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Dynamic Optimization on a Non-Convex Feasible Set: Some General Results for Non-Smooth Technologies

By

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I. Introduction

In the theory of optimal intertemporal allocation, the assumption of a convex feasible set has played a dominant role. In recent years, several contributions have focused on the implications for this theory, when the feasible set does not have the convexity property. (See, in particular, Skiba (1978), Majumdar and Mitra (1982, 1983), Dechert and Nishimura (1983), Majumdar and Nermuth (1982), and the much earlier insightful paper by Clark (1971)). These contributions have not only clarified the qualitative differences in the theory in convex and non-convex models, but they have also led to the development of new analytical techniques which have made some issues in the earlier theory in *convex* models simpler to address (see, for example, Mitra (1983)).

However, most of the contributions mentioned above have focused on particular types of non-convex feasible sets; that is, those generated by an S-shaped production function, exhibiting an initial phase of increasing returns, with diminishing returns setting in eventually. Majumdar and Nermuth (1982) work with a more general production function than this, but even there, for the development of the asymptotic stability theory of optimal programs, they have to impose some structure on the type of "non-concavities"

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that the production function can exhibit (see, in particular, assumption (A.3′) in their paper, (p. 347), and the statement of their turnpike theorem (p. 348)).

Furthermore, in developing many parts of the theory, the above contributions use differentiability of both the production and the utility functions in an essential way. To a certain extent, this is true about many of the earlier contributions which focused only on the theory for convex feasible sets. (See, for example, Cass (1965), Koopmans (1965), and Mitra (1979)).

The purpose of this paper is to develop some general results on two related aspects of the theory of optimal intertemporal allocation in an aggregative model, without the convexity assumption on the feasible set. In contrast to the contributions on non-convexity mentioned above, which have tended to emphasize the differences between the theory for convex and non-convex models, this paper tries to focus on the similarities between the two, and to provide a unifying theory in which the role of the convexity assumption on the feasible set is minimal. Of course, *some* stronger results can be obtained when the feasible set is convex (besides satisfying some other properties). We have tried to point out precisely what these stronger results are. Throughout, we refrain from making any differentiability assumptions on the production or utility functions. Also, we do not impose any structure at all on the types of non-concavities exhibited by the production functions.

The two issues we address are: (a) the monotonicity and insensitivity properties of optimal programs in finite-horizon models; (b) the existence of a non-trivial stationary optimal stock, and the asymptotic stability of optimal programs from arbitrary initial stocks in infinite-horizon models.

With respect to (a), the well-known results of Brock (1971) are generalized to frameworks where the production functions are not required to be concave, and where the utility functions are only assumed to be "weakly" concave. We think that our results are the most general possible for this class of models. We show, by means of an example, that stronger monotonicity results that have been obtained earlier depend crucially on the differentiability assumption on production functions.

With respect to (b), we look at the standard "quasi-stationary" model, where the production and the utility functions are stationary, and there is a positive discount factor, δ, less than one, at which future utilities are discounted. We show that when the production function is "δ-productive", there exists a non-trivial stationary optimal stock. An example is provided to show that this assumption is essential. (Even without this assumption, the set of stationary optimal stocks is non-empty, but then the set might consist of only a trivial stationary optimal stock, namely the zero stock). It is of interest to note that the non-trivial stationary optimal stock, (whose existence is proved) can be attained by decentralized profit-maximizing behavior of firms, and utility-maximizing behavior of consumers, under a competitive price system, even though the production function can display any type of non-concavity. We also note that the additional mileage one obtains by assuming the concavity of the production function allows one to conclude that "δ-productivity" is also a necessary condition for the existence of a non-trivial stationary optimal stock.

The rest of the paper is devoted to the stability of optimal programs and here we proceed *without the δ-productivity assumption on the production function.* First, we show that if the utility function is *strictly concave*, then optimal programs will converge, in terms of their input levels, to *some* stationary optimal stock. One may call this "system stability", a term coined by Arrow and Hurwicz (1958) to describe a similar phenomenon in the stability theory of general equilibrium models. An example is presented of a *concave* production function, and a *linear* utility function, for which this "system stability" does not hold, and an optimal program is seen to oscillate between two stationary optimal stocks. (This justifies the use of strict concavity of the utility function to prove "system stability").

The result on "system stability" and the literature on dynamic optimization, with a linear utility function, on convex (e. g. Malinvaud (1965), Clark (1971)) and non-convex (e. g. Clark (1971), Majumdar and Mitra (1983)) feasible sets suggest the need for a more general theory of asymptotic stability. The example just mentioned above reinforces this, if the oscillation observed there between two *stationary optimal stocks* is not accidental. Our result, (which attempts to provide such a theory) is that the distance between the optimal input stock and the *set* of stationary optimal stocks converges to zero asymptotically. (Strict concavity of the utility function is, of course, not used for this result). This result is in the same spirit as McKenzie's (1968) proposition that optimal programs converge to a "facet" of the production set, when future utilities are undiscounted, and the feasible set is convex.

We finally note that if f is concave, and there is a unique non-trivial stationary optimal stock, then optimal programs from arbitrary positive initial stocks converge asymptotically to the non-trivial stationary optimal stock.

2. The Model

2 a. Feasible Sets

The technology is given by a sequence of production functions $<f_t>$, with $f_t : \mathbb{R}_+ \to \mathbb{R}_+$. Throughout, we shall assume, for each $t \geq 0$,

(F.1) f_t is *increasing and continuous on* $\mathbb{R}_+$.

Additional assumptions will be made in a later section.

A *feasible program* $<x_t>$ from initial stock $s \geq 0$ is a sequence satisfying

$$x_0 = s, \quad 0 \leq x_{t+1} \leq f_t(x_t), \quad t \geq 0. \tag{2.1}$$

Associated with a feasible program $<x_t>$ from $s \geq 0$ is a *consumption sequence* $<c_t>$ defined by

$$c_{t+1} = f_t(x_t) - x_{t+1}, \quad t \geq 0. \tag{2.2}$$

A feasible program $<x_t>$ from s is *stationary* if

$$x_t = s, \quad t \geq 0. \tag{2.3}$$

In the context of finite horizon planning, we denote the time horizon by an integer $T \geq 1$, and the terminal stock by $b \geq 0$. Let $\xi = (T, s, b)$. A ξ*-feasible program* $<x_t>$ is a finite sequence $<x_t>_0{}^T$ satisfying (2.1) for all $t = 0, \ldots, T-1$, and in addition

$$x_T = b. \tag{2.1'}$$

Its associated consumption program is given by (2.2), for $t = 0, \ldots, T-1$.

The *pure accumulation program* is a feasible program $<\bar{x}_t>$ with

$$\bar{x}_{t+1} = f_t(\bar{x}_t), \quad t \geq 0. \tag{2.4}$$

We shall assume, to keep matters nontrivial,

(A.1) ξ *satisfies* $\bar{x}_T > b$.

2 b. Preferences

The planner's preferences are represented by a sequence of utility functions $<u_t>$, each mapping $\mathbb{R}_+$ to $\mathbb{R}$. Throughout, these are taken to satisfy, for each $t \geq 1$,

(U.1) u_t is *increasing and continuous on* $\mathbb{R}_+$,

(U.2) u_t is *concave on* $\mathbb{R}_+$.

In a later section, the sequence $<u_t>$ will be taken as $u_t = \delta^t u$, $t \geq 1$, where $\delta \in (0, 1)$ is the *discount factor,* and u satisfies (U.1), (U.2).

A feasible program $<x_t^*>$ from $s \geq 0$ is *optimal* if for every feasible program $<x_t>$ from s,

$$\limsup_{T \to \infty} \sum_{t=1}^{T} [u_t(c_t) - u_t(c_t^*)] \leq 0. \tag{2.5}$$

A ξ-feasible program $<x_t^*>$ is *ξ-optimal* if for every ξ-feasible program $<x_t>$,

$$\sum_{t=1}^{T} u_t(c_t) \leq \sum_{t=1}^{T} u_t(c_t^*). \tag{2.6}$$

A feasible program $<x_t>$ from $s \geq 0$ is a *stationary optimal program* if it is both stationary and optimal. If such a program exists from $s \geq 0$, we call s a *stationary optimal stock.* A stationary optimal program $<x_t>$ from $s \geq 0$ is *nontrivial* if $f_t(s) > s$ for $t \geq 0$. In this case, we also say that the stationary optimal stock, s, is non-trivial.

3. Monotonicity and Insensitivity Results in Finite Horizon Models

In this section, the monotonicity and insensitivity results of Brock (1971) are generalized to frameworks where the production functions are not required to be concave, and where the utility functions are only assumed to be weakly concave[1]. These results appear to be the most general that are possible for this class of models. Majumdar and Nermuth (1982) establish a stronger monotonicity result with strictly concave utilities[2], but this result is shown, by example, (Example 3.1 below) to be a consequence of their differentiability assumption on production functions.

3 a. Some Preliminary Results

Here, two results are stated, the proofs of which are standard and therefore omitted. These hold under the assumptions in Section 2.

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[1] Brock assumed concavity (and twice differentiability) of both utility and production functions, and strict concavity of utility functions.

[2] Majumdar and Nermuth also deal with the case of irreversible investment. All the proofs here can be generalized to include this phenomenon (see Ray (1983a)).

Lemma 3.1: There exists an ξ-optimal program.

Lemma 3.2: (Principle of Optimality): If $<x_t>_0{}^T$ is ξ-optimal, then $<x_t>_0{}^{T-k}$ is ξ_k-optimal, where $\xi_k = (T-k, s, x_{T-k})$, for $0 \le k < T$.

3 b. Monotonicity and Insensitivity

Theorem 3.1 (monotonicity): Let $\xi = (T, s, b)$, $\xi' = (T, s, b')$ with $b \le b'$. Then

(i) *For every ξ-optimal program $<x_t>$, there is a ξ'-optimal program $<x_t'>$ such that $x_t \le x_t'$ for all $t = 0, \ldots, T$.*

(ii) *For every ξ'-optimal program $<x_t'>$, there is a ξ-optimal program $<x_t>$ such that $x_t \le x_t'$ for all $t = 0, \ldots, T$.*

Proof: We prove Part (i), the proof of (ii) being analogous. To this end, let $<x_t>$ be ξ-optimal. Pick any ξ'-optimal program $<x_t''>$. Suppose there is τ with $x_\tau'' < x_\tau$. Clearly, $\tau < T$, since $x_T'' = b' \ge b = x_T$. Let s be the largest period with $x_s'' < x_s$. Then $x_t'' \ge x_t$ for all $t = s+1, \ldots, T$.

Now define $<x_t'>$ by

$$x_t' = x_t, \quad t \le s, \quad \text{and} \quad x_t' = x_t'', \quad t = s+1, \ldots, T. \tag{3.1}$$

Also define $<x_t>$ by

$$\hat{x}_t = x_t'', \quad t \le s, \quad \text{and} \quad \hat{x}_t = x_t, \quad t = s+1, \ldots, T. \tag{3.2}$$

We show that $<x_t'>$ is ξ'-optimal.

First note that $<x_t'>$ is ξ'-feasible, and $<\hat{x}_t>$ is ξ-feasible. This follows from the inequalities

$$f_s(x_s') - x_{s+1}' = f_s(x_s) - x_{s+1}'' \ge f_s(x_s'') - x_{s+1}'' \ge 0 \tag{3.3}$$

and

$$f_s(\hat{x}_s) - (\hat{x}_{s+1}) = f_s(x_s'') - x_{s+1} \ge f_s(x_s'') - x_{s+1}'' \ge 0. \tag{3.4}$$

Suppose that $<x_t'>$ is not ξ'-optimal. Then

$$\sum_{t=1}^{T} u_t(c_t'') + \sum_{t=1}^{T} u_t(c_t) > \sum_{t=1}^{T} u_t(c_t') + \sum_{t=1}^{T} u_t(\hat{c}_t). \tag{3.5}$$

Note that $\hat{c}_t = c_t''$, $t \le s$, $c_t = \hat{c}_t$, $t = s+2, \ldots, T$, $c_t' = c_t$, $t \le s$, $c_t' = c_t''$, $t = s+2, \ldots, T$. So cancelling common terms,

$$u_{s+1}(c_{s+1}'') + u_{s+1}(c_{s+1}) > u_{s+1}(c_{s+1}') + u_{s+1}(\hat{c}_{s+1}). \tag{3.6}$$

Define $x_{s+1}'' - x_{s+1} \equiv \varepsilon \ge 0$.

Then

$$c_{s+1}{}' = c_{s+1} - \varepsilon \tag{3.7}$$

$$c_{s+1} = c_{s+1}{}'' + \varepsilon. \tag{3.8}$$

Using this in (3.6),

$$u_{s+1}(c_{s+1}{}' + \varepsilon) - u_{s+1}(c_{s+1}{}') > u_{s+1}(c_{s+1}{}'' + \varepsilon) - u_{s+1}(c_{s+1}{}'') \tag{3.9}$$

and, since u_{s+1} is concave, this implies

$$c_{s+1}{}' < c_{s+1}{}''. \tag{3.10}$$

But (3.10) stands in contradiction to the chain of reasoning

$$\begin{aligned} c_{s+1}{}'' &= f_s(x_s{}'') - x_{s+1}{}'' \leq f_s(x_s) - x_{s+1}{}'' \\ &= f_s(x_s{}') - x_{s+1}{}' = c_{s+1}{}'. \quad || \end{aligned} \tag{3.11}$$

Remark: When optimal programs are not unique, note that (i) and (ii) are "independent" propositions.

Theorem 3.1 is the basic monotonicity result. Theorem 3.2 takes a step towards the insensitivity result by establishing monotonicity of finite horizon optimal programs, with zero terminal stock, when the *horizon* is changed.

Theorem 3.2: Let $\xi = (T, s, 0)$, $\xi' = (T+1, s, 0)$. *For every* ξ*-optimal program* $<x_t>$, *there is a* ξ'*-optimal program* $<x_t{}'>$ *with* $x_t \leq x_t{}'$, $t = 0, \ldots, T$.

Proof: Let $<x_t{}''>$ be any ξ'-optimal program. Then $x_T{}'' \geq 0$. Let $\eta = (T, s, x_T{}'')$. Then there is, by Theorem 3.1, an η-optimal program $<\hat{x}_t>$ with $\hat{x}_t \geq x_t$, $t = 0, \ldots, T$. By Lemma 3.2, $<x_t{}''>_0{}^T$ is η-optimal; hence $\sum_{t=1}^{T} u_t(c_t{}'') = \sum_{t=1}^{T} u_t(\hat{c}_t)$. So define $<x_t{}'>$ by $x_t{}' = \hat{x}_t$; $t = 0, \ldots, T$, and $x_{T+1}{}' = x_{T+1}{}''$. It should be clear, then, that $<x_t{}'>$ is ξ'-optimal. Also, $x_t{}' \geq x_t$, $t = 0, \ldots, T$. ||

Observe that the input levels of all feasible programs are bounded above by the pure accumulation program. This, coupled with Theorem 3.2, enables us to construct a sequence, as T varies, of optimal programs to $\xi^T = (T, s, 0)$, with inputs converging pointwise (as $T \to \infty$) to some infinite-horizon *limit program* $<\tilde{x}_t>$. Its feasibility is easily verified.

The insensitivity result is now established, as in Brock (1971), for programs with "not-too-large" terminal stocks (see statement of Theorem 3.3). For given (s, b), define $\xi_T \equiv (T, s, b)$.

Theorem 3.3 (Insensitivity): Consider a limit program $<\tilde{x}_t>$. *For each* $b < \liminf_{t\to\infty} \tilde{x}_t$, *there is a sequence of* ξ_T*-optimal programs* $\{<x_t^T>\}_{T=1}^{\infty}$ *such that* $\lim_{T\to\infty} x_t^T = \tilde{x}_t$ *for all* $t \geq 1$.

Proof of Theorem 3.3: Define $\hat{\xi}_T = (T, s, 0)$, and let $<x_t^T>$ be a sequence (in T) of $\hat{\xi}_T$-optimal programs, with $\hat{x}_t^{T+1} \geq \hat{x}_t^T$, and $\lim_{t\to\infty} \hat{x}_t^T = \tilde{x}_t$, for $t \geq 0$. Such a sequence exists, by Theorem 3.2 and definition of the limit program.

Now, there is $\varepsilon > 0$ and integer M such that $b \leq \tilde{x}_t - \varepsilon$ for all $t \geq M$. For $T \geq M$, pick an $\hat{\xi}_T$-optimal program $<x_t^T>$. Clearly, there exists N (depending on T), $N \geq T$, and an $\hat{\xi}_N$-optimal program $<\hat{x}_t^N>$ with $\hat{x}_T^N > b$.

By Theorem 3.1 (ii), there is a ξ_T-optimal program $<x_t'>$ with $x_t' \leq \hat{x}_t^N$, $t = 0, \ldots, T$. If, in addition, $x_t' \geq \hat{x}_t^T$, $t = 0, \ldots, T$, define $<x_t^T> \equiv <x_t'>$.

If, however, there is $s < T$ such that $x_s' < \hat{x}_s^T$, and $x_t' \geq \hat{x}_t^T$, $t = s+1, \ldots, T$, then define a ξ_T-feasible program $<x_t^T>$ by $x_t^T = \hat{x}_t^T$, $t \leq s$, $x_t^T = x_t'$, $t = s+1, \ldots, T$. Then, by an argument similar to the one in the proof of Theorem 3.1, $<x_t^T>$ is ξ_T-optimal. Also, $x_t^T \geq \hat{x}_t^T$, $t = 0, \ldots, T$.

Summarizing, it is possible to find a ξ_T-optimal program $<x_t^T>$ with

$$\hat{x}_t^T \leq x_t^T \leq \hat{x}_t^N, \quad t = 0, \ldots, T. \tag{3.12}$$

Repeat this for all $T \geq M$. Note that as $T \to \infty$, $N \to \infty$, and so, for all $t \geq 0$,

$$\tilde{x}_t = \lim_{T\to\infty} \hat{x}_t^T \leq \liminf_{T\to\infty} x_t^T \leq \limsup_{T\to\infty} x_t^T \leq \lim_{N\to\infty} \hat{x}_t^N = \tilde{x}_t. \tag{3.13}$$

Hence, $\lim_{T\to\infty} x_t^T = \tilde{x}_t$ for all $t \geq 0$. ||

Remark: This theorem establishes a variant of the Brock insensitivity result. Brock proves that optimal programs are "insensitive" to changes in target stocks when the horizon is large. Unlike Brock we do not necessarily have a unique optimal program. Our result, therefore, is that there *exist* optimal programs which are "insensitive" to changes in target stocks, for large enough horizons.

Majumdar and Nermuth (1982) establish the following version of Theorem 3.1, assuming that utility functions are *strictly* concave and production functions are differentiable. *All* optimal programs to a higher terminal stock exceed (in input levels, weakly) *all* optimal programs to a lower terminal stock. Their technique of proof is different from ours, and leans heavily on the differentiability of production functions[3]. It is interesting to inquire, therefore, whether this stronger result is driven by the strict concavity assumption on utility functions, or by the differentiability of production functions.

We provide an example where all their assumptions, except differentiability of production functions, are satisfied[4]. In this example, an optimal program to a higher terminal stock "crosses" an optimal program to a lower terminal stock, i. e., it exhibits lower input levels at some date, and higher input levels at another.

Example 3.1: $T=3$, and there are two terminal stocks: $b=0$, and $b=1$. Utility functions have the form $u_t(c)=u(c)=c^{1/2}$, $c \geq 0$. The production function $f_t(x)=f(x)$, where

$$\begin{aligned} f(x) &= 3x, \quad x \in [0, 1] \\ &= \frac{x}{3}+\frac{8}{3}, \quad x \in \left[1, \frac{5}{2}\right] \\ &= x+1, \quad x \in \left[\frac{5}{2}, 3\right] \\ &= \frac{x}{2}+\frac{5}{2}, \quad x \in [3, \infty). \end{aligned}$$

Finally, the initial stock, s, is given by $s=f^{-1}(d+1)$, where d is uniquely given by

$$d^{1/2}-(d-2)^{1/2}=3^{1/2}-2^{1/2}.$$

In the context of this example, it is possible to establish

Proposition 3.1:

(i) *There are exactly two optimal programs from* s *to* $b=0$, *given by the input sequences* $(s, 3, 1, 0)$ *and* $(s, 1, 1, 0)$.

(ii) *There are exactly two optimal programs from* s *to* $b=1$, *given by the input sequences* $(s, 3, 1, 1)$ *and* $(s, 1, 1, 1)$.

(iii) *The optimal programs* $(s, 3, 1, 0)$ *and* $(s, 1, 1, 1)$ *cross.*

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[3] The corresponding proof in this paper is considerably shorter.

[4] The example does not have irreversible investment. But the case of reversible investment is within the framework of Majumdar and Nermuth's paper, and so, therefore, is the example.

This provides the counterexample. The proof of this proposition is omitted; the reader is referred to Ray (1983b) for details.

4. Stationary Optimal Programs

In this section, and in the rest of the paper, we will consider a stationary model with discounting; that is, a model in which the production function, utility function and discount factor are all constant over time. Formally, we write $u_t(\cdot) = \delta^t u(\cdot)$, and $f_t(\cdot) = f(\cdot)$ for all t, and make the following additional assumptions, which will be maintained in the remaining part of the paper.

(F.2) $f(0) = 0$, *and there is* $k > 0$ *such that* $x > k$ *implies* $f(x) < x$.

(U.3) $0 < \delta < 1$.

Without loss of generality, we normalize $u(0) = 0$ [5].

Now, we introduce the concept of a *modified golden rule*, which embodies a well-known duality concept dealing with the attainment of given programs by decentralized agents making maximizing decisions on the basis of prices. A *modified golden rule* is defined to be a pair (x^*, p^*) such that $x^* \geq 0$, $p^* \geq 0$, and

(i) $f(x^*) - x^* > 0$,

(ii) $p^* [\delta f(x^*) - x^*] \geq p^* [\delta f(x) - x], \quad x \geq 0$,

(iii) $u(f(x^*) - x^*) - p^* (f(x^*) - x^*) \geq u(c) - p^* c, \quad c \geq 0$.

Observe that condition (ii) implies that "profits" are maximized at x^* if a price of p^* is charged for inputs and δp^* for outputs. Simultaneously, p^* has the property that it "supports" consumption $(f(x^*) - x^*)$ in the sense of (iii): any consumption affording more utility must be more expensive.

In this section, we shall examine some aspects of stationary optimal programs, and their relationship to modified golden rules. After some preliminary results (section 4 a), we describe some general properties of the *set* of all stationary optimal stocks (section 4b).

[5] The case $u(0) = -\infty$ is essentially ruled out by the assumption that u maps $\mathbb{R}_+$ to $\mathbb{R}$. Our analysis can, however, be easily extended to include this case. Some modifications are necessary. For instance, the existence of an optimal program now requires an additional assumption such as: there is a feasible program with $\sum_{t=1}^{\infty} \delta^{t-1} u(c_t) > -\infty$. For a rigorous treatment of such issues, see Ekeland and Scheinkman (1983).

Next, we provide conditions under which a nontrivial stationary optimal stock exists (section 4 c). Finally, we observe an equivalence between the concepts of a modified golden rule and a stationary optimal stock.

4 a. Some Preliminary Results

In this section, we present some standard results, without proof, which are helpful in subsequent discussions.

Lemma 4.1: If $<x_t>$ *is a feasible program from* $s \geq 0$, *then* $x_t \leq \hat{k}$ and $c_{t+1} \leq \hat{k}$ *for* $t \geq 0$, *where* $\hat{k} = \max(k, s)$.

Lemma 4.2: There exists an optimal program from every $s \geq 0$.

Now define a *value function* $V : \mathbb{R}_+ \rightarrow \mathbb{R}$ as follows:

$$V(s) = \sup \{ \sum_{t=1}^{\infty} \delta^{t-1} u(c_t) : <x_t> \text{ is a feasible program from } s \}.$$

This is well-defined, by Lemma 4.1 and (U.3). By Lemma 4.2, there is a feasible program $<x_t^*>$ from s such that $V(s) = \sum_{t=1}^{\infty} \delta^{t-1} \cdot u(c_t^*)$. Using this, one can state some implications of the Principle of Optimality.

Lemma 4.3: If $<x_t>$ *is a feasible program from s, then*

$$V(s) \geq u(c_1) + \delta V(x_1).$$

Lemma 4.4: If $<x_t>$ *is an optimal program from s, then* (i) *for* $T \geq 1$, *the sequence* $<x_t'>$ *defined by* $x_t' = x_{t+T}$ *for* $t \geq 0$, *is an optimal program from* x_T, *and* (ii) $V(s) = u(c_1) + \delta V(x_1)$.

Lemma 4.5: The value function V has the following properties: (i) *V is increasing on* $\mathbb{R}_+$; (ii) *V is continuous on* $\mathbb{R}_+$; (iii) $V(s) \leq u(\hat{k})/(1-\delta)$; (iv) $V(0) = 0$, $V(s) > 0$ *for* $s > 0$.

Lemma 4.6: If $<x_t>$ *is a feasible program from s, and*

$$V(x_t) = u(c_{t+1}) + \delta V(x_{t+1}) \quad \textit{for} \quad t \geq 0$$

then $<x_t>$ *is an optimal program from s.*

Lemma 4.7: If (i) $s \geq 0$; (ii) $f(s) \geq s$, *and* (iii) $V(s) = u[f(s) - s] + \delta V(s)$, *then s is a stationary optimal stock, and* $<x_t>$ *given by* $x_t = s$ *for* $t \geq 0$ *is a stationary optimal program from s.*

Lemma 4.8: If s is a stationary optimal stock, then (i) $s \geq 0$; (ii) $f(s) \geq s$, *and* (iii) $V(s) = u[f(s) - s] + \delta V(s)$.

Lemma 4.9: If (x, p) is a modified golden rule, then $<x_t>$ given by $x_t = x$ for $t \geq 0$ is a nontrivial stationary optimal program from x, and x is a stationary optimal stock. For a proof of Lemma 4.9, see e. g. Peleg and Ryder (1972, 1974).

4 b. The Set of Stationary Optimal Stocks

Let G denote the set of all stationary optimal stocks. It is clearly nonempty, since $0 \in G$.

Given (F.1) and (F.2), it is easy to show that

$$k^* \equiv \min \{s : f(s) - s \geq f(x) - x \text{ for all } x \geq 0\}$$

is well defined. This is the smallest capital stock with the property that it affords maximal net output compared to all other capital stocks.

Proposition 4.1: G is a non-empty compact subset of $[0, k^*]$.

Proof: Clearly G is non-empty since 0 belongs to G. We can next show that G is a subset of $[0, k^*]$. Suppose, on the contrary, there is some $x > k^*$ which belongs to G. Now, $[f(x) - x] \leq [f(k^*) - k^*]$. The stationary program $<x_t>$ given by $x_t = x$ for $t \geq 0$ has $c_t = f(x) - x$ for $t \geq 1$. The sequence $<x_t'>$ given by $x_0' = x$, $x_t' = k^*$ for $t \geq 1$ is a feasible program from x, and has $c_1' = f(x) - k^* = [f(x) - x] + [x - k^*] > [f(x) - x]$; $c_t' = f(k^*) - k^* \geq f(x) - x$ for $t \geq 2$. Thus $<x_t>$ is not an optimal program, so x is not in G. This shows that G is a subset of $[0, k^*]$.

We have checked that G is bounded. To note that G is closed, consider a sequence x^n in G, $x^n \to x$ as $n \to \infty$. Then, for each n, by Lemma 4.8, (i) $x^n \geq 0$, (ii) $f(x^n) \geq x^n$, and (iii) $V(x^n) = u[f(x^n) - x^n] + \delta V(x^n)$. Using $x^n \to x$ as $n \to \infty$, we have (i) $x \geq 0$, (ii) $f(x) \geq x$, and (iii) $V(x) = u[f(x) - x] + \delta V(x)$, using (F.1), (U.1) and Lemma 4.5. So by Lemma 4.7, x is in G. ||

Remarks: (i) Note that k^* could be zero (for example, if $f(x) = x/(1+x)$ for $x \geq 0$). In this case G degenerates automatically to the single point, 0.

(ii) A more interesting case arises when $k^* > 0$. In this case, by its very definition, $f(k^*) > k^*$ [since $f(0) = 0$], so that there is a stationary program from k^*, with stationary positive consumption

$[f(k^*) - k^*]$. We then interpret k^* as a "golden-rule stock"; that is, a stock for which the corresponding stationary program has the maximum stationary positive consumption among all possible stationary programs. In fact, k^* is the smallest of all stocks with this property. (It is a matter of convention only not to call k^* a "golden-rule stock" when k^* happens to be zero).

(iii) It is not possible to refine the interval $[0, k^*]$ further in Proposition 4.1, in view of the following example.

Example 4.1: Let $f(x) = 3x$ for $0 \leq x \leq 1$; $f(x) = \frac{1}{2}x + \frac{5}{2}$ for $x \geq 1$; $u(c) = c/(1+c)$ for $c \geq 0$; $\delta = 1/2$. Here $k^* = 1$, and this is also the only non-trivial stationary optimal stock.

The example also shows that G is, in general, not convex. Here G consists of precisely two points, 0 and 1.

4 c. Existence of a Non-Trivial Stationary Optimal Stock

We noted in Section 4b, that zero always belongs to G, the set of stationary optimal stocks. The zero stock, however, can justifiably be called a "trivial" one. Note that this "trivial" stock might be the *only* one for many production functions (for example, $f(x) = x$ for $x \geq 0$). We, therefore, have to impose an additional condition on f to ensure that a non-trivial stationary optimal stock exists.

In looking for this condition on f, it is worth observing that it should involve δ in an essential way. For instance, imposing the condition that there is some positive stock, x, for which $f(x) > x$ will only ensure the existence of a stationary program from x with positive consumption. But, with δ appropriately chosen, one might still not have a non-trivial stationary *optimal* stock, as the following example shows.

Example 4.2: Let $f(x) = 2x/(1+x)$; $\delta = 1/2$; $u(c) = c/(1+c)$. Clearly $f\left(\frac{1}{2}\right) = 2/3 > 1/2$, while $\delta f(x) = x/(1+x) < x$ for all $x > 0$.

Suppose there is some $s \geq 0$, such that there is a non-trivial stationary optimal program $< x_t >$ from s. Then $f(s) > s$, so $0 < s < 1$.

Then,

$$\sum_{t=1}^{\infty} \delta^{t-1} u(c_t) = u[f(s) - s]/(1-\delta).$$

Now,

$$f(s) - s = [2s/(1+s)] - s = s(1-s)/(1+s).$$

So

$$u\,[f\,(s)-s]=s\,(1-s)/(1+2\,s-s^2)$$

and

$$\sum_{t=1}^{\infty} \delta^{t-1}\,u\,(c_t)=2s\,(1-s)/(1+2\,s-s^2). \tag{4.1}$$

Consider the sequence $<x_t'>$ given by $x_0'=s$, and $x_t'=0$ for $t\geq 1$. Then $<x_t'>$ is a feasible program and

$$\sum_{t=1}^{\infty} \delta^{t-1}\,u\,(c_t')=u\,[f\,(s)].$$

Now,

$$f\,(s)=2s/(1+s), \quad \text{so} \quad u\,[f\,(s)]=2s/(1+3s).$$

Comparing this with (4.1) contradicts the optimality of the stationary program.

Call the production function *δ-productive*[6] if there is $\bar{k}>0$ with $\delta f\,(\bar{k})\geq \bar{k}$. Under this condition, we establish the existence of a modified golden rule, and hence (by Lemma 4.9) the existence of a nontrivial stationary optimal stock.

Theorem 4.1: If f is δ-productive, then there exists a modified golden rule $(\hat{x}, \hat{p})$ with $\hat{x}>0$, $\hat{p}>0$.

Proof: Given (F.1) and (F.2), it is easy to see that

$$F\equiv\{s : \delta f\,(s)-s\geq \delta f\,(x)-x \quad \text{for all} \quad x\geq 0\}$$

is nonempty. Moreover, by the δ-productivity of f, we can pick $\hat{x}\in F$, $\hat{x}>0$. Clearly,

$$\delta f\,(\hat{x})-\hat{x}\geq \delta f\,(x)-x \quad \text{for all} \quad x\geq 0. \tag{4.2}$$

Since $0<\delta<1$ and $\hat{x}>0$, $f\,(\hat{x})-\hat{x}\equiv\hat{c}>0$. Since u is concave, it has a right-hand derivative at $\hat{c}$; call this $\hat{p}$. Then

$$u\,(c)-u\,(\hat{c})\leq \hat{p}\,c-\hat{p}\,\hat{c} \quad \text{for all} \quad c\geq 0 \tag{4.3}$$

or

$$u\,(\hat{c})-\hat{p}\,\hat{c}\geq u\,(c)-\hat{p}\,c \quad \text{for all} \quad c\geq 0. \tag{4.4}$$

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[6] This definition is somewhat weaker than that of Peleg and Ryder (1974), who require the existence of some $x\geq 0$ with $\delta f\,(x)>x$. The production function $f\,(x)=2\,x$, $x\geq 0$ is δ-productive in our sense, if $\delta=1/2$, but not in theirs.

Since u is increasing, we have $\hat{p} > 0$. So, using (4.2)

$$\hat{p}\,[\delta f(\hat{x}) - \hat{x}] \geq \hat{p}\,[\delta f(x) - x] \quad \text{for all} \quad x \geq 0. \tag{4.5}$$

Therefore, $(\hat{x}, \hat{p})$ is a modified golden rule. ||

Theorem 4.2: If f is δ-productive, then there exists a non-trivial stationary optimal stock.

Proof: Use Theorem 4.1 and Lemma 4.9. ||

This result can also be proved as in Dechert and Nishimura (1983). Their proof goes through with our weaker assumptions.

The results here also provide some insight into the relationships between the concepts of δ-productivity, modified golden rule, and non-trivial stationary optimal stock. By Theorem 4.1 and Lemma 4.9, we have the chain of implications (i) f δ-productive implies the existence of a modified golden rule, and (ii) if (x, p) is a modified golden rule then x is a non-trivial stationary optimal stock. Now, the converse of (i) is true, while that of (ii) is not, in general.

That the converse of (i) is true follows from the definition of a modified golden rule and (U.1). That the converse of (ii) is not, in general, true is given by the following example.

Example 4.3: Let

$$\begin{aligned} f(x) &= 2x^{1/2}, \quad x \in [0, 9] \\ &= 15\,(x-9)^{1/2} + 6, \quad x \in [9, \infty) \\ u(c) &= c/(1+c), \quad c \geq 0 \\ \delta &= \frac{1}{2}. \end{aligned}$$

Note that f is not concave. We show that (i) $k \equiv 1/4$ is a non-trivial stationary optimal stock, and (ii) there is no $p \geq 0$ such that (k, p) is a modified golden rule.

It is easily seen that $\delta f(k) - k \geq \delta f(x) - x$ for $x \in [0, 4]$. Now $c^* \equiv f(k) - k = 3/4$. Define $p = u'(3/4) = 16/49$. Then $u(c^*) - pc^* \geq u(c) - pc$ for all $c \geq 0$. Hence (using the fact that any feasible program $< x_t >$ from k must satisfy $x_t \leq 4$ for $t \geq 0$) by the Peleg-Ryder argument (1972, p. 168) k is a stationary optimal stock (clearly non-trivial).

However, note that for $x = 18$, $f(x) = 51$, so $\delta f(x) - x = 7\,{}^1\!/_2$. Also, $\delta f(k) - k = 1/4$. So $\delta f(x) - x > \delta f(k) - k$, hence there is no $p \geq 0$ for which (k, p) is a modified golden rule.

Remark: When f is concave, it is possible to establish the equivalence of the concepts of modified golden rule and stationary optimal stock. We omit a detailed discussion here[7], but note that this yields the added equivalence of the two conditions (i) f is δ-productive, and (ii) there exists a nontrivial stationary optimal stock.

5. Asymptotic Stability of Optimal Programs when the Utility Function is Strictly Concave

In this section, we will establish that optimal programs from arbitrary initial stocks asymptotically approach *some* stationary optimal program. (In another context, Arrow and Hurwicz (1958) have called this kind of asymptotic behavior "system stability".) This seems to be the general stability property that can be established for the aggregative model, when the utility function is strictly concave.

We will, in fact, show that optimal programs are monotone over time in input levels (in a weak sense). This allows us to conclude that optimal programs must converge, and by utilizing the Principle of Optimality in the limit, the asymptotic stock must be a stationary optimal stock[8].

For our purpose, we will assume that the utility function is *strictly concave* on $\mathbb{R}_+$. The case of a weakly concave utility function is somewhat more subtle and explored in the next section.

The monotonic convergence of optimal programs to a stationary optimal stock is an easy consequence of the following lemmas.

Lemma 5.1: If u *is strictly concave,* $\langle x_t \rangle$ *is an optimal program from* s, $\langle x_t' \rangle$ *is an optimal program from* s', and $s > s'$, *then* $x_1 \geq x_1'$.

Proof: See Mitra and Ray (1983), or Dechert and Nishimura (1983), noting that the proof in the latter paper goes through under our weaker assumptions.

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[7] However, since we use this fact in the proof of Lemma 6.3 below, the interested reader is referred to Mitra and Ray (1983) for further details.

[8] Somewhat stronger statements about optimal programs have been established by Dechert and Nishimura (1983) and Majumdar and Nermuth (1982), but these use explicitly the differentiability of both utility and production functions. Example 3.1 above has already suggested that without differentiability of production functions, these results might not be valid.

Lemma 5.2: Suppose u is strictly concave. If $<x_t>$ *is an optimal program from s, then* (i) $x_1 > s$ *implies* $x_{t+1} \geq x_t$ *for* $t \geq 1$, (ii) $x_1 < s$ *implies* $x_{t+1} \leq x_t$ *for* $t \geq 1$.

Proof: See Mitra and Ray (1983), or, with some modifications, Dechert and Nishimura (1983).

Lemma 5.3: Suppose u is strictly concave. If $<x_t>$ *is an optimal program from s, then* (i) $<x_t>$ *is monotonic* (ii) $<x_t>$ *converges to some number, z, in* $[0, k]$, *with* $f(z) \geq z$.

Proof: (i) If $x_t = x_{t+1}$ for $t \geq 0$, then we are done. If not, let r be the first time period for which $x_r \neq x_{r+1}$. If $x_r < x_{r+1}$, then $x_t \leq x_{t+1}$ for $t \geq r$ by Lemma 5.2. If $x_r > x_{r+1}$, then $x_t \geq x_{t+1}$ for $t > r$ by Lemma 5.2. Hence $<x_t>$ is monotonic.

(ii) Since $0 \leq x_t \leq \hat{k}$ for $t \geq 0$, so by using (i), x_t converges to some number, z. Since $x_t \geq 0$ for $t \geq 0$, so $z \geq 0$. Since $f(x_{t-1}) - x_t \geq 0$ for $t \geq 0$, so $f(z) - z \geq 0$, and $z \leq k$, by (F.2). Thus, z is in $[0, k]$. ||

Theorem 5.1: Suppose u is strictly concave. If $<x_t>$ *is an optimal program from s, then* x_t *converges to a stationary optimal stock.*

Proof: By Lemma 5.3, x_t converges to some z in $[0, k]$ with $f(z) \geq z$. Note that for $t \geq 0$

$$V(x_t) = u[f(x_t) - x_{t+1}] + \delta V(x_{t+1}).$$

Since $x_t \to z$ as $t \to \infty$, and u and V are continuous, so

$$V(z) = u[f(z) - z] + \delta V(z).$$

By Lemma 4.7, z is a stationary optimal stock. ||

6. A General Stability Result for Optimal Programs

Although the stability result obtained in Section 5 is fairly general, there are two reasons to remain unsatisfied with it. First, in the context of a non-convex technology set (with a special structure) and a *linear* utility function, Majumdar and Mitra (1983) have shown that optimal programs converge asymptotically to some stationary optimal program. This result *cannot* be viewed as a special case of Theorem 5.1. Some unification should surely be possible for the class of all concave utility functions. Second, "system stability" is in general not true in models with concave utility functions. As an example, let $f(x)=3x$, for $0 \leq x \leq 1$, $f(x)=2x+1$ for $1 \leq x \leq 4$, $f(x)=\frac{1}{2}x+7$ for $x \geq 4$, $u(c)=c$ and $\delta=\frac{1}{2}$. Let $s=2$. Then it is easy to check that the program generated by the input sequence $<x_t>$, given by $x_t=2$, t odd, and $x_t=3$, t even, is optimal. Clearly, no system stability is to be had here. On the other hand, the example also indicates that the oscillating optimal program stays close to the *set* of stationary optimal stocks (in fact at a zero distance from this set, since 2 and 3 both belong to G). Thus, it does not seem outrageous to conjecture that the general stability property of optimal programs should be that the distance from the optimal input level at date t to the *set* of stationary optimal stocks (G), converges to zero as t goes to infinity. In this section, we provide an analysis which confirms this conjecture.

Lemma 6.1: If $<x_t>$ *is an optimal program from* s, *and* $<x_t'>$ *is an optimal program from* s', *and* $s>s'$, *and* $x_1<x_1'$, *then* $(s, x_1', x_2', \ldots)$ *and* $(s', x_1, x_2, \ldots)$ *are optimal programs from* s *and* s' *respectively.*

Proof: The method is basically the same as that used in Lemma 5.1. By Lemma 4.4, we have

$$V(s)=u(c_1)+\delta V(x_1) \tag{6.1}$$

and

$$V(s')=u(c_1')+\delta V(x_1'). \tag{6.2}$$

Consider the sequence $<\bar{x}_t>$ given by $\bar{x}_0=s$, $\bar{x}_t=x_t'$ for $t \geq 1$. Then $f(\bar{x}_0)-\bar{x}_1=f(s)-x_1'>f(s')-x_1'=c_1'$; and $f(\bar{x}_t)-\bar{x}_{t+1}=c'_{t+1}$ for $t \geq 1$. So $<\bar{x}_t>$ is a feasible program from s.

Similarly, consider the sequence $<\bar{x}_t'>$ given by $\bar{x}_0'=s'$, $\bar{x}_t'=x_t$ for $t \geq 1$. Then $f(\bar{x}_0')-\bar{x}_1'=f(s')-x_1>f(s')-x_1'=c_1'$; and $f(\bar{x}_t')-\bar{x}_{t+1}'=c_{t+1}$ for $t \geq 1$. So $<\bar{x}_t'>$ is a feasible program from s'. Suppose, contrary to the Lemma, either $<\bar{x}_t>$ or $<\bar{x}_t'>$ is not optimal. Then, using Lemma 4.3, we have

$$V(s) \geq u(\bar{c}_1)+\delta V(x_1') \tag{6.3}$$

and

$$V(s') \geq u(\bar{c}_1')+\delta V(x_1) \tag{6.4}$$

with strict inequality either in (6.3) or in (6.4). Thus, combining (6.1)—(6.4) we have

$$u(c_1)+u(c_1')>u(\bar{c}_1)+u(\bar{c}_1'). \tag{6.5}$$

Now, $c_1+c_1'=f(s)-x_1+f(s')-x_1'$; and $\bar{c}_1+\bar{c}_1'=f(s)-x_1'+f(s')-x_1$. So $c_1+c_1'=\bar{c}_1+\bar{c}_1'$. Also, $\bar{c}_1=f(s)-x_1'>f(s')-x_1'=c_1'$, while $\bar{c}_1=f(s)-x_1'<f(s)-x_1=c_1$. Thus, there is $0<\theta<1$, such that $\bar{c}_1=\theta c_1+(1-\theta)\,c_1'$. Then, $\bar{c}_1'=c_1+c_1'-\bar{c}_1=c_1+c_1'-\theta c_1-(1-\theta)\cdot c_1'=\theta c_1'+(1-\theta)\,c_1$. So using concavity of u, we have

$$\left.\begin{aligned} u(\bar{c}_1) &= u(\theta c_1+(1-\theta)\,c_1')\geq \theta u(c_1)+(1-\theta)\,u(c_1') \\ u(\bar{c}_1') &= u(\theta c_1'+(1-\theta)\,c_1)\geq \theta u(c_1')+(1-\theta)\,u(c_1) \end{aligned}\right\}. \tag{6.6}$$

It follows from (6.6) that

$$u(\bar{c}_1)+u(\bar{c}_1')\geq u(c_1)+u(c_1') \tag{6.7}$$

which contradicts (6.5), and establishes the result. ||

Remark: Note that the proof of Theorem 3.1 uses a similar argument for a change in *target* stocks.

Lemma 6.2: If $<x_t>$ *is an optimal program, then* (i) $x_t=x_{t+1}$ *for some t implies* x_t *is in G.* (ii) $x_{t-1}<x_t$, $x_t>x_{t+1}$ *for some t implies* x_t *is in G.* (iii) $x_{t-1}>x_t$, $x_t<x_{t+1}$ *for some t implies* x_t *is in G.*

Proof: To prove (i), note that by Lemma 4.4, $V(x_t)=u\,[f(x_t)-x_{t+1}]+\delta V(x_{t+1})$. So using $x_t=x_{t+1}=s$ (say), we know that s is a stationary optimal stock by Lemma 4.7.

To prove (ii), note that $(x_t, x_{t+1}, \ldots)$ is an optimal program from x_t, and $(x_{t-1}, x_t, x_{t+1}, \ldots)$ is an optimal program from x_{t-1}. Since $x_t>x_{t-1}$ and $x_{t+1}<x_t$, so by Lemma 5.1, $(x_t, x_t, x_{t+1}, \ldots)$ is an optimal program from x_t. Now, using (i), x_t is a stationary optimal stock.

The proof of (iii) is similar to (ii), and is therefore omitted. ||

We now introduce the concept of "distance" that is required for our next result. For x, x' in $\mathbb{R}_+$, define the *distance between x and x'* as $d(x, x')\equiv|x-x'|$. Now, for a non-empty set $H\subset\mathbb{R}_+$, and x in $\mathbb{R}_+$, define the *distance between x and H* as $d(x, H)=\inf_{y\in H} d(x, y)$. Since $d(x, y)\geq 0$, for all x, y in $\mathbb{R}_+$, so $d(x, H)$ is well-defined. Note that if x is in H, then $d(x, H)=0$.

Theorem 6.1: If $<x_t>$ *is an optimal program from* $s\geq 0$, *then* $d(x_t, G)\rightarrow 0$ *as* $t\rightarrow\infty$.

Proof: Suppose, on the contrary, there is $\varepsilon>0$ and a subsequence of x_t for which $d(x_t, G)\geq\varepsilon$ for all x_t in the subsequence. There is then a convergent subsequence (of this subsequence), by Lemma 4.1. Call this $<x_{t_i}>$, $i=1, 2, \ldots$, and call its limit z. Since $d(x_{t_i}, G) \geq \varepsilon$, so $d(z, G) \geq \varepsilon$ also. Consequently, there is $\alpha > 0$, such that $z-\alpha > 0$, and the interval $I=[z-\alpha, z+\alpha]$ contains no element of G. Since $x_{t_i} \to z$ as $i \to \infty$, so there is i^*, such that $i \geq i^*$ implies x_{t_i} is in I.

Pick any $i > i^*$. Now, $x_{t_i-1} \neq x_{t_i}$; for if $x_{t_i-1} = x_{t_i}$ then by Lemma 6.2, $x_{t_i} \in G$, a contradiction. So either $x_{t_i-1} > x_{t_i}$ or $x_{t_i-1} < x_{t_i}$. We will suppose that $x_{t_i-1} > x_{t_i}$ (the rest of the proof for the other case follows similar lines).

Now, we claim that for $t_i \leq t \leq t_{i+1}$, $x_t \in I$. If not, let T_1 be the first period $(t_i < T_1 < t_{i+1})$ for which x_{T_1} is not in I. Then, $x_{t_i}, \ldots$ $\ldots, x_{T_1-1}$ are all in I. Since $x_{t_i-1} > x_{t_i}$, so $x_t > x_{t+1}$ for $t_i \leq t \leq T_1-1$ by Lemma 6.2. [If the inequality was violated for some t, with $t_i \leq t \leq T_1-1$, then $x_t \in G$ for the first such t, a contradiction to $x_t \in I$.] Thus, we can conclude that $x_{T_1} < z-\alpha$.

Let T_2 be the first period $(T_1 < T_2 \leq t_{i+1})$ for which x_{T_2} is again in I. [Since $x_{t_{i+1}} \in I$, and x_{T_1} is not in I, T_2 is well-defined.] Then, x_{T_2-1} is not in I; that is, either $x_{T_2-1} > z+\alpha$, or $x_{T_2-1} < z-\alpha$.

If $x_{T_2-1} > z+\alpha$, then since $x_{T_1} < z-\alpha$, we know that $T_2-1 > T_1$, and there is some period T_3 $(T_1 \leq T_3 < T_2-1)$ such that $x_{T_3} < z-\alpha$ while $x_{T_3+1} > z+\alpha$. Then $x_{t_i} \geq z-\alpha > x_{T_3}$, while $x_{t_i+1} < x_{t_i} \leq z+\alpha < x_{T_3+1}$. Thus, $(x_{t_i-1}, x_{t_i}, x_{T_3+1}, x_{T_3+2}, \ldots)$ is an optimal program from x_{t_i-1}, by using Lemma 6.1. Since $x_{t_i-1} > x_{t_i}$, and $x_{T_3+1} > z+\alpha \geq x_{t_i}$, so by Lemma 6.2, $x_{t_i} \in G$, a contradiction. Thus, $x_{T_2-1} > z+\alpha$ is not possible. We conclude therefore that $x_{T_2-1} < z-\alpha$. Since $x_{T_2} \in I$, so we must have $x_{T_2} > x_{T_2-1}$.

Now, if $x_{T_2+1} \leq x_{T_2}$, then $x_{T_2} \in G$ by Lemma 6.2, a contradiction to $x_{T_2} \in I$. So we also must have $x_{T_2+1} > x_{T_2}$.

Finally, we compare x_{T_1-1} and x_{T_2} (i. e., the values of x just before "leaving" the interval I, and just after "re-entering" the interval I). We know that $x_{T_1} < z-\alpha \leq x_{T_1-1}$ while $x_{T_2} \geq z-\alpha > x_{T_2-1}$. Thus, we have $x_{T_1-1} > x_{T_2-1}$ while $x_{T_1} < x_{T_2}$. So applying Lemma 6.1, $(x_{T_1-1}, x_{T_2}, x_{T_2+1}, \ldots)$ is an optimal program from x_{T_1-1}, and $(x_{T_1-2}, x_{T_1-1}, x_{T_2}, x_{T_2+1}, \ldots)$ is an optimal program from x_{T_1-2}. If $x_{T_1-1} = x_{T_2}$, then $x_{T_1-1} \in G$ by Lemma 6.2, a contradiction to $x_{T_1-1} \in I$. If $x_{T_1-1} > x_{T_2}$, then since $x_{T_2+1} > x_{T_2}$, so $x_{T_2} \in G$ by Lemma 6.2, a contradiction to $x_{T_2} \in I$. And, if $x_{T_1-1} < x_{T_2}$, then since

$x_{T_1-2} > x_{T_1-1}$, so by Lemma 6.2, $x_{T_1-1} \in G$, a contradiction to $x_{T_1-1} \in I$. Since these are the only possibilities, we have established our claim that $x_t \in I$ for $t_i \leq t \leq t_{i+1}$. (As mentioned before, if we supposed $x_{t_i-1} < x_{t_i}$, we would establish this claim along similar lines.)

Since $x_t \in I$ for $t_i \leq t \leq t_{i+1}$, and $x_{t_i-1} > x_{t_i}$, so $x_{t-1} > x_t$ for all $t_i \leq t \leq t_{i+1}$. In particular, $x_{t_{i+1}-1} > x_{t_{i+1}}$. So the argument can be repeated for all successive t_i's to get $x_{t-1} > x_t$ for all $t \geq t_i$. But, then x_t converges (being monotonically decreasing, bounded below). And since $x_{t_i} \to z$ as $i \to \infty$ so $x_t \to z$ as $t \to \infty$. Since $V(x_t) = u(f(x_t) - x_{t+1}) + \delta V(x_{t+1})$ for $t \geq 0$, so $V(z) = u(f(z) - z) + \delta V(z)$. By Lemma 4.7, z is a stationary optimal stock. That is, $z \in G$, which contradicts the fact that $z \in I$. This establishes the result. ||

If the production function is concave, and there exists only one nontrivial stationary optimal stock, more powerful stability results may be obtained[9].

Lemma 6.3: Suppose that there is a unique nontrivial stationary optimal stock, x^*, and that f is concave, then $\delta f(x^*) > x^*$.

Proof: By the remark following Theorem 4.2, x^* is associated with a modified golden rule. So $\delta f(x^*) - x^* \geq \delta f(x) - x$ for $x \geq 0$. So $\delta f(x^*) \geq x^*$. If $\delta f(x^*) = x^*$, then $\delta f([1/2]\, x^*) \geq [1/2]\, x^*$, by the concavity of f. Since $\delta f(x^*) = x^*$, this implies that $[1/2]\, x^*$ is also associated with a modified golden rule, and is therefore a stationary optimal stock, a contradiction. ||

Lemma 6.4: Suppose that there is a unique nontrivial stationary optimal stock, and f is concave. Then, there exists $(\tilde{x}, \tilde{\eta}) \gg 0$ such that for all $(x, \eta) \leq (\tilde{x}, \tilde{\eta})$ with $x \geq 0$ and $\eta > 0$,

$$\frac{\delta\,[f(x+\eta) - f(x)]}{\eta} > 1.$$

Proof: Pick $\tilde{\eta} > 0$ such that

$$\frac{\delta\,[f(\tilde{\eta}) - f(0)]}{\tilde{\eta}} > 1.$$

[9] Given a concave production function, the uniqueness of a nontrivial stationary optimal stock is equivalent to assuming a strict concavity property in the neighborhood of some nontrivial stationary optimal stock.

Such $\tilde{\eta}$ exists, given Lemma 6.3 and $f(0)=0$. By the continuity of f, there is $\tilde{x}>0$ such that

$$\frac{\delta\,[f(\tilde{x}+\tilde{\eta})-f(\tilde{x})]}{\tilde{\eta}}>1.$$

Now pick any $(x,\eta)\leq(\tilde{x},\tilde{\eta})$ with $x\geq 0$, $\eta>0$. Then, by the concavity of f,

$$\frac{\delta\,[f(x+\eta)-f(x)]}{\eta}\geq\frac{\delta\,[f(\tilde{x}+\eta)-f(\tilde{x})]}{\eta}\geq$$
$$\geq\frac{\delta\,[f(\tilde{x}+\tilde{\eta})-f(\tilde{x})]}{\tilde{\eta}}>1. \quad \| \qquad (6.7)$$

Lemma 6.5: Suppose that there is $(\tilde{x},\tilde{\eta})$ such that the result in Lemma 6.4 holds. Let $<x_t>$ from $s>0$ be a monotone optimal program. Then inf $x_t>0$.

Proof: Suppose not; then, since $<x_t>$ is monotone, x_t decreases to 0 as $t\to\infty$. If $x_t=0$ for some t, let T be the first period (≥ 1) when this happens. Then, clearly, $c_T>c_{T+1}$. Otherwise, $x_t>0$ for all t. In this case, define T such that $x_{T-1}<\min(\tilde{x},\tilde{\eta})$, and $c_T>c_{T+1}$. Then $x_T<\tilde{x}$, and $f(x_{T-1})>x_{T-1}\geq x_T$ (using Lemma 6.4 with $x_{T-1}=\eta$, and $x=0$). Now pick $\hat{\eta}>0$, such that $\hat{\eta}<\min(\tilde{\eta}, f(x_{T-1})-x_T)$, and $c_{T+1}+f(x_T+\hat{\eta})-f(x_T)<c_T$. Noting that $c_T-\hat{\eta}=f(x_{T-1})-x_T-\eta>0$, we have

$$V(x_{T-1})\geq u(c_T-\hat{\eta})+\delta u(c_{T+1}+f(x_T+\hat{\eta})-f(x_T))+ +\delta^2 V(x_{T+1}). \qquad (6.8)$$

Also,

$$V(x_{T-1})=u(c_T)+\delta u(c_{T+1})+\delta^2 V(x_{T+1}) \qquad (6.9)$$

so, combining (6.8) and (6.9),

$$u(c_T)-u(c_T-\hat{\eta})\geq\delta\,[u(c_{T+1}+f(x_T+\hat{\eta})-f(x_T))-u(c_{T+1})]$$

or

$$\frac{u(c_T)-u(c_T-\hat{\eta})}{\hat{\eta}}\geq\frac{\delta[f(x_T+\hat{\eta})-f(x_T)]}{\hat{\eta}}\left\{\frac{u(c_{T+1}+f(x_T+\hat{\eta})-f(x_T))-u(c_{T+1})}{f(x_T+\hat{\eta})-f(x_T)}\right\}$$
$$>\frac{u(c_{T+1}+f(x_T+\hat{\eta})-f(x_T))-u(c_{T+1})}{f(x_T+\hat{\eta})-f(x_T)} \qquad (6.10)$$

using Lemma 6.4, noting that $(x_T,\hat{\eta})\leq(\tilde{x},\tilde{\eta})$.

But noting that $c_T>c_{T+1}+f(x_T+\hat{\eta})-f(x_T)>c_{T+1}+\hat{\eta}$, (6.10) contradicts the concavity of u. Hence, inf $x_t>0$. $\|$

Theorem 6.2: Suppose that there is a unique nontrivial stationary optimal stock, x^*, and that f is concave. Then, for all $s > 0$, optimal programs $< x_t >$ from s are monotone in inputs, and $x_t \to x^*$ as $t \to \infty$.

Proof: If $x_t = x_{t+1}$ for all $t \geq 0$, we are done. Otherwise, consider the first period T for which $x_T \neq x_{T+1}$. Then either (i) $x_{T+1} < x_T$, or (ii) $x_{T+1} > x_T$.

In case (i), we claim that $x_{t+1} \leq x_t$ for $t > T$. If not, let s be the first period for which $x_{s+1} > x_s$. Then $x_s \leq x_{s-1}$. If $x_s = x_{s-1}$, then x_s is a stationary optimal stock, by Lemma 6.2. If $x_s < x_{s-1}$, then, again by Lemma 6.2, x_s is a stationary optimal stock. Since $x_{s+1} > x_s$, $x_s > 0$, so $x_s = x^*$. Now, clearly $x_{t+1} > x_t$ for $t \geq s$. If not, so that there is a smallest $\tau > s$ for which $x_{\tau+1} \leq x_\tau$, we have x_τ a nontrivial stationary optimal stock, and $x_\tau > x^*$, contradicting our assumption of uniqueness. Since $x_t \leq \hat{k}$ for all t, $x_t \to \bar{x} > x^*$. But then $\bar{x}$ is a nontrivial stationary optimal stock, a contradiction.

So, in Case (i), $x_{t+1} \leq x_t$ for all $t \geq 0$. By Lemma 6.5, $\inf x_t > 0$. So $x_t \to \hat{x} > 0$. Clearly, x is a nontrivial stationary optimal stock, so $\hat{x} = x^*$.

In Case (ii), we claim that $x_{t+1} \geq x_t$ for $t > T$. Following the "mirror argument" of Case (i), we can show that if $x_{s+1} < x_s$ for some first $s \geq T$, then (a) $x_s = x^*$, and (b) $x_{t+1} < x_t$ for $t \geq s$. The program $(x_s, x_{s+1}, \ldots)$ is clearly monotone and optimal from $x_s = x^*$; hence, by Lemma 6.5, $\inf x_t > 0$. But $x_s > x_{s+1} > x_{s+2}, \ldots$. So $\lim x_t = \bar{x} > 0$, with $\bar{x} < x^*$. But $\bar{x}$ must be a nontrivial stationary optimal stock, a contradiction. So in Case (ii), $x_{t+1} \geq x_t$ for all $t \geq 0$, and as in Case (i), $x_t \to x^*$ as $t \to \infty$. ||

Remarks: Note that the proof of Theorem 6.2, which establishes very strong properties of optimal programs, depends only on the fact that a unique nontrivial optimal stationary stock exists, and that the production function satisfies the property stated in Lemma 6.4. Concavity (apart from establishing this property in Lemma 6.4) is nowhere used. So the result is more general, handling, for example, the nonconvex technologies in Majumdar and Mitra (1982, 1983), for discount factors close to one.

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