K}}(s) - x_{0\mathbf{K}})) \geqq \alpha\right) \geqq 2\delta \quad \text{for all } x \in \mathbf{S} .$$

Since $|v| = 1$, $v \cdot (Z_{\mathbf{K}}(s) - x_{0\mathbf{K}}) \leqq |Z_{\mathbf{K}}(s) - x_{0\mathbf{K}}|$. By combining this with (6.5) and (6.8), we obtain for all $x \in \mathbf{S}$ satisfying $|x - x_0| \leqq \varepsilon/2$,

$$(6.9) \qquad P_x\left(\max_{0 \leqq s \leqq t_\delta} |Z_{\mathbf{K}}(s) - x_{0\mathbf{K}}| \geqq \alpha, \quad \max_{0 \leqq s \leqq t_\delta} |Z_{|\mathbf{K}}(s) - x_{0|\mathbf{K}}| < \gamma/4\right) \geqq \delta .$$

Now let

$$\Upsilon = \{x \in \mathbf{S} : |x_{\mathbf{K}} - x_{0\mathbf{K}}| = \alpha \text{ and } |x_{|\mathbf{K}} - x_{0|\mathbf{K}}| \leqq \gamma/4$$
$$\text{or} \quad |x_{\mathbf{K}} - x_{0\mathbf{K}}| < \alpha \text{ and } |x_{|\mathbf{K}} - x_{0|\mathbf{K}}| = \gamma/4\} ,$$

and

$$\Upsilon^* = \{x \in \mathbf{S} : |x_{\mathbf{K}} - x_{0\mathbf{K}}| = \alpha \text{ and } |x_{|\mathbf{K}} - x_{0|\mathbf{K}}| \leqq \gamma/4\} .$$

Let $\sigma_\Upsilon = \inf\{t \geqq 0 : Z(t) \in \Upsilon\}$. Then from (6.9), for all $x \in \mathbf{S}$ satisfying $|x - x_0| \leqq \frac{\varepsilon}{2} \wedge \alpha$ we have

$$P_x(Z(\sigma_\Upsilon) \in \Upsilon^*) \geqq \delta .$$

Now, for each $u \in \Upsilon^*$, $d(u, F) \geqq 3\gamma/2$ and there is $\mathbf{L}_0 \subset \{0, 1, \ldots, d\}$ such that $0 \in \mathbf{L}_0$, $|\mathbf{L}_0| \leqq k$, and $u \in (\bigcap_{i \in \mathbf{L}_0} F_i) \backslash (\bigcup_{j \in \mathbf{L}_0^c} F_j)$. By the induction hypothesis and the fact that $z(\cdot \wedge \tau)$ under Q_x° is equivalent in law to $Z(\cdot \wedge \hat{\tau})$ under P_x for each $x \in \mathbf{S}$, it follows that there exists $\kappa(u) > 0$, $\gamma(u) \in (0, \gamma/4)$, and $\beta(u) \in (0, \frac{1}{2})$, such that for all $x \in \mathbf{S}$ satisfying $|x - u| \leqq \beta(u)\gamma(u)$,

$$P_x(Z(\zeta(u)) \in \mathscr{I}_{\gamma(u)}) \geqq \kappa(u) > 0 ,$$

where $\zeta(u) = \inf\{t \geqq 0 : |Z(t) - u| \geqq \gamma(u)\}$. Since Υ^* is compact, finitely many of the balls $B_u = \{x \in \mathbf{S} : |x - u| < \beta(u)\gamma(u)\}$, $u \in \Upsilon^*$, cover Υ^*. Let $B_{u_1}, B_{u_2}, \ldots, B_{u_n}$

be such a covering, and with each point $x \in \Upsilon^*$ associate a point $u(x) \in \{u_1, u_2, \ldots, u_n\}$ such that $x \in B_{u(x)}$. Define a stopping time $\hat{\sigma}$ for Z as follows:

$$\hat{\sigma} = \begin{cases} \sigma_\Upsilon & \text{if } Z(\sigma_\Upsilon) \notin \Upsilon^* \text{ or } \sigma_\Upsilon = +\infty \\ \sigma_\Upsilon + \zeta(u(Z(\sigma_\Upsilon))) \circ Z(\cdot + \sigma_\Upsilon)) & \text{if } Z(\sigma_\Upsilon) \in \Upsilon^* \text{ and } \sigma_\Upsilon < +\infty \ . \end{cases}$$

Thus if Z first hits Υ^* on exiting Υ, then $\hat{\sigma}$ is the first time after that time that Z exits the ball B_u with center $u = u(Z(\sigma_\Upsilon))$. Let $U = \bigcup_{i=1}^n (\mathscr{I}_{\gamma(u_i)} \cap \partial B_{u_i})$ and $\kappa^* = \min_{1 \leqq i \leqq n} \kappa(u_i)$. Then by the strong Markov property of Z under P_x (which follows from Corollary 3.5), we conclude that for all $x \in \mathbf{S}: |x - x_0| \leqq \frac{\varepsilon}{2} \wedge \alpha$,

$$\begin{aligned} P_x(Z(\hat{\sigma}) \in U) &\geqq P_x(Z(\sigma_\Upsilon) \in \Upsilon^*, P_{Z(\sigma_\Upsilon)}(Z(\zeta(u_\Upsilon)) \in \mathscr{I}_{\gamma(u_\Upsilon)})) \\ &\geqq \delta\kappa^* > 0 \ , \end{aligned}$$

where $u_\Upsilon = u(Z(\sigma_\Upsilon))$. Observe that if $\gamma^* = \min_{1 \leqq i \leqq n} \gamma(u_i)$, then $d(U, \partial\mathbf{S}) > \gamma^*/8d$. Also, for $0 \leqq t \leqq \hat{\sigma}$,

$$|Z(t) - x_0| \leqq \sup_{u \in \Upsilon} (|u - x_0|) + \gamma/4 \leqq \alpha + \gamma/4 + \gamma/4 \leqq 3\gamma/4 \ .$$

For $r > 0$, $B(x_0, r) \equiv \{x \in \mathbf{S}: |x - x_0| < r\}$. Since $\mathscr{I}_\gamma \cap \partial B(x_0, \gamma)$ has positive surface measure as a subset of $\partial B(x_0, \gamma)$, and Z under P_x behaves in $\mathbf{S}^\circ$ like a d-dimensional Brownian motion, there exists $\delta' > 0$ such that

$$\inf_{x \in U} P_x(Z(\hat{\zeta}_\gamma) \in \mathscr{I}_\gamma) \geqq \delta' > 0 \ ,$$

where $\hat{\zeta}_\gamma = \inf\{t \geqq 0: |Z(t) - x_0| \geqq \gamma\}$. Then for all $x \in \mathbf{S}: |x - x_0| \leqq \frac{\varepsilon}{2} \wedge \alpha$, by the strong Markov property of Z under P_x,

$$\begin{aligned} P_x(Z(\hat{\zeta}_\gamma) \in \mathscr{I}_\gamma) &\geqq P_x(Z(\hat{\sigma}) \in U, P_{Z(\hat{\sigma})}(Z(\hat{\zeta}_\gamma) \in \mathscr{I}_\gamma)) \\ &\geqq \delta\kappa^*\delta' > 0 \ . \qquad (6.10) \end{aligned}$$

Setting $\kappa = \delta\kappa^*\delta'$ and $\beta = (\frac{\varepsilon}{2} \wedge \alpha)\frac{1}{\gamma} = \frac{1}{8(c+1)} \wedge \frac{\alpha}{\gamma}$, for all $x \in \mathbf{S}: |x - x_0| \leqq \beta\gamma$, we have

$$P_x(Z(\hat{\zeta}_\gamma) \in \mathscr{I}_\gamma) \geqq \kappa > 0 \ . \qquad (6.11)$$

The transition from (6.11) to that with $r \in (0, \gamma]$ in place of γ is achieved by scaling as follows. By the uniqueness in law of the SRBM Z under P_x, it follows in a similar manner to that in Theorem 4.7 that for each $\lambda > 0$, $x \in \mathbf{S}^{\mathrm{K}}$, the process $\lambda^{-1}(Z(\lambda^2\cdot) - x_0) + x_0$ under $P_{x_0 + \lambda x}$ is equivalent in law to $Z(\cdot)$ under $P_{x_0 + x}$. By combining this with (6.11) and the facts that $d(\partial B(x_0, \gamma), F) \geqq \gamma$ and that x_0 only has non-zero components in the directions indexed by $\mathbf{K}^c$, we conclude that for each $0 < r \leqq \gamma$ and $x \in \mathbf{S}: |x - x_0| \leqq \beta r$, we have

$$P_x(Z(\hat{\zeta}_r) \in \mathscr{I}_r) \geqq \kappa \ .$$

By the equivalence of $Z(\cdot \wedge \hat{\tau})$ under P_x to $z(\cdot \wedge \tau)$ under Q_x°, (6.2) follows, and our induction argument is complete. □

Lemma 6.3 *Let* $\Lambda = \{x \in \mathbf{S}: |x| = 1\}$. *Fix* $x_0 \in \Lambda$ *and for each* $r \geqq 0$, *let* $\zeta_r \equiv \inf\{t \geqq 0: |z(t) - x_0| \geqq r\}$. *There are constants* $\kappa > 0, \gamma \in (0, \frac{1}{2}], \beta \in (0, \frac{1}{4})$ *such that for each r satisfying* $0 < r \leqq \gamma$ *and* $x \in \mathbf{S}$ *satisfying* $|x - x_0| \leqq \beta r$, *we have*

$$Q_x^\circ(z(\zeta_r) \in A_r) \geqq \kappa , \tag{6.12}$$

whenever $A_r \subset \mathbf{S} \cap \partial B(x_0, r)$ *such that* $|A_r| \geqq \frac{1}{2}|\mathbf{S} \cap \partial \mathbf{B}(x_0, r)|$. *Here* $|\cdot|$ *denotes surface measure on* $\partial B(x_0, r)$. *The constants* κ, γ, *and* β *can be chosen to be independent of r.*

Proof. For $x_0 \in \Lambda$, by Lemma 6.2, there exists $\kappa_1 > 0, \gamma \in (0, \frac{1}{2}], \beta \in (0, \frac{1}{2})$ such that for all $0 < r \leqq \gamma$ and $x \in \mathbf{S}: |x - x_0| \leqq \beta r/2$,

$$Q_x^\circ(z(\zeta_{r/2}) \in \mathscr{I}_{r/2}) \geqq \kappa_1 .$$

Let $\mathbf{K}^c = \{i \in \mathbf{J}: x_0 \notin F_i\}$ and $\mathbf{K} = \mathbf{J} \backslash \mathbf{K}^c$. By the proof of Lemma 6.2, we may assume that $d(x_0, F_i) \geqq 2\gamma$ for all $i \in \mathbf{K}^c$. Let $\alpha \in (0, \frac{1}{32d})$ such that for all $r \in (0, \gamma)$,

$$|\{x \in \mathbf{S} \cap \partial B(x_0, r): d(x, \partial \mathbf{S}) \leqq \alpha r\}| \leqq \tfrac{1}{4}\,|\mathbf{S} \cap \partial B(x_0, r)| . \tag{6.13}$$

Note that for fixed r, such an α exists because the left member above tends to zero as $\alpha \downarrow 0$. By Euclidean scaling, α can be chosen independent of $r \in (0, \gamma)$. Let $U_r = \{x \in B(x_0, r) \cap \mathbf{S}: d(x, \partial \mathbf{S}) > \alpha r\}$. Then, for $0 < r \leqq \gamma$,

$$U_r = \{(r(x - x_0)/\gamma) + x_0 : x \in U_\gamma\} , \tag{6.14}$$

$\partial B(x_0, r/2) \cap \mathscr{I}_{r/2} \subset U_r$, and $\partial B(x_0, r/2) \cap \mathscr{I}_{r/2}$ is at least distance $r/32d$ from ∂U_r. Suppose that $A_r \subset \mathbf{S} \cap \partial B(x_0, r)$ is such that $|A_r| \geqq \frac{1}{2}|\mathbf{S} \cap \partial B(x_0, r)|$. Then using (6.13) we obtain

$$|\partial U_r \cap A_r| \geqq \tfrac{1}{4}\,|\mathbf{S} \cap \partial B(x_0, r)| \geqq c|\partial U_r| ,$$

where $c > 0$ is independent of r by scaling. Now, from a fixed point in U_γ, harmonic measure on ∂U_γ is bounded below by a constant times a power of the surface measure on ∂U_γ [5, Corollary 3]. It follows from this, together with Harnack's inequality, Brownian scaling and the scaling properties of U_r and $\mathscr{I}_{r/2}$, that there is a constant $\kappa_2 > 0$, independent of A_r and r such that for all $0 < r \leqq \gamma$ and $\sigma_r \equiv \inf\{t \geqq 0: z(t) \notin U_r\}$,

$$\inf_{x \in \partial B(x_0, r/2) \cap \mathscr{I}_{r/2}} Q_x^\circ(z(\sigma_r) \in A_r) \geqq \kappa_2 .$$

Combining the above with the strong Markov property for z under Q_x°, we conclude that for all $0 < r \leqq \gamma$ and $x \in \mathbf{S}: |x - x_0| \leqq \beta r/2$,

$$\begin{aligned} Q_x^\circ(z(\zeta_r) \in A_r) &\geqq Q_x^\circ(z(\zeta_{r/2}) \in \mathscr{I}_{r/2}, Q_{z(\zeta_{r/2})}^\circ(z(\zeta_r) \in A_r)) \\ &\geqq \kappa_1 \kappa_2 > 0 , \end{aligned}$$

where $\kappa \equiv \kappa_1 \kappa_2$ does not depend on A_r or r. Relabelling $\beta/2$ as β yields the desired result. □

Lemma 6.4 *For each* $r > 0$, *let* $\tau_r = \inf\{t \geqq 0: |z(t)| = r\}$. *There is a finite constant C such that for each* $r > 0$, *any* $x \in \mathbf{S}$ *satisfying* $|x| \leqq r$, *and* P_x *a probability measure induced on* $(\mathbf{C}, \mathscr{M})$ *by an* SRBM *and its attendant pushing process for the data* $(\mathbf{S}, 0, \Gamma, R)$ *and starting point x, we have*

$$E^x[\tau_r] \leqq Cr^2 . \tag{6.15}$$

Proof. For each $i \in \mathbf{J}$, let v^i denote the i^{th} column of the reflection matrix R. Since R' is completely-$\mathscr{S}$, there is $v \in \mathbb{R}^d$ such that $v > 0$ and $v \cdot v^i > 0$ for all $i \in \mathbf{J}$. Let $g(\alpha) = \alpha^2/2$ for all $\alpha \in \mathbb{R}_+$ and define $f(x) = g(v \cdot x)$ for all $x \in \mathbf{S}$. Note that $v^i \cdot \nabla f(x) = (v^i \cdot v) g'(v \cdot x) \geqq 0$ for all $x \in \mathbf{S}$, $i \in \mathbf{J}$. Fix $x \in \mathbf{S}$ and let P_x be as described in the statement of the lemma. By applying Itô's formula to the SRBM z under P_x and letting $b = z - Ry$ we obtain P_x-a.s. for all $t \geqq 0$,

$$\begin{aligned} f(z(t)) &= f(z(0)) + \int_0^t \nabla f(z(s)) \cdot db(s) \\ &\quad + \sum_{i=1}^d \int_0^t v^i \cdot \nabla f(z(s)) dy_i(s) + \int_0^t Lf(z(s)) ds , \end{aligned} \tag{6.16}$$

where

$$L = \frac{1}{2} \sum_{i,j=1}^d \Gamma_{ij} \frac{\partial^2}{\partial x_i \partial x_j} .$$

Now, b is a continuous zero drift Brownian motion $\{\mathscr{M}_t\}$-martingale under P_x, and so the stochastic integral with respect to b above defines an almost surely continuous local $\{\mathscr{M}_t\}$-martingale under P_x. By the assumptions on y under P_x and f, the term in (6.16) involving the sum over $i \in \{1, \ldots, d\}$ defines an almost surely continuous non-decreasing $\{\mathscr{M}_t\}$-adapted process under P_x. Thus, the continuous process

$$\left\{ f(z(t)) - f(z(0)) - \int_0^t Lf(z(s))\, ds, \mathscr{M}_t, t \geqq 0 \right\}$$

is a P_x-local submartingale. For $a \geqq v \cdot x$, let $\sigma_a = \inf\{s \geqq 0 : v \cdot z(s) \geqq a\}$. Since f and its derivatives are bounded on the compact set $\{u \in \mathbf{S} : v \cdot u \leqq a\}$, when the above local submartingale is stopped at σ_a it yields a P_x-submartingale, and so

$$E^{P_x}[g(v \cdot z(t \wedge \sigma_a))] - g(v \cdot x) \geqq \frac{1}{2} E^{P_x}\left[\int_0^{t \wedge \sigma_a} v' \Gamma v g''(v \cdot z(s)) ds \right] .$$

Since $0 \leqq g(v \cdot u) \leqq a^2/2$ for $u \in \mathbf{S}$ satisfying $0 \leqq v \cdot u \leqq a$, and $g'' = 1$, this yields

$$a^2/2 \geqq \frac{1}{2} v' \Gamma v E^{P_x}[t \wedge \sigma_a] .$$

Since $v > 0$ and Γ is strictly positive definite, we have $v' \Gamma v > 0$. Letting $t \uparrow \infty$ in the above, we obtain

$$E^{P_x}[\sigma_a] \leqq \frac{a^2}{v' \Gamma v} = ca^2 ,$$

where $c = (v' \Gamma v)^{-1}$. Then, for fixed $r > 0$ and $x \in \mathbf{S} : |x| \leqq r$, we have

$$E^{P_x}[\tau_r] \leqq E^{P_x}[\sigma_{|v|r}] \leqq c|v|^2 r^2 .$$

Thus, (6.15) holds with $C = (v' \Gamma v)^{-1} |v|^2$. □

6.3 An ergodic property

Let $\Lambda \equiv \{x \in \mathbf{S}: |x| = 1\}$. For each $x \in \Lambda$, define the sub-probability measure $Q(x, \cdot)$ on the Borel σ-field $\mathscr{B}(\Lambda)$ of Λ by

$$Q(x, A) = Q_x^\circ(z(\tau_2)/2 \in A, \tau_2 < \tau_0) \quad \text{for all } A \in \mathscr{B}(\Lambda)\,, \tag{6.17}$$

where $\tau_r = \inf\{t \geqq 0: |z(t)| = r\}$ for $r \geqq 0$. We now prove several properties of Q.

Lemma 6.5 *For $x \in \mathbf{S}\backslash\{0\}$ and $r = |x|$,*

$$Q_x^\circ(z(\tau_{2r})/2r \in A, \tau_{2r} < \tau_0) = Q\left(\frac{x}{|x|}, A\right) \quad \textit{for all } A \in \mathscr{B}(\Lambda)\,.$$

Proof. This scaling property is an immediate consequence of Theorem 4.7. □

Let $C(\Lambda)$ denote the space of (bounded) continuous real-valued functions defined on Λ endowed with the sup norm topology. For each $f \in C(\Lambda)$, define

$$(Qf)(x) = \int_\Lambda Q(x, dy) f(y) \quad \text{for all } x \in \Lambda\,.$$

Lemma 6.6 *For each $f \in C(\Lambda)$, $Qf \in C(\Lambda)$. Moreover, Q is a compact operator on $C(\Lambda)$.*

Proof. This lemma is proved in the same manner as Theorem 3.2 of Kwon and Williams [19], except that Lemma 6.3, β, ζ_r of this paper take the place of Lemma 3.3, $\frac{1}{4}$, η_r in [19]. □

We now prove the main result of this subsection.

Lemma 6.7 *Suppose G and H are continuous real-valued functions on Λ such that $H \geqq 0$ and $H \not\equiv 0$. Let $\{\nu_n\}$ be a sequence of probability measures on $(\Lambda, \mathscr{B}(\Lambda))$. Then*

$$\frac{\int_\Lambda (Q^n G)(x)\nu_n(dx)}{\int_\Lambda (Q^n H)(x)\nu_n(dx)} \to C(G, H) \quad \textit{as } n \to \infty, \tag{6.18}$$

where $C(G, H)$ is a finite constant depending only on Q, G, H, and not on the sequence $\{\nu_n\}$.

Proof. This can be proved in an analogous manner to that in Kwon and Williams [19, Theorem 3.3] and Bass and Pardoux [1, Theorem 5.4]. All that needs to be verified for our particular situation here is that Q satisfies the hypotheses of the Krein–Rutman theorem [17, Theorem 6.3].

By Lemma 6.6, $Q: C(\Lambda) \to C(\Lambda)$ is compact. It is therefore enough to prove that Q is strongly positive on the cone $K = \{f \in C(\Lambda): f \geqq 0 \text{ on } \Lambda\}$. For this, let $f \in K, f \not\equiv 0$. Then there is $\bar{x} \in \Lambda\backslash\partial\mathbf{S}$, $\varepsilon \in (0, \frac{1}{4})$ and $c > 0$ such that $f(x) > c$ whenever $x \in \Lambda \cap B(\bar{x}, \varepsilon)$. By Lemma 6.2, for each $x_0 \in \Lambda$ there are constants $\kappa(x_0) > 0$, $\gamma(x_0) \in (0, \frac{1}{2}]$, and $\beta(x_0) \in (0, \frac{1}{2})$ such that for all $x \in \Lambda$: $|x - x_0| \leqq \beta(x_0)\gamma(x_0)$, we have

$$Q_x^\circ(z(\zeta_{\gamma(x_0)}) \in \mathscr{I}_{\gamma(x_0)}) \geqq \kappa(x_0)\,,$$

where $\zeta_{\gamma(x_0)} = \inf\{t \geqq 0 : |z(t) - x_0| \geqq \gamma(x_0)\}$ and $\mathscr{I}_{\gamma(x_0)} = \{u \in \mathbf{S} : d(u, \partial\mathbf{S}) > \gamma(x_0)/8d\}$. Since Λ is compact, there is a finite subcollection of the open balls $\{B(x_0, \beta(x_0)\gamma(x_0)) : x_0 \in \Lambda\}$ that covers Λ. Let $\{B_{u_i}, i = 1, \ldots, n\}$ be such a collection where the center of B_{u_i} is at u_i. To each $x \in \Lambda$, associate $u(x) \in \{u_1, \ldots, u_n\}$ such that $x \in B_{u(x)}$. Then, for all $x \in \Lambda$,

$$Q_x^\circ(z(\zeta_{\gamma(u(x))}) \in \mathscr{I}_{\gamma(u(x))}) \geqq \kappa ,$$

where $\kappa \equiv \min\{\kappa_i, i = 1, \ldots, n\}$. Let $U = \bigcup_{i=1}^n (\mathscr{I}_{\gamma(u_i)} \cap \partial B_{u_i})$. Then U is a positive distance from $\partial\mathbf{S}$ and $U \subset \{u \in \mathbf{S} : |u| \leqq \frac{3}{2}\}$, and so since under Q_x°, z behaves like a Brownian motion in $\mathbf{S}^\circ$ until it reaches $\partial\mathbf{S}$, there is $\delta > 0$ such that

$$\inf_{x \in U} Q_x^\circ(z(\tau_2) \in B(2\bar{x}, 2\varepsilon), \tau_2 < \tau_0) \geqq \delta .$$

Using the strong Markov property of z under Q_x°, we then obtain for all $x \in \Lambda$,

$$Q_x^\circ(z(\tau_2)/2 \in B(\bar{x}, \varepsilon), \tau_2 < \tau_0) \geqq \kappa\delta > 0 .$$

Hence, for each $x \in \Lambda$,

$$\begin{aligned}(Qf)(x) &\geqq cQ_x^\circ(z(\tau_2)/2 \in B(\bar{x}, \varepsilon), \tau_2 < \tau_0)\\ &\geqq c\kappa\delta > 0 .\end{aligned}$$

This completes the verification that Q is strongly positive and then the desired result follows as in Bass and Pardoux [1, Theorem 5.4]. □

6.4 Uniqueness

Observe that in order to prove the uniqueness statement in Theorem 1.3, since we have existence of an SRBM for $(\mathbf{S}, \theta, \Gamma, R)$ starting from each $x \in \mathbf{S}$, it suffices to prove uniqueness of a family of probability measures $\{P_x, x \in \mathbf{S}\}$, where for each $x \in \mathbf{S}$, P_x is a probability measure induced on $(\mathbf{C}, \mathscr{M})$ by an SRBM and its attendant pushing process for the data $(\mathbf{S}, \theta, \Gamma, R)$ and starting point x. Now recall from the Remark in Sect. 6.1 that it suffices to consider $\theta = 0$ and to prove that z together with $\{P_x, x \in \mathbf{S}\}$ is unique. Further, by a Markov selection theorem, we may assume that z together with the $\{P_x, x \in \mathbf{S}\}$ has the strong Markov property. (This can be proved in a similar manner to that in Theorems 12.2.4, 12.2.3 of [26]. The key is to verify the analogous hypotheses of the supporting Lemmas 12.2.1 and 12.2.2 in [26]. This can be done using the tightness for the laws of SRBM's proved in Theorem 5.4 and the time homogeneity of our problem.) Assuming this strong Markov property, we see that it suffices to show uniqueness of the family of resolvents $\{R_\lambda, \lambda > 0\}$ defined on $C_b(\mathbf{S})$, the space of bounded continuous real-valued functions on $\mathbf{S}$, where

$$(R_\lambda f)(x) = E^{P_x}\left[\int_0^\infty e^{-\lambda t} f(z(t))dt\right] \quad \text{for all } x \in \mathbf{S},\ f \in C_b(\mathbf{S}), \lambda > 0 .$$

Now for each $r \geqq 0$, let $\tau_r = \inf\{t \geqq 0 : |z(t)| = r\}$, $\mathbf{S}_r = \{x \in \mathbf{S} : |x| \leqq r\}$ and

$$(R_\lambda^r f)(x) = E^{P_x}\left[\int_0^{\tau_r} e^{-\lambda t} f(z(t))dt\right] \quad \text{for all } x \in \mathbf{S}_r,\ f \in C_b(\mathbf{S}_r), \lambda > 0 .$$

By Lemma 6.4, the following is also well defined and finite for all $f \in C_b(\mathbf{S}_r)$ and $x \in \mathbf{S}_r$,

$$(R_0^r f)(x) \equiv E^{P_x}\left[\int_0^{\tau_r} f(z(t))dt\right].$$

Lemma 6.8 *The family $\{R_\lambda, \lambda > 0\}$ is unique on $C_b(\mathbf{S})$ if and only if for each $r > 2$, $(R_0^r f)(0)$ is unique for each $f \in C_b(\mathbf{S}_r)$ that vanishes in a neighborhood of the origin.*

Proof. The "only if" part is clear. For the "if" part, note that $\tau_r \to \infty$ as $r \uparrow \infty$, and so by dominated convergence, it suffices to prove the uniqueness of $(R_\lambda^r f)(x)$ for all $x \in \mathbf{S}_r$, $f \in C_b(\mathbf{S}_r)$, $\lambda > 0$ and $r > 2$. By Lemma 6.4, for $x \in \mathbf{S}_r$ and $f \in C_b(\mathbf{S}_r)$,

$$|(R_0^r f)(x)| \leqq \|f\|_r C r^2 ,$$

where $\|f\|_r = \sup_{u \in \mathbf{S}_r} |f(u)|$. It then follows from the proof of Theorem V.5.10 of [3] that to show uniqueness of the family $\{R_\lambda^r, \lambda > 0\}$ defined on functions in $C_b(\mathbf{S}_r)$, it suffices to show uniqueness of $(R_0^r f)(x)$ for all $x \in \mathbf{S}_r$ and $f \in C_b(\mathbf{S}_r)$.

Now, by the strong Markov property of z under $\{P_x, x \in \mathbf{S}\}$, and the fact that the law of $(z(\cdot \wedge \tau_0), y(\cdot \wedge \tau_0))$ under P_x is equal to Q_x°, for $f \in C_b(\mathbf{S}_r)$ and $x \in \mathbf{S}_r$ we have

$$\begin{aligned}
(R_0^r f)(x) &= E^{P_x}\left[\int_0^{\tau_r} f(z(t))dt\right] \\
&= E^{P_x}\left[\int_0^{\tau_r \wedge \tau_0} f(z(t))dt\right] + E^{P_x}\left[E^{P_0}\left[\int_0^{\tau_r} f(z(t))dt\right]; \tau_0 < \tau_r\right] \\
&= E^{Q_x^\circ}\left[\int_0^{\tau_r \wedge \tau_0} f(z(t))dt\right] + Q_x^\circ(\tau_0 < \tau_r) E^{P_0}\left[\int_0^{\tau_r} f(z(t))dt\right] \\
&= E^{Q_x^\circ}\left[\int_0^{\tau_r \wedge \tau_0} f(z(t))dt\right] + Q_x^\circ(\tau_0 < \tau_r)(R_0^r f)(0) . \qquad (6.19)
\end{aligned}$$

Thus the value of $(R_0^r f)(x)$ is determined by Q_x° and $(R_0^r f)(0)$. Only the latter needs to be shown to be unique. Let $\{\phi_n\}$ be a sequence of functions in $C_b(\mathbf{S}_r)$ such that for each n, $\phi_n \equiv 0$ in some neighborhood of the origin, and $0 \leqq \phi_n \uparrow 1_{\mathbf{S}_r \setminus \{0\}}$ on $\mathbf{S}_r$ as $n \to \infty$. Then

$$(R_0^r f)(0) = \lim_n R_0^r(f\phi_n)(0)$$

by dominated convergence, using Lemma 6.4 and (2.1). Thus it suffices to prove the uniqueness of $(R_0^r f)(0)$ for all $f \in C_b(\mathbf{S}_r)$ that vanish in a neighborhood of the origin. □

The proof of the following is modeled on that of Bass and Pardoux [1, Theorem 5.5].

Theorem 6.9 *For each $r > 2$ and $f \in C_b(\mathbf{S}_r)$ that vanishes in a neighborhood of the origin, $(R_0^r f)(0)$ is unique.*

Proof. Let $r > 2, f \in C_b(\mathbf{S}_r)$, and suppose that f vanishes in $\{x \in \mathbf{S}: |x| \leqq 2\delta\}$ for some $\delta \in (0, 1)$. Then for any $0 < \varepsilon < \delta$, by the strong Markov property of z under $\{P_x, x \in \mathbf{S}\}$,

$$\begin{aligned}(R_0^r f)(0) &= E^{P_0}\left[E^{P_{z(\tau_\varepsilon)}}\left[\int_0^{\tau_r} f(z(t))dt\right]\right] \\ &= E^{P_0}\left[E^{P_{z(\tau_\varepsilon)}}\left[\int_0^{\tau_r \wedge \tau_0} f(z(t))dt\right]\right] \\ &\quad + E^{P_0}\left[E^{P_{z(\tau_\varepsilon)}}\left[\tau_0 < \tau_r;\, E^{P_0}\left[\int_0^{\tau_r} f(z(t))dt\right]\right]\right] \\ &= E^{P_0}\left[E^{P_{z(\tau_\varepsilon)}}\left[\int_0^{\tau_r \wedge \tau_0} f(z(t))dt\right]\right] \\ &\quad + E^{P_0}[P_{z(\tau_\varepsilon)}(\tau_0 < \tau_r)]\,(R_0^r f)(0).\end{aligned} \tag{6.20}$$

In the last line above, $P_{z(\tau_\varepsilon)}$ can be replaced by $\overset{\circ}{Q}_{z(\tau_\varepsilon)}$. Observe that $E^{P_0}[\overset{\circ}{Q}_{z(\tau_\varepsilon)}(\tau_r < \tau_0)] = 1 - E^{P_0}[\overset{\circ}{Q}_{z(\tau_\varepsilon)}(\tau_0 < \tau_r)] > 0$, by continuity of paths, the strong Markov property, and since $P_0(\tau_r < \infty) = 1$ by Lemma 6.4. Therefore, (6.20) yields

$$(R_0^r f)(0) = \frac{E^{P_0}\left[E^{\overset{\circ}{Q}_{z(\tau_\varepsilon)}}\left[\int_0^{\tau_r \wedge \tau_0} f(z(t))dt\right]\right]}{E^{P_0}[\overset{\circ}{Q}_{z(\tau_\varepsilon)}(\tau_r < \tau_0)]}. \tag{6.21}$$

We now let $g(x) = E^{\overset{\circ}{Q}_x}[\int_0^{\tau_r \wedge \tau_0} f(z(t))dt]$ for $x \in \mathbf{S}\setminus\{0\}$, and define $G(x) = g(\delta x)$ for all $x \in \Lambda$. For $\sigma_\gamma \equiv \inf\{t \geqq 0: |z(t) - z(0)| \geqq \gamma\}$ where $\gamma \in (0, \delta/2)$, we have

$$g(x) = E^{\overset{\circ}{Q}_x}[g(z(\sigma_\gamma))] \quad \text{for } x \in \mathbf{S}: \frac{\delta}{2} \leqq |x| \leqq \frac{3\delta}{2}, \tag{6.22}$$

by the strong Markov property of z under $\overset{\circ}{Q}_x$ and since $f \equiv 0$ on $\{u \in \mathbf{S}: |u| \leqq 2\delta\}$. Setting $h(x) = \overset{\circ}{Q}_x(\tau_r < \tau_0)$ for $x \in \mathbf{S}\setminus\{0\}$, we have

$$h(x) = E^{\overset{\circ}{Q}_x}[\overset{\circ}{Q}_{z(\sigma_\gamma)}(\tau_r < \tau_0)] = E^{\overset{\circ}{Q}_x}[h(z(\sigma_\gamma))] \quad \text{for } x \in \mathbf{S}: \frac{\delta}{2} \leqq |x| \leqq \frac{3\delta}{2}. \tag{6.23}$$

Define $H(x) = h(\delta x)$ for $x \in \Lambda$.

To show that G and H are continuous on $\{x \in \mathbf{S}: |x| = 1\}$, it suffices to show that g and h are continuous on $\{x \in \mathbf{S}: |x| = \delta\}$. This continuity can be shown in a similar manner to that in the proof of Theorem 3.2 of [19], using Lemma 6.3, the representations (6.22) and (6.23), and the boundedness of g, h. Observe that $h(x) \not\equiv 0$ on $\{x \in \mathbf{S}: |x| = \delta\}$, otherwise the denominator in (6.21) would be zero. Hence, $H \not\equiv 0$ on Λ.

Now, by the strong Markov property of z under Q_x° and Lemma 6.5, for $x \in \mathbf{S}$ satisfying $|x| = 2^{-n}\delta$ and $\lambda \in \Lambda$, we have

$$
\begin{aligned}
&Q_x^\circ(z(\tau_\delta)/\delta \in d\lambda;\ \tau_\delta < \tau_0)\\
&\quad = \int_\Lambda Q_x^\circ(z(\tau_{\delta/2})/(\delta/2) \in du;\ \tau_{\delta/2} < \tau_0) Q^\circ_{\frac{u\delta}{2}}(z(\tau_\delta)/\delta \in d\lambda;\ \tau_\delta < \tau_0)\\
&\quad = \int_\Lambda Q_x^\circ(z(\tau_{\delta/2})/(\delta/2) \in du;\ \tau_{\delta/2} < \tau_0) Q(u, d\lambda)\\
&\quad \ \vdots\\
&\quad = \int_{\Lambda^n} Q\left(\frac{x}{|x|}, du_1\right) Q(u_1, du_2) \cdots Q(u_n, d\lambda)\\
&\quad = Q^n\left(\frac{x}{|x|}, d\lambda\right),
\end{aligned}
$$

where Q is the sub-probability measure defined on $(\Lambda, \mathscr{B}(\Lambda))$ in (6.17). Then for $x \in \mathbf{S}$ satisfying $|x| = 2^{-n}\delta$, we have

$$
\begin{aligned}
g(x) &= E^{Q_x^\circ}\left[E^{Q^\circ_{z(\tau_\delta)}}\left[\int_0^{\tau_r \wedge \tau_0} f(z(t))dt \right]; \tau_\delta < \tau_0 \right]\\
&= E^{Q_x^\circ}[g(z(\tau_\delta));\ \tau_\delta < \tau_0]\\
&= E^{Q_x^\circ}[G(z(\tau_\delta)/\delta);\ \tau_\delta < \tau_0]\\
&= \int_\Lambda G(\lambda) Q_x^\circ(z(\tau_\delta)/\delta \in d\lambda;\ \tau_\delta < \tau_0)\\
&= (Q^n G)\left(\frac{x}{|x|}\right).
\end{aligned}
$$

Similarly,

$$h(x) = (Q^n H)\left(\frac{x}{|x|}\right).$$

We now let v_n be the distribution of $z(\tau_\varepsilon)/\varepsilon$ under P_0, when $\varepsilon = 2^{-n}\delta$. Since $\tau_\varepsilon < \infty$ P_0-a.s., v_n is a probability mesure on Λ, and by (6.21) and the above,

$$(R_0^r f)(0) = \frac{\int_\Lambda (Q^n G)(\lambda) v_n(d\lambda)}{\int_\Lambda (Q^n H)(\lambda) v_n(d\lambda)}. \tag{6.24}$$

It then follows by Lemma 6.7 that the limit as $n \to \infty$ of the right member of (6.24) equals a finite constant that depends only on Q, g, h and not on $\{v_n\}$. Consequently, $(R_0^r f)(0)$ equals a constant not depending on P_0, since P_0 entered only through $\{v_n\}$ because Q, h, and g depend only on $\{Q_x^\circ, x \in \mathbf{S}\}$ and f. We conclude that $(R_0^r f)(0)$ is unique. □

The uniqueness part of Theorem 1.3 has now been proved. In particular, for each $x \in \mathbf{S}$, the probability measure Q_x defined in Sect. 5 is the law of the SRBM and its attendant pushing process for the data ($\mathbf{S}$, θ, Γ, R) and starting point x. The Feller continuity and strong Markov property of z together with $\{Q_x, x \in \mathbf{S}\}$ follows by standard arguments (cf. [26, Corollary 4.6]) using the uniqueness in law and the tightness of these laws proved in Theorem 5.4. When combined with Corollary 5.3, we see that this yields Theorem 1.3 in dimension d. This completes the induction step and so Theorem 1.3 holds for all $d \geqq 1$.

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