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# Generalized Ginzburg-Landau Equations, Slaving Principle and Center Manifold Theorem

A. Wunderlin and H. Haken
Institut für Theoretische Physik der Universität Stuttgart, Federal Republic of Germany

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The Generalized Ginzburg-Landau equations, introduced by one of us (H.H.), are considered in a simplified version to clarify their relation to the center manifold theorem.

## I. Introduction

The concept of *order parameters* and *slaving* (or adiabatic elimination) in non-equilibrium systems [1–3], has proved to be a powerful tool to analyze instabilities far from thermal equilibrium. Furthermore it has elucidated important analogies between systems of quite different disciplines. Our equations of motion for the order parameters, which we called Generalized Ginzburg-Landau equations [1–3], are obtained via an elimination scheme which may be summarized as follows.
Usually one starts from a set of equations of the general type [1–3]

$$\dot{U}_\mu = G_\mu(\mathbf{U}, \nabla, \{\sigma\}) + D_\mu \nabla^2 U_\mu + F_\mu(t). \tag{1.1}$$

$U_\mu$ denotes the $\mu$-th component of the state vector $\mathbf{U}$ which is assumed to describe the system under consideration. $G_\mu$ contains all the kinetics and may depend on gradients (indicated symbolically by $\nabla$) as well as on a certain set of external parameters $\{\sigma\}$. $D_\mu$ is an element of the diffusion matrix. Finally $F_\mu(t)$ represents the $\mu$-th component of the fluctuating forces. For the following purposes we shall confine ourselves to a simplified version of (1.1), namely

$$\dot{U}_\mu = G_\mu(\mathbf{U}, \{\sigma\}), \tag{1.2}$$

i.e., we omit the fluctuating forces and the dependence of spatial variations. We note, however, that it is straight forward to incorporate a spatial dependence of the state vector $\mathbf{U}$ into the following calculations also (compare [1, 2]).
The analysis of (1.2) usually starts from a steady state $\mathbf{U}_0$ (more general situations are treated in [4, 5]). Here and in the following the steady state is considered as time independent. In a second stage the stability of the steady state is probed as a function of the external parameters $\{\sigma\}$. The ansatz

$$\mathbf{U} = \mathbf{U}_0 + \hat{\mathbf{q}} \tag{1.3}$$

separates $\mathbf{G}$ into linear and nonlinear deviations from the stationary state

$$\dot{\hat{\mathbf{q}}} = K(\{\sigma\})\,\hat{\mathbf{q}} + \mathbf{N}(\hat{\mathbf{q}}, \{\sigma\}). \tag{1.4}$$

$K$ is a matrix independent of $\hat{\mathbf{q}}$, $\mathbf{N}$ contains all nonlinear terms. Now linear stability analysis in (1.4) yields a set of eigenvalues $\lambda_j(\{\sigma\})$ and the corresponding set of right and left hand eigenvectors $\mathbf{O}_j$ and $\bar{\mathbf{O}}_j$, respectively, which together form a biorthogonal set. Eventually the hypothesis

$$\hat{\mathbf{q}} = \sum_j \xi_j \mathbf{O}_j \tag{1.5}$$

allows to transform (1.4) into a set of equations for the $\xi$. Near an instability point the following observation is crucial: Linear stability analysis devides the set of the $\{\xi\}$ into two parts, the long living modes called $\xi_u$ which finally play the role of order parameters and a set of strongly damped modes $\xi_s$:

$$\xi \begin{array}{l} \nearrow \xi_u \\ \searrow \xi_s. \end{array} \tag{1.6}$$

They obey equations of the form

$$\dot{\xi}_u = \Lambda_u \xi_u + \mathbf{q}(\xi_u, \xi_s), \tag{1.7}$$

$$\dot{\xi}_s = \Lambda_s \xi_s + \mathbf{p}(\xi_u, \xi_s), \tag{1.8}$$

where $\Lambda_u$ and $\Lambda_s$ are diagonal matrices or at least of the Jordan canonical form; $\mathbf{p}$ and $\mathbf{q}$ fulfill the condition

$$\|\mathbf{q}\|,\ \|\mathbf{p}\| = O(\|\xi_u\|^2). \tag{1.9}$$

These considerations together lead to the conclusion

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that the long living modes will dominate the behaviour of the whole system. Therefore the motion of the system on time scales prescribed by the unstable modes may be analyzed in the subspace spanned by the unstable modes. To achieve this we formally integrate (1.8) to obtain

$$\xi_s = \left(\frac{d}{dt} - \Lambda_s\right)^{-1} \mathbf{p}(\xi_u, \xi_s). \tag{1.10}$$

Equation (1.10) is an implicit relation between the stable and the unstable modes, where different time arguments occur on the rhs.

In order to solve (1.10), i.e. to express $\xi_s$ as a functional of $\xi_u$, one of us [2] developed an iteration procedure in powers of the unstable modes $\xi_u$ up to arbitrary orders. The resulting expressions are exact, but still complicated, because the unstable modes must be taken at all previous times and an appropriate time integration has to be performed. Therefore the question arises, whether $\xi_s(t)$ can be expressed by $\xi_u(t)$ at the *same* time. One possibility for this rests on the adiabatic elimination hypothesis [1-5]. In this case all contributions to the different powers in the unstable modes occur with the *same* time argument. The result may therefore expressed in the form

$$\xi_s = \left(\frac{d}{dt} - \Lambda_s\right)^{-1}_{(0)} \mathbf{p}_{\mathrm{ad}}(\xi_u, \xi_s). \tag{1.11}$$

The operators $\left(\frac{d}{dt} - \Lambda_s\right)^{-1}$ and $\left(\frac{d}{dt} - \Lambda_s\right)^{-1}_{(0)}$ have already been defined in [1-5] and will be given explicitly in the following section. The suffix "ad" indicates that the expression has to be evaluated in the adiabatic approximation. (1.11) can now be solved yielding

$$\xi_s(t) = \xi_s(\xi_u(t)). \tag{1.12}$$

In the present paper we shall show that the general iteration method of [2] can be reduced to an "equal time" iteration in the unstable modes if the conditions mentioned in the beginning are met. The final result will be of the general form (1.12) again, but no use of the adiabatic approximation need be made. Or, in other words, the procedure fully takes into account the nonadiabatic contributions inherent in (1.10). The price to be paid for the exact "equal time" relation is the omission of fluctuations. In a forthcoming paper we shall show how our former iteration scheme [2] can be simplified in such a case. Our results allow us to make contact with the center manifold theorem.

## II. Description and Extension of the Elimination Procedure

As usual [1-3] we shall assume that the nonlinear part in the equations of motion for the stable modes may be written in the following way

$$\begin{aligned}\mathbf{p}(\xi_u, \xi_s) &= A_{\mathbf{suu}}:\mathbf{u}:\mathbf{u} + 2A_{\mathbf{sus}}:\mathbf{u}:\mathbf{s} \\ &+ A_{\mathbf{sss}}:\mathbf{s}:\mathbf{s} + B_{\mathbf{suuu}}:\mathbf{u}:\mathbf{u}:\mathbf{u} + 3B_{\mathbf{suus}}:\mathbf{u}:\mathbf{u}:\mathbf{s} \\ &+ 3B_{\mathbf{suss}}:\mathbf{u}:\mathbf{s}:\mathbf{s} + B_{\mathbf{ssss}}:\mathbf{s}:\mathbf{s}:\mathbf{s} + \dots\end{aligned} \tag{2.1}$$

To abbreviate expression (2.1) we used the notation of [2] which furthermore allows for generalization to situations which contain spatial variations. **s** is a short hand notation for the stable, **u** for the unstable modes. Similarly we shall take into account the motion of the stable modes via

$$\begin{aligned}\mathbf{q}(\xi_u, \xi_s) &= a_{\mathbf{uuu}}:\mathbf{u}:\mathbf{u} + 2a_{\mathbf{uus}}:\mathbf{u}:\mathbf{s} \\ &+ a_{\mathbf{uss}}:\mathbf{s}:\mathbf{s} + b_{\mathbf{uuuu}}:\mathbf{u}:\mathbf{u}:\mathbf{u} + 3b_{\mathbf{uuus}}:\mathbf{u}:\mathbf{u}:\mathbf{s} \\ &+ 3b_{\mathbf{uuss}}:\mathbf{s}:\mathbf{s}:\mathbf{u} + b_{\mathbf{usss}}:\mathbf{s}:\mathbf{s}:\mathbf{s} + \dots\end{aligned} \tag{2.2}$$

To begin with the elimination of the stable modes we write (1.8) in the form

$$\xi_s = \int_{-\infty}^{t} \exp(\Lambda_s(t-t'))\, \mathbf{p}\,(\xi_u(t'), \xi_s(t'))\, dt', \tag{2.3}$$

or in a more formal way [1-3]

$$\xi_s = \left(\frac{d}{dt} - \Lambda_s\right)^{-1} \mathbf{p}. \tag{2.4}$$

Equations (2.3) and (2.4) together define the operator $\left(\frac{d}{dt} - \Lambda_s\right)^{-1}$. It is our aim to solve (2.4) iteratively. Up to lowest order in $\xi_u$, i.e. $O(\|\xi_u\|^2)$, we explicitly obtain for the stable modes

$$\xi_s(t) = \int_{-\infty}^{t} \exp(\Lambda_s(t-t'))\, A_{\mathbf{suu}}\, \xi_u \xi_u\, dt'. \tag{2.5}$$

In the following we denote by $\xi_s^{(i)}$ the $i$-th component of the vector $\xi_s$ and by $A_{\mathbf{suu}}^{(i,j,k)}$ an element of $A_{\mathbf{suu}}$. Confining ourselves to cases where the matrices $\Lambda_s$ and $\Lambda_u$ can be diagonalized completely, relations of the type

$$\Lambda_s^{(i,j)} = \gamma_i \delta_{ij}; \quad \Lambda_u^{(i,j)} = i\omega_j \delta_{ij}, \tag{2.6}$$

where

$$\mathrm{Re}\,\gamma_i < 0,$$

hold. $\Lambda_s^{(i,j)}$, $\Lambda_u^{(i,j)}$ are the elements of the matrices $\Lambda_s$ and $\Lambda_u$, respectively. The $\omega_j$ are assumed to be real, i.e., we evaluate $\Lambda_u$ in (2.5) at the critical point.

Splitting $\xi_u^{(j)}$ into its fast ($\propto \exp(i\omega_j t)$) and slowly varying parts ($\tilde{\xi}_u^{(j)}$), i.e.

$$\xi_u^{(j)} = \exp(i\omega_j t)\, \tilde{\xi}_u^{(j)}, \tag{2.7}$$

we obtain from (2.5) after a partial integration

$$\begin{aligned}\xi_s^{(i)} &= [i(\omega_j+\omega_k)-\gamma_i]^{-1} A_{\mathbf{suu}}^{(i,j,k)} \xi_u^{(j)} \xi_u^{(k)} \\ &- \int_{-\infty}^{t} \exp[\gamma_i(t-t') + i(\omega_j+\omega_k)t'] \cdot [i(\omega_j+\omega_k)-\gamma_i]^{-1} \\ &\cdot A_{\mathbf{suu}}^{(i,j,k)} (\tilde{\xi}_u^{(j)} \tilde{\xi}_u^{(k)})^{\cdot}\, dt',\end{aligned} \tag{2.8}$$

where summation runs over dummy indices. The symbol $^{\cdot}$ means time derivative of the slowly varying part of the $\xi_u$ only. Therefore the second term in Eq. (2.8) is at least of order $\|\xi_u\|^3$ in the unstable modes $\xi_u$. From these considerations we conclude that for the lowest order, $n=2$, the stable modes $\xi_s(2)$ are given by

$$\xi_s(2) = \left(\frac{d}{dt} - \Lambda_s\right)_{(0)}^{-1} A_{\mathbf{suu}} : \mathbf{u} : \mathbf{u}. \tag{2.9}$$

Here we have introduced the operator $\left(\frac{d}{dt} - \Lambda_s\right)_{(0)}^{-1}$. Its definition follows from a comparison of (2.8) and (2.9). The result (2.9) is in fact well known from calculations based on the adiabatic elimination method [1-5]. In this order of approximation one obtains an equation of motion for the order parameters of the Landau type.

Our interest, however, consists in calculating corrections to the higher order terms of the adiabatic procedure. To this end we shall consider in (2.1) as well as in (2.8) the terms of order $\|\xi_u\|^3$. (2.1) yields

$$2A_{\mathbf{sus}} : \mathbf{u} : \mathbf{s} + B_{\mathbf{suuu}} : \mathbf{u} : \mathbf{u} : \mathbf{u}, \tag{2.10}$$

where we have to replace $\mathbf{s}$ by the result (2.9). We note that these terms are already taken into account in the adiabatic domain when one takes into account the higher order non-linearities in the equations of motion of the stable modes (compare [2]). In order to obtain equal time contributions in the unstable modes only, we now additionally include terms of the same order which stem from the second term of the rhs of (2.8).

The final result can be put into the form

$$\begin{aligned}\xi_s(3) &= \left(\frac{d}{dt} - \Lambda_s\right)_{(0)}^{-1} [2A_{\mathbf{sus}} : \mathbf{u} : \mathbf{s} + B_{\mathbf{suuu}} : \mathbf{u} : \mathbf{u} : \mathbf{u}] \\ &+ \left(\frac{d}{dt} - \Lambda_s\right)_{(1)}^{-1} A_{\mathbf{suu}} : \mathbf{u} : \mathbf{u}.\end{aligned} \tag{2.11}$$

In (2.11) we have introduced the operator $\left(\frac{d}{dt} - \Lambda_s\right)_{(1)}^{-1}$ which can be explicitly contructed as follows: We insert the time derivative of the slowly varying amplitudes $\tilde{\xi}_u$ into the second term of (2.8)

$$\begin{aligned}&- \int_{-\infty}^{t} \exp[\gamma_i(t-t') + i(\omega_j+\omega_k)t'] \cdot \{i(\omega_j+\omega_k)-\gamma_i\}^{-1} \\ &\cdot A_{\mathbf{suu}}^{(i,j,k)} (\tilde{\xi}_u^{(j)} \tilde{\xi}_u^{(k)})^{\cdot}\, dt'.\end{aligned} \tag{2.12}$$

After having performed another partial integration we combine all the resulting terms of $O(\|\xi_u\|^3)$. The explicit result for $\xi_s(3)$ then reads

$$\begin{aligned}\xi_s^{(i)}(3) &= \{[-\gamma_i + i(\omega_j+\omega_k+\omega_l)]^{-1} B_{\mathbf{suuu}}^{(i,j,k,l)} \\ &+ 2[(-\gamma_m + i(\omega_l+\omega_k))(-\gamma_i + i(\omega_j+\omega_l+\omega_k)]^{-1} \\ &\cdot A_{\mathbf{sus}}^{(i,j,m)} A_{\mathbf{suu}}^{(m,l,k)} - 2[(-\gamma_i + i(\omega_k+\omega_l+\omega_j)) \\ &\cdot (-\gamma_i + i(\omega_m+\omega_k))]^{-1} A_{\mathbf{suu}}^{(i,k,m)} a_{\mathbf{uuu}}^{(m,l,j)}\} \cdot \xi_u^{(j)} \xi_u^{(k)} \xi_u^{(l)}.\end{aligned} \tag{2.13}$$

It is just the last term in the sum of the rhs of (2.11) or (2.13) which goes beyond the adiabatic domain. We recall that adiabatic elimination means that the stable modes follow the motion of the unstable ones instantaneously. The term which formally results from an expansion of the operator $\left(\frac{d}{dt} - \Lambda_s\right)^{-1}$ additionally takes care of the motion of the unstable modes. In other words, this term measures the influence of the motion of the unstable modes on the behaviour of the stable ones which is neglected in (1.11).

To complete the discussion of the order $O(\|\xi_u\|^3)$ we shall derive a more formal expression for $\left(\frac{d}{dt} - \Lambda_s\right)_{(1)}^{-1}$. We introduce the notation

$$\left(\dot{\mathbf{u}}^{(m+1)} \cdot \frac{\partial}{\partial \mathbf{u}}\right) \cdot f(\mathbf{u}) + \left(\frac{d}{dt}\right)_{(m)} f(\mathbf{u}), \tag{2.14}$$

where $f$ means a differentiable but otherwise arbitrary function of the unstable modes. $\dot{\mathbf{u}}^{(m+1)}$ is a short hand notation for the term of order $m+1$ in the equation of motion for the unstable modes. Denoting the corresponding term of $\mathbf{q}$ (compare (1.7) and (2.2)) as $\mathbf{q}^{(m+1)}$, we have

$$\dot{\mathbf{u}}^{(m+1)} = \mathbf{q}^{(m+1)}. \tag{2.15}$$

Thus the lhs of (2.12) defines $(d/dt)_{(m)}$. Using this operator we obtain

$$\left(\frac{d}{dt} - \Lambda_s\right)_{(1)}^{-1} = \left(\frac{d}{dt} - \Lambda_s\right)_{(0)}^{-1} \left(-\frac{d}{dt}\right)_{(1)} \cdot \left(\frac{d}{dt} - \Lambda_s\right)_{(0)}^{-1} \tag{2.16}$$

as will be shown explicitly in the appendix.

## III. Generalization to Arbitrary Orders

Formally it is now straight forward to generalize our iteration scheme to arbitrary orders $n$. Adopting the notation of [2] we introduce $\mathbf{C}^{(n)}$ via

$$\mathbf{s}=\sum_{m=2}^{n}\mathbf{C}^{(m)}(\mathbf{u}) \tag{3.1}$$

which contains all contributions up to order $n$. Abbreviating the $n$-th order term of (2.1) (compare [2]) by $\mathbf{p}^{(n)}(\mathbf{u})$, we obtain by generalizing (2.11) for the $n$-th order contribution to the stable modes

$$\mathbf{C}^{(n)}=\sum_{m=0}^{n-2}\left(\frac{d}{dt}-\Lambda_s\right)^{-1}_{(m)}\mathbf{p}^{(n-m)}(\mathbf{u}). \tag{3.2}$$

In order to construct $\left(\frac{d}{dt}-\Lambda_s\right)^{-1}_{(m)}$ we shall first derive a recursion formula for $\mathbf{C}^{(n)}$. To this end we note that (2.11) may be equally well written as

$$\mathbf{C}^{(3)}=\left(\frac{d}{dt}-\Lambda_s\right)^{-1}_{(0)}\{B_{\mathbf{suuu}}:\mathbf{u}:\mathbf{u}:\mathbf{u}+2A_{\mathbf{sus}}:\mathbf{u}:\mathbf{C}^{(2)}(\mathbf{u})\}$$
$$-\left(\frac{d}{dt}-\Lambda_s\right)^{-1}_{(0)}\left\{\left(\frac{d}{dt}\right)_{(1)}\mathbf{C}^{(2)}(\mathbf{u})\right\} \tag{3.3}$$

or in a more condensed form

$$\mathbf{C}^{(3)}=\left(\frac{d}{dt}-\Lambda_s\right)^{-1}_{(0)}\left\{\mathbf{p}^{(3)}-\left(\frac{d}{dt}\right)_{(1)}\mathbf{C}^{(2)}\right\}. \tag{3.4}$$

Indeed the first two terms in (3.3) result from the non-linearities in the equations of motion for the stable modes yielding $\mathbf{p}^{(3)}$. At this stage we may generalize (3.4) to arbitrary orders $n$

$$\mathbf{C}^{(n)}=\left(\frac{d}{dt}-\Lambda_s\right)^{-1}_{(0)}\left\{\mathbf{p}^{(n)}-\sum_{m=1}^{n-2}\left(\frac{d}{dt}\right)_{(m)}\mathbf{C}^{(n-m)}\right\}. \tag{3.5}$$

To prove (3.5) we have to demonstrate that this expression is also correct to order $n+1$ by performing the various steps which have been described in the foregoing section. Obviously there are three different contributions if we proceed to the next order. The first stems from the next order term in the nonlinearities of the stable modes and reads

$$\left(\frac{d}{dt}-\Lambda_s\right)^{-1}_{(0)}\mathbf{p}^{(n+1)}. \tag{3.6a}$$

The second is obtained by passing to the next order term in the equation of motion for the unstable modes, i.e., $\left(\frac{d}{dt}\right)_{(m)}\rightarrow\left(\frac{d}{dt}\right)_{(m+1)}$. Explicitly one obtains the sum

$$-\left(\frac{d}{dt}-\Lambda_s\right)^{-1}_{(0)}\sum_{m=1}^{n-2}\left(\frac{d}{dt}\right)_{(m+1)}\mathbf{C}^{(n-m)}. \tag{3.6b}$$

Finally we have to perform the partial integration as we did in passing from (2.12) to (2.13). The explicit calculation is presented in the appendix, the result reads

$$-\left(\frac{d}{dt}-\Lambda_s\right)^{-1}_{(0)}\left[\left(\frac{d}{dt}\right)_{(1)}\mathbf{C}^{(n)}\right]. \tag{3.6c}$$

Adding the different contributions (3.6a–c) we indeed obtain the desired result to order $n+1$ which proves the hypothesis (3.5).

Equation (3.5) contains the main result of this section. It is now possible to construct the operator $\left(\frac{d}{dt}-\Lambda_s\right)^{-1}_{(m)}$ and by comparing (3.2) and (3.5) we observe that we only have to rearrange terms in the following way: combine all operations on $\mathbf{p}^{(n-m)}$ which contribute to $\mathbf{C}^{(n)}$. Because we shall not need the expression in the following we only mention the final result

$$\left(\frac{d}{dt}-\Lambda_s\right)^{-1}_{(m)}=\left(\frac{d}{dt}-\Lambda_s\right)^{-1}_{(0)}$$
$$\cdot\sum_{\substack{\text{all}\\ \text{prod}}}\prod_{i(\Sigma i=m)}\left\{\left(-\frac{d}{dt}\right)_{(i)}\left(\frac{d}{dt}-\Lambda_s\right)^{-1}_{(0)}\right\}. \tag{3.7}$$

The product has to be taken in such a way that the sum of the orders $i$ is equal to $m$. Finally the sum indicates summation over all different products of order $m$.

At this stage it might become instructive to treat explicitly the equations discussed in [2]. There just the terms written explicitly in (2.1) and (2.2) are kept. To arbitrary orders $n$ we obtain as solution for the stable modes

$$\mathbf{C}^{(n)}=\left(\frac{d}{dt}-\Lambda_s\right)^{-1}_{(0)}(\{\ldots\}-[\ldots]\}, \tag{3.8}$$

where

$$\{\ldots\}=2A_{\mathbf{sus}}:\mathbf{u}:\mathbf{C}^{(n-1)}+(1-\delta_{n,3})\sum_{m=2}^{n-2}A_{\mathbf{sss}}:\mathbf{C}^{(m)}:\mathbf{C}^{(n-m)}$$
$$+\delta_{n,3}B_{\mathbf{suuu}}:\mathbf{u}:\mathbf{u}:\mathbf{u}+3(1-\delta_{n,3})B_{\mathbf{suus}}:\mathbf{u}:\mathbf{u}:\mathbf{C}^{(n-2)}$$
$$+3\prod_{i=3}^{4}(1-\delta_{n,i})\sum_{m=2}^{n-3}B_{\mathbf{suss}}:\mathbf{u}:\mathbf{C}^{(m)}:\mathbf{C}^{(n-1-m)}$$
$$+\prod_{i=3}^{5}(1-\delta_{n,i})\sum_{\substack{m_1,m_2,m_3\geq 2\\ m_1+m_2+m_3=n}}B_{\mathbf{ssss}}:\mathbf{C}^{(m_1)}:\mathbf{C}^{(m_2)}:\mathbf{C}^{(m_3)} \tag{3.9}$$

and

$$[\ldots]=\mathbf{C}^{(n-1)'}:a_{\mathbf{uuu}}:\mathbf{u}:\mathbf{u}\,(1-\delta_{n,2})$$
$$+2\sum_{m=2}^{n-2}\mathbf{C}^{(n-m)'}:a_{\mathbf{uus}}:\mathbf{u}:\mathbf{C}^{(m)}\prod_{i=2}^{3}(1-\delta_{n,1})$$
$$+\sum_{\substack{m,m_1,m_2\geqq 2\\(m-1)+m_1+m_2=n}}\mathbf{C}'^{(m)}:a_{\mathbf{uss}}:\mathbf{C}^{(m_1)}:\mathbf{C}^{(m_2)}\prod_{i=2}^{4}(1-\delta_{n,i})$$
$$+\mathbf{C}^{(n-2)'}:\mathbf{b}_{\mathbf{uuuu}}:\mathbf{u}:\mathbf{u}:\mathbf{u}\cdot\prod_{i=2}^{3}(1-\delta_{n,i})$$
$$+3\sum_{l=2}^{n-3}\mathbf{C}^{(l)'}:\mathbf{b}_{\mathbf{uuus}}:\mathbf{u}:\mathbf{u}:\mathbf{C}^{(n-l-1)}\cdot\prod_{i=2}^{4}(1-\delta_{n,i})$$
$$+3\sum_{\substack{m,m_1,m_2\geqq 2\\(m-1)+m_1+m_2=n}}\mathbf{C}^{(m)'}:b_{\mathbf{uuss}}:\mathbf{u}:\mathbf{C}^{(m_1)}:\mathbf{C}^{(m_2)}\prod_{i=2}^{5}(1-\delta_{n,i})$$
$$+\sum_{\substack{m,m_1,m_2,m_3\geqq 2\\(m-1)+m_1+m_2+m_3=n}}\mathbf{C}^{(m)'}:b_{\mathbf{usss}}:\mathbf{C}^{(m_1)}:\mathbf{C}^{(m_2)}:\mathbf{C}^{(m_3)}$$
$$\cdot\prod_{i=2}^{6}(1-\delta_{n,i}). \tag{3.10}$$

The first part (3.9) has the same structure as the result of [2], however the unstable modes are taken at equal times. (3.10) contains all nonadiabatic contributions again taken at the same time. Therefore, as one would expect, the result of the adiabatic elimination method is contained in (3.5) or (3.2) and might formally be written (compare the appendix)

$$\mathbf{C}_{(\mathrm{ad})}^{(n)}=\left(\frac{d}{dt}-\Lambda_s\right)_{(0)}^{-1}\mathbf{p}_{(\mathrm{ad})}^{(n)}, \tag{3.11}$$

where the suffix (ad.) refers to the adiabatic elimination. $\mathbf{p}_{(\mathrm{ad})}^{(n)}$ means that one takes into account $\mathbf{C}_{(\mathrm{ad})}^{(m)}$ to order $n$ $(m<n)$ which originate from the nonlinearities of the stable modes only.

## IV. Connection with the Center Manifold

In the vicinity of a critical point there exists in general a whole class of invariant manifolds [6, 7]. If we concentrate our attention to self-organizing systems the so-called center manifold is the relevant one. To define this manifold in the special case of small $\xi_u$, we start from Eqs. (1.7), (1.8) with eigenvalues of $\Lambda_u$ which are purely imaginary and eigenvalues of $\Lambda_s$ which give rise to damping and dissipation (compare (2.6)). We first note that this assumption does not restrict our results to the critical point only, but also allows us to consider a certain range below and above threshold. In order to show this we shall resort to a well-known trick (e.g. [7, 8]). Confining ourselves to one external parameter $\sigma$ and denoting by $\sigma_c$ its value at the critical point we may define $\varepsilon=\sigma-\sigma_c$ and formally take into account by

$$\dot{\varepsilon}=0 \tag{4.1}$$

which we interpret as an additional equation for an unstable mode. Equation (1.7), for example, may now be rewritten in the form

$$\dot{\xi}_u=\Lambda_u(0)\,\xi_u+(\Lambda_u(\varepsilon)-\Lambda_u(0))\,\xi_u+\mathbf{q}, \tag{4.2}$$

where we explicitly took into account the external parameter $\varepsilon$ and used that now $\Lambda_u=\Lambda_u(\varepsilon)$. A similar equation holds for $\xi_s$. The second term on the rhs of (4.2) can formally be treated as a nonlinearity and the method discussed in the preceding sections can be applied to this extended set of equations. Then, after having performed the elimination procedure, we may reconsider $\varepsilon$ as an external parameter.

To explain the idea of a center manifold we start from $\mathbf{p}$ and $\mathbf{q}$ which are $C^k$ $(2\leqq k<\infty)$ in a certain neighborhood of the origin. The statement is (center manifold theorem) that there exists in a certain neighborhood $|\xi_u|<\delta$ around the origin a $C^{k-1}$ invariant manifold

$$\xi_s=\xi_s(\xi_u) \tag{4.3}$$

and $\|\xi_s\|=O(\|\xi_u\|^2)$. (For a more precise presentation of the theorem compare [6–8]). Evidently (4.3) has the same formal structure as (1.12). Whereas the center manifold theorem is an existence proof, our previous results [2] and in particular the above description provide us with a construction procedure.

Another construction procedure was recently given by [7], which, however, rests on the existence theorem, who start from the formal relation (4.3) and differentiate it with respect to time

$$\dot{\xi}_s(t)=\frac{\partial\xi_s}{\partial\xi_u}:\dot{\xi}_u(t). \tag{4.4}$$

Then by inserting (1.7), (1.8) one may solve (4.4) iteratively with respect to succeeding powers of $\xi_u$. These authors give explicit results up to the fifth order. Using the ansatz (3.1) it becomes straight forward to group together terms of the same order and to identify the different contributions with the corresponding ones in (3.5). Thus we may conclude that our elimination scheme discussed in Section II yields a method to construct this manifold iteratively. Incidentally, it transpires from our results that the method [2] is an exact one and the criticism of [7] that [2] neglects nonadiabatic terms is unjustified. On the other hand (compare Sects. II, III) the adiabatic method for higher order contributions than $O(\|\xi_u\|^2)$ means partial summation of the nonlinear terms caused by the stable modes up to arbitrary order $n$.

We finally remark that this procedure can be extended to more complex situations as described in [4, 5].

## Appendix

In order to derive the formal expression (2.16) and to complete the proof of Sect. III we start from the expression for the stable modes

$$\mathbf{s}=\int_{-\infty}^{t} \exp(A_s(t-t'))\,\mathbf{p}(\exp(A_u t')\,\tilde{\mathbf{u}}(t'), \mathbf{s}(t'))\,dt' \qquad (\text{A}.1)$$

which directly follows from (2.3) by a slight change of notation which has been explained in the text. Furthermore the fast and the slow part in the time dependence of the unstable modes $\mathbf{u}$ has been separated. The hypothesis

$$\mathbf{s}=\sum_{n=2}^{l} \mathbf{C}^{(n)}(\mathbf{u}) \qquad (\text{A}.2)$$

which yields a power series expansion of the stable modes in terms of the unstable ones may be inserted to (A.1). The result reads

$$\mathbf{s}=\int_{-\infty}^{t} \exp(A_s(t-t'))\cdot\mathbf{p}(\exp(A_u t')\,\tilde{\mathbf{u}}(t'), \sum_n \mathbf{c}^{(n)}(e^{A_u t'}\,\tilde{\mathbf{u}}(t'))\,dt'. \qquad (\text{A}.3)$$

Comparing (2.1) and (A.2) we observe that the terms in (A.3) may be rearranged as a power series in the unstable modes

$$\mathbf{s}=\int_{-\infty}^{t} \exp(A_s(t-t'))\sum_n P^{(n)}(t')\underbrace{:\tilde{\mathbf{u}}(t'):\tilde{\mathbf{u}}(t')\dots}_{n}\,dt', \qquad (\text{A}.4)$$

where the fast part in the time dependence of the unstable modes has been absorbed into $P^{(n)}$. We are not interested in the explicit form of $P^{(n)}$ which can be obtained in a lengthy but straightforward calculation. The main point is, however, that (A.4) allows for a separation of the terms which one keeps performing an adiabatic approximation from these which are neglected. To achieve this goal we use a partial integration in (A.4) to get

$$\mathbf{s}=\sum_n \int_{-\infty}^{t} dt' \exp(A_s(t-t'))\,P^{(n)}(t')\underbrace{:\tilde{\mathbf{u}}(t'):\tilde{\mathbf{u}}(t)\dots}_{n}$$
$$-\sum_n \int_{-\infty}^{t} dt'\left[\int_{-\infty}^{t'} dt'' \exp(A_s(t'-t''))\,P^{(n)}(t'')\right]:\underbrace{(\tilde{\mathbf{u}}(t'):\tilde{\mathbf{u}}(t')\dots)^{\cdot}}_{n}. \qquad (\text{A}.5)$$

It is the first term on the rhs which is treated in the adiabatic elimination method, whereas the second one contains all non-adiabatic contributions.

To derive (2.16) we note that the equation for $\mathbf{C}^{(3)}$ is given by (3.3). Using the explicit form of $\mathbf{C}^{(2)}$ we obtain for the non-adiabatic contribution in (3.3)

$$-\left(\frac{d}{dt}-A_s\right)^{-1}_{(0)}\left(\frac{d}{dt}\right)_{(1)}\cdot\left(\frac{d}{dt}-A_s\right)^{-1}_{(0)} A_{\mathbf{suu}}:\mathbf{u}:\mathbf{u} \qquad (\text{A}.6)$$

which we have to compare with

$$\left(\frac{d}{dt}-A_s\right)^{-1}_{(1)} A_{\mathbf{suu}}:\mathbf{u}:\mathbf{u}. \qquad (\text{A}.7)$$

The result is (2.12).

To complete the proof of (3.2) we have to consider the expression

$$\int_{-\infty}^{t} \exp(A_s(t-t'))\left\{\mathbf{p}^{(n)}(t')-\sum_{m=1}^{n-2}\left(\frac{d}{dt'}\right)_{(m)}\mathbf{C}^{(n-m)}(t')\right\}dt'. \qquad (\text{A}.8)$$

By the same arguments which we used to arrive from (A.1) at (A.2) we may rewrite (A.8) in the form

$$(\text{A}.8)=\int_{-\infty}^{t} dt' \exp(A_s(t-t'))\,Q^{(n)}(t'):\underbrace{\tilde{\mathbf{u}}(t'):\tilde{\mathbf{u}}(t')\dots}_{n} \qquad (\text{A}.9)$$

with some coefficient $Q^{(n)}(t)$. Partial integration in (A.9) yields

$$(\text{A}.8)=\int_{-\infty}^{t} dt' \exp(A_s(t-t'))\,Q^{(n)}(t'):\underbrace{\tilde{\mathbf{u}}(t):\tilde{\mathbf{u}}(t)\dots}_{n}$$
$$-\int_{-\infty}^{t} \exp(A_s(t-t'))\,dt'\left(\dot{\tilde{\mathbf{u}}}^{(2)}\frac{\partial}{\partial\tilde{\mathbf{u}}(t')}\right)\cdot\left\{\int_{-\infty}^{t'} dt'' \exp(A_s(t'-t'')\cdot Q^{(n)}(t'')\right\}\underbrace{:\tilde{\mathbf{u}}(t'):\tilde{\mathbf{u}}(t'):\dots}_{n}.$$

(We considered terms of order $n+1$ only.) Obviously the first term in (A.10) is $\mathbf{C}^{(n)}$. The second term therefore can be written as

$$-\left(\frac{d}{dt}-A_s\right)^{-1}_{(0)}\left(\frac{d}{dt}\right)_{(1)}\cdot\mathbf{C}^{(n)}+\text{h.o.} \qquad (\text{A}.11)$$

which is just (3.6c).

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A. Wunderlin
H. Haken
Institut für Theoretische Physik
Universität Stuttgart
Pfaffenwaldring 57/IV
D-7000 Stuttgart 80
Federal Republic of Germany