ERRORS IN MEASURING DISTANCES FROM POPULATIONS TO SERVICE CENTERS

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Abstract

A common practice in spatial analysis is to represent the population of a spatial unit, such as a county or census tract, by a single point, and to use this point when measuring the distance between the population and other places such as service centers. In theoretical spatial systems, distance measurements obtained under this practice may differ from true distances by as much as eight percent, and the difference may be greater for real spatial systems. The presence and magnitude of these measurement errors have important implications for spatial analysis, and particularly for evaluating alternative facility location plans.

I. Introduction

This paper investigates a simplifying operational definition made in many locational analyses. These analyses frequently require the measurement of the distance between members of a dispersed population and some point, such as a service center. The analysis may be greatly simplified if the region to be analyzed contains many small spatial units and if the population of each unit is known. The analyst may then aggregate the individual members of the population residing within each unit, and treat these individuals as though they all were located at a single point. This permits the estimation of $\underline{D}$, the true average distance between the unit's population and the service center, by $\underline{E}$, the distance between the unit's aggregation point and the service center.

The use of $\underline{E}$ to estimate $\underline{D}$ is common in locational analysis. As the first section of this paper demonstrates, however, $\underline{E}$ contains errors from three sources. Some of these errors, which we call Source A errors, are simply inherent in the measurement of the distance to a service center from an aggregation point instead of from the dispersed population. Other errors arise when a service center occurs at an aggregation point or, when several centers are

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involved, if the estimate causes distances from part of the population to be measured to the wrong service center. We call these Source B and Source C errors respectively.

Although other researchers have acknowledged the existence of such errors (2, 7), to our knowledge no one has attempted to examine the three sources separately, or to measure their size. Accordingly, we document the existence and sign of these errors in the next section of this paper; then we compute their size in three theoretical spatial systems. In the final section, we discuss the implications of these errors for locational analysis, with special attention given to facility location planning.

II. Sources of Error in Estimating Average Distance

Source A

Source A errors are inherent in the use of $\underline{E}$, the distance between an aggregation point and a service center, to estimate $\underline{D}$, the true average distance between a population and a service center. To demonstrate the existence of Source A errors, we refer to Figure 1. Assume that the spatial unit in the figure contains a uniformly distributed population P, which for purposes of analysis has been aggregated to point q, the mean center of the population (6). Further assume that a service center is located at k. For convenience, let k lie at the origin of an x-y coordinate system, with the x-axis passing through q. In this case, $\underline{E}$ is simply x_q, the x-coordinate of the aggregation point. Now consider an infinitely thin slice of population dP drawn perpendicular to the x-axis.

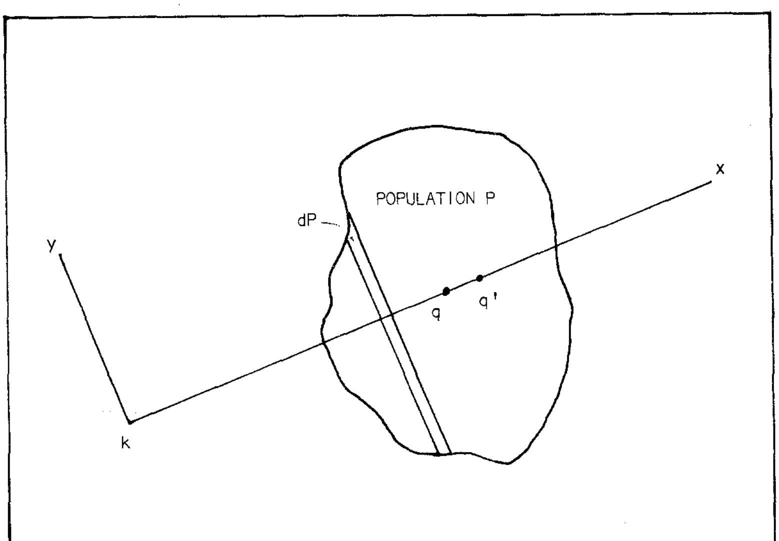

FIGURE 1. EXISTENCE OF SOURCE A ERROR.

The distance from the service center, at k, to the point where the slice intersects the x-axis is simply x_s, the x-coordinate of the slice. The distance r_s, between k and any portion of the slice not on the x-axis, is greater than x_s, or

(1) $r_s > x_s$.

Integrating both sides of this inequality over P and then dividing by P yields

(2) $\frac{1}{P} \int r \, dP > \frac{1}{P} \int x \, dP.$

The left-hand side of this inequality is the average distance from the population P to the service center at k, or D . The right-hand side is the mean value of the x-coordinates of P, or E, since q is the mean center of P. Thus, if the mean center is used as the aggregation point, E will underestimate D. If some other aggregation point is used such as q', E will underestimate D by an even greater amount if the service center is located to the right (Figure 1). On the other hand, E measured from q' may underestimate, equal, or overestimate D for service centers located to the left. In general, E will not equal D for most service center locations; therefore, Source A errors occur if E is used as an estimate of D. The mean center is an attractive choice for an aggregation point because then E will never overestimate D and the Source A errors will always be negative.

Source B

Errors from Source B are much simpler to examine, in part because they are special cases of Source A errors. We consider Source B errors separately because they do not occur in all analyses where the more general Source A errors occur. Source B errors occur if a region has been divided into many small spatial units, and if service centers are assumed to occur at the aggregation points of some of these units. When a service center occurs at the aggregation point of one of these units, then E, the distance from the aggregation point to the service center, is zero. However, if the population is dispersed throughout the unit, then D, must be greater than zero for the unit; therefore, E will always underestimate D, regardless of the location of the aggregation point.

Source C

Source C errors, like Source B errors, do not occur in all locational analyses. Rather, they occur when E is used to estimate D*, the average distance between the population in a spatial unit and its nearest, or most preferred, or other "best" center. When E is used to estimate D*, some individuals will be allocated to the wrong center causing an error in E. We call such errors Source C errors and examine them with help of Figure 2. D and E can be measured from the population in the area to two service centers, located at m and n. Since the solid vertical line in the figure is equidistant from these two

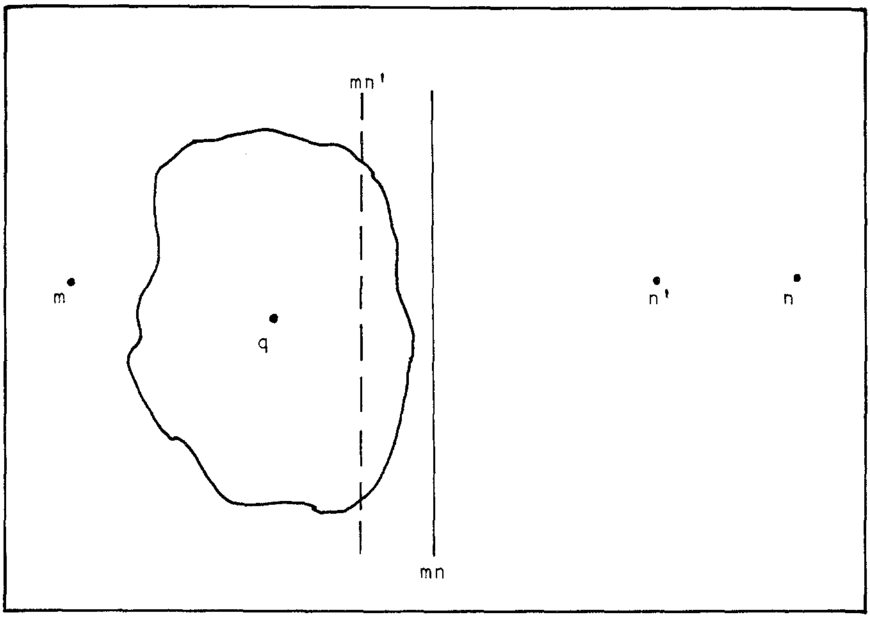

FIGURE 2. EXISTENCE OF SOURCE C ERROR.

centers, D measured to m is equal to D*. E, the estimate of both D and D*, is simply the distance from q to m. Moving the center at n to n', however, makes the vertical dashed line equidistant from the two service centers. Individuals to the right of this line are nearer to n' than to m, and D* will decrease. E, measured from q to m, will remain constant, however, because it is an estimate of the distance from all individuals in the area to the center at m. This mis-allocation of individuals between neighboring centers causes what we call Source C errors. Source C errors are always positive, regardless of the sign of any Source A errors which also may be present in E.

III. Measurement of Errors in E

From the preceding section, it follows that D = E + (Source A error) + (Source B error), and that D* = E + (Source A error) + (Source B error) + (Source C error). Because Source A $\gtrless$ O, Source B < O, and Source C > O, the three errors may either cancel or reinforce each other. In this section, we measure the total error in E as an estimate of D* for service centers in three theoretical spatial systems composed of hexagonal, square, and triangular spatial units. We describe the method of measurement in detail for the hexagonal units, generalize it for use with squares and triangles, and then present and analyze the measurement results.

We assume a population distributed uniformly over a plane, with population density d. We then lay a grid of regular hexagons over the plane and aggregate the population of each hexagon to its mean center. If the radius R of a hexagon is defined as the distance from the mean center to a vertex, then each hexagon contains $3(3)^{1/2}R^2d/2$ individuals. This aggregation procedure converts the uniform population of the plane into equal sized concentrations, located on a regular hexagonal lattice of points.

Next, we locate service centers at regularly spaced points on the lattice in accordance with a Löschian system (5). Any regular spacing of centers on the lattice will give each center a regular, hexagonal, proximal service area containing K lattice or aggregation points, where K varies with the spacing between centers. To compute $\underline{E}$, the estimated average distance between the center and the population of the service area, we use the equation

(3) $$\underline{E} = \frac{1}{K} \sum_{i=1}^{K} s_i$$

where s_i is the distance to the service center from the ith lattice point in the service area.

To compute $\underline{D}^*$ for the same service area, we divide the area into 2n triangular segments (Figure 3), where n is the number of sides of the service area, area, or 6 for a hexagon. Because the 2n triangles are identical, $\underline{D}^*$ for the service area will equal $\underline{D}^*$ for any one of the 2n triangles. Letting $A = \pi/n$, $\underline{D}^*$ is computed as the total distance traveled within a triangle, divided by the population of the triangle, or

(4) $$\underline{D}^* = \frac{d \int_{a=0}^{A} \int_{r=0}^{RK^{1/2} \cos(A) \sec(a)} r^2 \, dr \, da}{(1/2)\, R^2K \cos(A) \sin(A)\, d}$$

Integrating over r gives

(5) $$\underline{D}^* = \frac{2R^3K^{3/2} \cos^3(A)}{3R^2K \cos(A) \sin(A)} \int_{a=0}^{A} \sec^3(a)\, da.$$

Integrating over a and combining terms gives

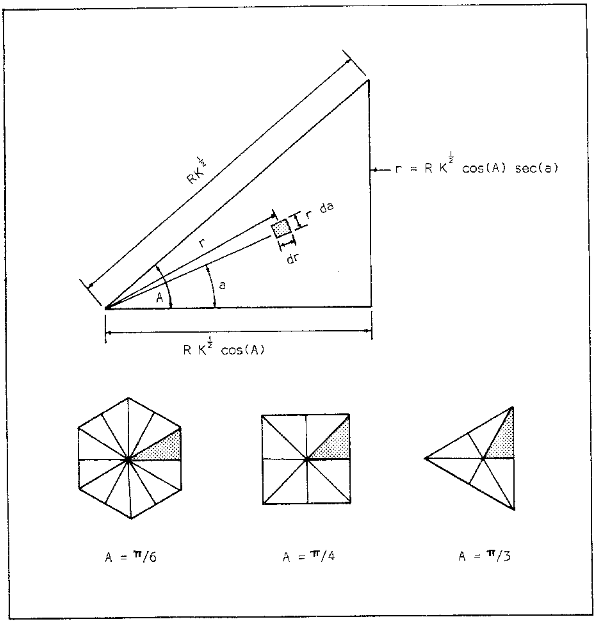

FIGURE 3. MEASUREMENT OF $\underline{D}^*$

(6) $$\underline{D}^* = \frac{RK^{1/2}\cos^2(A)}{3\sin(A)}\left[[\sec(A)\tan(A)] + \log_e[\sec(A)\tan(A)]\right].$$

For hexagons, $A = \pi/6$, and

(7) $$\underline{D}^* = .6080RK^{1/2}.$$

Given equations (3) and (6) and the definition of the radius R of a polygon, the calculation of $\underline{E}$ and $\underline{D}^*$ for grids of squares and triangles is straightforward. These systems have different possible regular spacings of centers and, hence, different values of K, but equation (3) may still be used to compute $\underline{E}$. For squares, $A = \pi/4$ in equation (6) and

(8) $$\underline{D}^* = .5411RK^{1/2}.$$

For triangles, $A = \pi/3$ and

(9) $$\underline{D}^* = .4601RK^{1/2}.$$

Tables 1, 2, and 3 contain the results of our measurements on the hexagonal, square, and triangular systems respectively. Column 1 of each table

TABLE 1

ERRORS IN E FOR THE HEXAGONAL LATTICE SYSTEM

K (1)	E (in units of R) (2)	D* (in units of R) (3)	Total Error (E-D*) (4)	% Total Error (5)	% Source B Error (6)	% Source A + C Error (7)
1	0.000	0.608	-0.608	-100.00	-100.00	0.00
3	1.155	1.053	0.102	9.65	-19.25	28.90
4	1.299	1.216	0.083	6.83	-12.50	19.33
7	1.485	1.609	-0.124	-7.71	-5.40	-2.31
9	1.821	1.824	-0.003	-0.14	-3.70	3.56
12	2.193	2.106	0.087	4.14	-2.41	6.55
13	2.184	2.192	-0.008	-0.37	-2.13	1.76
16	2.462	2.432	0.030	1.27	-1.56	2.78
19	2.588	2.650	-0.062	-2.34	-1.21	-1.13
21	2.778	2.786	-0.008	-0.29	-1.04	0.75
25	3.067	3.040	0.027	0.89	-0.80	1.69
27	3.225	3.159	0.065	2.07	-0.71	2.78
28	3.229	3.217	0.012	0.38	-0.67	1.05
31	3.360	3.385	-0.025	-0.73	-0.58	-0.15
36	3.660	3.648	0.012	0.33	-0.46	0.79
37	3.658	3.698	-0.040	-1.09	-0.44	-0.64
39	3.791	3.797	-0.006	-0.16	-0.41	0.25
43	3.985	3.987	-0.002	-0.05	-0.35	0.30
48	4.264	4.212	0.052	1.23	-0.30	1.53
49	4.262	4.256	0.006	0.13	-0.29	0.42

Columns 5 - 7 are expressed as a percentage of Column 3.

lists the possible values of K for systems through 50.[1] Columns 2 and 3 give E and D* respectively, for each value of K. The difference between D* and E,

[1]In the system based on square units, two different configurations of centers have K values of 25. These two cases appear separately in Table 2. In the triangular system, two different configurations are possible for K values of 49; these two cases appear separately in Table 3.

Spatial irregularities in the triangular lattice of aggregation points occasionally permit regular service areas of equal size to contain different numbers of lattice points. For example, when K = 9, each service area equals the area of nine small triangular units, but each service area contains either 7-1/2 or 9-1/2 lattice points. To compute E for these cases, we have averaged the E values. These irregularities do not affect D*.

TABLE 2

ERRORS IN E FOR THE SQUARE LATTICE SYSTEM

K	E (in units of R)	D* (in units of R)	Total Error (E-D*)	% Error	% Source B Error	% Source A + C Error	% Source A Error (Est.)	% Source C Error (Est.)
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1	0.000	0.541	-0.541	-100.00	-100.00	0.00	0.00*	0.00*
2	0.707	0.765	-0.058	-7.59	-35.36	27.76	-5.53	33.30
4	1.207	1.082	0.125	11.55	-12.50	24.05	-4.75	28.80
5	1.131	1.210	-0.079	-6.49	-8.94	2.45	-4.20	6.66
8	1.561	1.530	0.030	1.98	-4.42	6.40	-3.06	9.47
9	1.517	1.623	-0.106	-6.52	-3.70	-2.81	-2.81*	0.00*
10	1.682	1.711	-0.029	-1.70	-3.16	1.46	-2.60	4.06
13	1.921	1.951	-0.030	-1.54	-2.13	0.59	-2.12	2.71
16	2.248	2.164	0.083	3.85	-1.56	5.41	-1.79	7.20
17	2.213	2.231	-0.018	-0.81	-1.42	0.62	-1.70	2.32
18	2.326	2.296	0.030	1.31	-1.31	2.62	-1.62	4.24
19	2.313	2.358	-0.046	-1.94	-1.21	-0.73	-1.55	0.81
25	2.651	2.705	-0.055	-2.02	-0.80	-1.22	-1.22*	0.00*
25A	2.690	2.705	-0.016	-0.58	-0.80	0.22	-1.22	1.44
26	2.745	2.759	-0.014	-0.50	-0.75	0.25	-1.18	1.43
27	2.821	2.812	0.009	0.32	-0.71	1.04	-1.14	2.18
32	3.087	3.061	0.026	0.86	-0.55	1.41	-0.98	2.39
34	3.146	3.155	-0.009	-0.29	-0.50	0.21	-0.93	1.14
36	3.307	3.246	0.060	1.85	-0.46	2.32	-0.88	3.20
37	3.284	3.291	-0.007	-0.21	-0.44	0.24	-0.86	1.10
40	3.420	3.422	-0.002	-0.07	-0.40	0.33	-0.80	1.13
41	3.455	3.465	-0.010	-0.28	-0.38	0.10	-0.78	0.88
45	3.616	3.630	-0.013	-0.37	-0.33	-0.03	-0.72	0.69
46	3.661	3.670	-0.009	-0.25	-0.32	0.07	-0.71	0.78
49	3.751	3.788	-0.036	-0.96	-0.29	-0.67	-0.67*	0.00*
50	3.818	3.826	-0.008	-0.22	-0.28	0.07	-0.65	0.72

Columns 5 - 9 are expressed as a percentage of Column 3.
K = 25A, See Footnote 1.
*Actual, not estimated, error.

in column 4, is the total error in E, or the sum of the three sources of error. This is expressed as a percentage of D* in column 5.

For each grid type, the percent total error in column 5 is large for small

TABLE 3

ERRORS IN E FOR THE TRIANGULAR LATTICE SYSTEM

K (1)	E (in units of R) (2)	D* (in units of R) (3)	Total Error (E-D)* (4)	% Total Error (5)	% Source B Error (6)	% Source A+C Error (7)	% Source A Error (Est.) (8)	% Source C Error (Est.) (9)
1	0.000	0.460	-0.460	-100.00	-100.00	0.00	0.00*	0.00*
3	0.789	0.797	-0.008	-1.05	-19.25	18.22	-6.45	24.67
4	0.750	0.920	-0.170	-18.49	-12.50	-5.99	-5.99*	0.00*
7	1.171	1.217	-0.046	-3.81	-5.40	1.59	-3.86	5.46
9	1.411	1.380	0.031	2.21	-3.70	5.91	-3.20	9.11
12	1.625	1.594	0.031	1.94	-2.40	4.34	-2.54	6.88
13	1.642	1.659	-0.017	-1.05	-2.13	1.09	-2.38	3.47
16	1.776	1.840	-0.066	-3.57	-1.56	-2.01	-2.01*	0.00*
19	1.985	2.005	-0.020	-1.00	-1.20	0.21	-1.73	1.94
21	2.097	2.109	-0.012	-0.55	-1.04	0.49	-1.58	2.07
25	2.251	2.300	-0.051	-2.16	-0.80	-1.36	-1.36*	0.00*
27	2.416	2.391	0.025	1.07	-0.71	1.78	-1.27	3.05
28	2.419	2.434	-0.015	-0.63	-0.68	0.05	-1.23	1.28
31	2.556	2.561	-0.006	-0.23	-0.58	0.35	-1.12	1.47
36	2.774	2.760	0.014	0.51	-0.46	0.97	-0.98	1.95
37	2.788	2.798	-0.010	-0.39	-0.44	0.06	-0.95	1.01
39	2.861	2.873	-0.012	-0.41	-0.41	0.00	-0.91	0.91
43	3.013	3.017	-0.003	-0.11	-0.35	0.24	-0.83	1.08
48	3.209	3.187	0.021	0.67	-0.30	0.97	-0.75	1.72
49	3.187	3.220	-0.033	-1.03	-0.29	-0.73	-0.73*	0.00*
49A	3.215	3.220	-0.006	-0.17	-0.29	0.12	-0.74	0.86

Columns 5 - 9 are expressed as a percentage of Column 3.
K = 49A, See Footnote 1.
*Actual, not estimated, error.

values of K, and declines in magnitude as K increases. This decrease in percent total error is independent of the number of individuals in the service area, because equations (3) and (6) are functions only of K, the number of spatial units in the service area. Thus, as one would expect, the use of many small spatial units within a service area tends to reduce the relative size of the error in E.

It is possible to separate the percent total error into its component parts.

Source B errors are the easiest to isolate because they are the entire error in E when K = 1, and are constant for other values of K. As one would expect, the relative importance of Source B errors decreases as K, the number of spatial units in the service area, increases (column 6). Source B errors are negative and they mask some of the effects of Source A and C errors.

Because we have aggregated the population of each small spatial unit to its mean center, all Source A errors in the three systems are negative. Thus where the combined errors from Sources A and C in column 7 are positive, the Source C errors, which are always positive, must predominate. We expect the percent combined error in column 7 to be dominated by Source C errors for K values with a high proportion of aggregation points on the service area boundaries. K values of 3, 4, 12, 27, and 48 in the hexagonal system are good illustrations of this (Figure 4a), as are K values of 2, 4, 8, and 16 in the square system, and 3, 9, and 12 in the triangular system. The combined error for these cases in column 7 is both large and positive; the Source C errors alone must be even larger. Where the proportion of aggregation points on or near service area boundaries is low, as for K values of 7 and 19 in the hexagonal system (Figure 4b), Source C errors should be small and Source A errors should and do predominate.

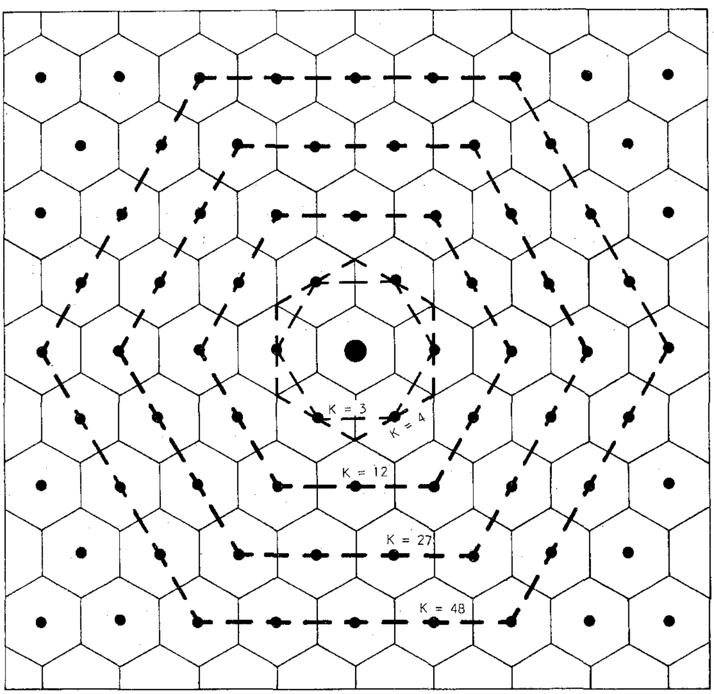

FIGURE 4A. HEXAGONAL CONFIGURATIONS WITH HIGH SOURCE C ERRORS.

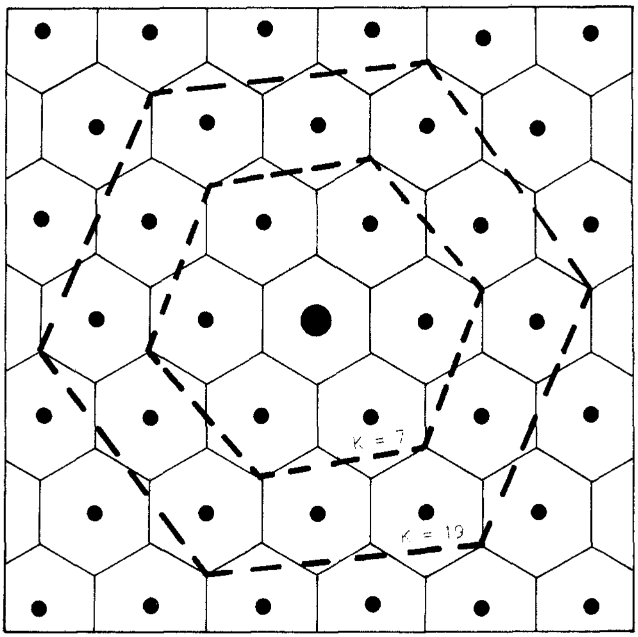

FIGURE 4B. HEXAGONAL CONFIGURATIONS WITH LOW SOURCE C ERRORS.

In the square system, K values of 1, 9, 25 and 49 have no Source C errors at all because the service area boundaries coincide with the boundaries of the small square units. This enabled us to make a detailed analysis of Source A errors for the service area configurations of these four K values. Our analysis indicated that the Source A error in total distance traveled, e_A, is linearly related to the square root of K according to the equation

(10) $$e_A = (.4144 - .4128\ K^{1/2})\ R^3\ d\ .$$

Although we cannot prove that this equation holds for other values of K, the equation does offer a method of separating the combined Source A and Source C errors in column 7 of Table 2. To obtain the Source A error for each value of K, we divided e_A by the service area population and expressed the result as a percentage of $\underline{D}^*$ (Column 8). Column 9 is the difference between columns 7 and 8, and is the estimated Source C error.

A similar equation to (10) can be derived for the triangular system, using K values of 1, 4, 16, 25, and 49 where no Source C errors occur. This equation,

(11) $$e_A = (.2306 - .2488\ K^{1/2})\ R^2\ d\ ,$$

was used to estimate Source A errors (column 8) and Source C errors (column 9) for the triangular system.

It is not possible to develop a similar relationship for the hexagonal system because service area boundaries coincide with the small hexagons only when K = 1. The estimates from equations (10) and (11) are consistent with our earlier conclusions about Source C errors; K values with a high proportion of aggregation points on service area boundaries have high estimated Source C errors. In both the square and triangular systems, the estimated Source C errors tend to be larger than either the Source A or Source B errors.

IV. Implications of Error in E

From the preceding section, it is clear that the total error in E as an estimate of D* is poorly behaved; it may equal ± 2% of D* in relatively large service areas, and ± 8% in small service areas, where the size of a service area is measured by the number of units which it contains. It seems reasonable to expect errors larger than those in Tables 1, 2 and 3 for the irregular spatial units found in real spatial systems. Moreover, in the analysis of actual spatial systems, the location of a spatial unit's mean population center is usually unknown; this would also increase the size and variation of the errors in E.

Leaving aside the presumed errors in E for real spatial systems, even errors as small as those in the theoretical systems can have considerable effect on some kinds of locational analyses. Average distance from a population to its nearest service center, or D*, is a common measure of the quality of a plan of facility locations (1); there is a substantial location-allocation literature devoted to generating and comparing location patterns to find those patterns which minimize D* (4). Accurate comparisons of locational patterns are very important in this literature, but the use of E instead of D* introduces errors into the comparisons. For any given region, many patterns of service centers may have D* values within a few percent of the minimum value (3). If we estimate D* by E, we can also find a pattern of centers with a minimum value of E and many patterns with E values within a few percent of this minimum. Because of the errors in E, if we follow the common practice of the literature and only measure E, we cannot say with certainty which pattern has the minimum D*. In this situation, we might do well to assume that all patterns with low E values are essentially equal, and use some measure other than E to distinguish among them.

Our discussion has focused on the implications of errors in E for facility location planning and location-allocation modeling because the accurate estimation of D* is crucial in these fields. In addition, the use of E to estimate D, without Source C errors, is important in spatial interaction and analysis of the influence of spatial structure on processes which operate within it. The results presented here suggest that when E is measured from the population in the townships of a typical Midwestern county to the county seat (K = 16 or K = 25), E could underestimate D by three to five percent, even if the mean centers of the township populations were known. We believe that this error is large enough to potentially affect the methodological and substantive interpretations of the results of research on such spatial systems. The errors in E may also affect research on other spatial systems. We hope that these errors will receive greater attention in the near future.

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APPENDIX

Derivation of equations (4) through (9).

1. Total distance for one segment (Figure 6) $= d \int_{a=0}^{A} \int_{r=0}^{RK^{1/2}\cos A \sec(a)} r^2 \, dr \, da$

2. Segment population $= (RK^{1/2} \cos A)(RK^{1/2} \cos A \tan A)(1/2)(d)$

   $= (RK^{1/2} \cos A)(RK^{1/2} \sin A)(1/2)(d)$

3. Average distance for one segment, D* $= \dfrac{\text{Total distance}}{\text{Population}}$

$$D^* = \frac{d \int_{a=O}^{A} \int_{r=O}^{RK^{1/2}\cos A \sec(a)} r^2 \, dr \, da}{(RK^{1/2}\cos A)(RK^{1/2}\sin A)(1/2)(d)} \qquad \text{(Equation 4)}$$

$$D^* = \frac{2 \int_{a=O}^{A} \int_{r=O}^{RK^{1/2}\cos A \sec(a)} r^2 \, dr \, da}{R^2 K \cos A \sin A}$$

4. Integrating over r,

$$D^* = \frac{2R^3 (K^{1/2})^3 \cos^3 A \int_{a=O}^{A} \sec^3 (a) \, da}{3R^2 \, K \cos A \sin A} \qquad \text{(Equation 5)}$$

5. Integrating over a,

$$D^* = \frac{2RK^{1/2} \cos^2 A}{3 \sin A} \left[1/2 \left\{ \sec(a)\tan(a) + \log_e(\sec(a) + \tan(a)) \right\} \right]_{a=O}^{A}$$

6. Evaluating at a=O and a=A; sec(O) = 0, tan(O) = 1

$$D^* = \frac{RK^{1/2} \cos^2 A}{3 \sin A} [\sec A \tan A + \log_e(\sec A + \tan A) - \sec O \tan O - \log_e (\sec O + \tan O)]$$

$$D^* = \frac{RK^{1/2} \cos^2 A}{3 \sin A} [(\sec A \tan A) + \log_e (\sec A + \tan A)] \qquad \text{(Equation 6)}$$

7. For hexagons; $A = \pi/6$, $\sin A = 1/2$, $\cos A + 3^{1/2}/2$, $\tan A = 1/(3)^{1/2}$, $\sec A = 2/(3)^{1/2}$

$$D^* = \frac{RK^{1/2}(3/4)}{3(1/2)} \left[\frac{2}{3^{1/2}} \; \frac{1}{3^{1/2}} + \log_e \left(\frac{2}{3^{1/2}} + \frac{1}{3^{1/2}} \right) \right. = .607986 \, RK^{1/2} \qquad \text{(Equation 7)}$$

8. For squares; $A = \pi/4$, $\sin A = \frac{1}{2^{1/2}}$, $\cos A = \frac{1}{2^{1/2}}$, $\tan A = 1$, $\sec A = 2^{1/2}$

$$D^* = \frac{RK^{1/2}(1/2)}{3/(2^{1/2})} \left[(2^{1/2})(1) + \log_e(2^{1/2} + 1) \right] = .541075RK^{1/2}$$

(Equation 8)

9. For equalateral triangles; $A = \pi/3$, $\sin A = 1/2(3^{1/2})$, $\cos A=1/2$, $\tan A= 3^{1/2}$, $\sec A = 2$

$$D^* = \frac{RK^{1/2}(1/4)}{3(3^{1/2}/2)} \cdot \left[(2)\ (3^{1/2}) + \log_e (2 + 3^{1/2}) \right] = .460058RK^{1/2}$$

(Equation 9)