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Symmetries and Scattering Relations in Plane-Stratified Anisotropic, Gyrotropic, and Bianisotropic Media

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Abstract. We consider plane-wave propagation in uniaxial anisotropic, gyrotropic or bianisotropic plane-stratified media, characterized by 6×6 constitutive tensors $\underset{\sim}{\mathbf{K}}$ which relate the wave fields **D** and **B** to **E** and **H**. Biorthogonality of the given and adjoint eigenmodes is derived for all media. Seven different 6×6 diagonal matrices $\underset{\sim}{\mathbf{P}}$ are considered, which either transform $\underset{\sim}{\mathbf{K}}$ into its transpose ($\underset{\sim}{\mathbf{P}}\underset{\sim}{\mathbf{K}}\underset{\sim}{\mathbf{P}}=\underset{\sim}{\tilde{\mathbf{K}}}$), or leave it unchanged, each transformation being applicible to at least one of the media discussed. By applying the transformation to the corresponding adjoint propagation equations, it is shown that the solution of a given propagation problem leads to the formulation and solution of a "conjugate" problem, in which either, or both, of the tangential components of the propagation vector are reversed in sign. Some of the transformations converting $\underset{\sim}{\mathbf{K}}$ to $\underset{\sim}{\tilde{\mathbf{K}}}$ lead to a reciprocity-type scattering relation, with positive-going waves in the given problem being related to negative-going waves in the conjugate problem. Some of the transformations leaving $\underset{\sim}{\mathbf{K}}$ unchanged ($\underset{\sim}{\mathbf{P}}\underset{\sim}{\mathbf{K}}\underset{\sim}{\mathbf{P}}=\underset{\sim}{\mathbf{K}}$) lead to an equivalence relationship between scattering matrices in the two problem.
Interesting consequences with regard to the formulation of Lorentz-type reciprocity relations between currents and fields are envisaged.

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In two earlier papers [1, 2] a scattering theorem was derived for a plane-stratified gyrotropic medium, which was based essentially on a particular transformation of the electric permittivity tensor for a gyrotropic medium. A much more general procedure is here adopted. We consider the 6×6 constitutive tensor $\underset{\sim}{\mathbf{K}}'$ which relates the field vectors **D** and **B** to **E** and **H** at a point:

$$\begin{bmatrix}\mathbf{D}\\ \mathbf{B}\end{bmatrix}=\begin{bmatrix}\underset{\sim}{\boldsymbol{\varepsilon}} & \underset{\sim}{\boldsymbol{\xi}}\\ \underset{\sim}{\boldsymbol{\eta}} & \underset{\sim}{\boldsymbol{\mu}}\end{bmatrix}\begin{bmatrix}\mathbf{E}\\ \mathbf{H}\end{bmatrix}\equiv\underset{\sim}{\mathbf{K}}'\begin{bmatrix}\mathbf{E}\\ \mathbf{H}\end{bmatrix}, \tag{1}$$

where $\underset{\sim}{\boldsymbol{\varepsilon}}$ and $\underset{\sim}{\boldsymbol{\mu}}$ are the 3×3 electric permittivity and magnetic permeability tensors. In anisotropic and gyro-

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tropic media, $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$ are zero, but differ from zero in bianisotropic media. We shall consider seven basic transformations of the tensor $\underset{\sim}{K}'$ each of which will be applicable to at least one of the media discussed and then show that the solution of a given propagation problem will lead to the formulation and solution of a "conjugate" problem, depending on the particular transformation of the $\mathbf{K}'$ matrix adopted, and consequently of the field vectors. A comparison between the given and the conjugate problem will then yield a corresponding scattering relation, in which the reflection and transmission matrices in the two problems are connected.

1. The Given and Adjoint Eigenmodes

We consider plane-wave propagation in a plane-stratified multilayer system. The fields are assumed to be time-harmonic, varying as $\exp(i\omega t)$. The z-axis is taken normal to the stratification, and all fields in a given layer will vary spatially as

$$\mathbf{E},\mathbf{H} \sim \exp[-ik_0(S_1x+S_2y+qz)], \tag{2}$$

where $k_0=\omega/c$, and S_1 and S_2, because of the stratification, are constants for all layers. The homogeneous Maxwell's equations, with the 6×6 constitutive tensor given in (1), may be written in the form

$$\left\{\begin{bmatrix} \underset{\sim}{\varepsilon}/\varepsilon_0 & c\underset{\sim}{\xi} \\ c\underset{\sim}{\eta} & \underset{\sim}{\mu}/\mu_0 \end{bmatrix} - \frac{i}{k_0}\begin{bmatrix} 0 & -\nabla\times\underset{\sim}{I} \\ \nabla\times\underset{\sim}{I} & 0 \end{bmatrix}\right\}\begin{bmatrix}\mathbf{E}\\ \mathscr{H}\end{bmatrix}=0 \tag{3}$$

or, in condensed notation,

$$\left[\underset{\sim}{K}-\frac{i}{k_0}\left\{\underset{\sim}{U}_1\frac{\partial}{\partial x}+\underset{\sim}{U}_2\frac{\partial}{\partial y}+\underset{\sim}{U}_3\frac{\partial}{\partial z}\right\}\right]\mathbf{e}=0 \tag{4}$$

in which $\tilde{\mathbf{e}}=[\tilde{\mathbf{E}},\tilde{\mathscr{H}}]$ and the magnetic field vector, for convenience, is taken as $\mathscr{H}=(\mu_0/\varepsilon_0)^{1/2}\mathbf{H}$; the off-diagonal (coupling) tensors, $\underset{\sim}{\xi}$ and $\underset{\sim}{\eta}$, will be taken to be equal in the later discussion, in which we shall consider the so-called Tellegen (magneto-electric) media [4] and moving media [3]. The symmetric matrices $\underset{\sim}{U}_1$, $\underset{\sim}{U}_2$, and $\underset{\sim}{U}_3$ are given by

$$\underset{\sim}{U}_1=\left[\begin{array}{ccc|ccc} & & & 0 & 0 & 0\\ & 0 & & 0 & 0 & 1\\ & & & 0 & -1 & 0\\ \hline 0 & 0 & 0 & & & \\ 0 & 0 & -1 & & 0 & \\ 0 & 1 & 0 & & & \end{array}\right],$$

$$\underset{\sim}{U}_2=\left[\begin{array}{ccc|ccc} & & & 0 & 0 & -1\\ & 0 & & 0 & 0 & 0\\ & & & 1 & 0 & 0\\ \hline 0 & 0 & 1 & & & \\ 0 & 0 & 0 & & 0 & \\ -1 & 0 & 0 & & & \end{array}\right],$$

$$\underset{\sim}{U}_3=\left[\begin{array}{ccc|ccc} & & & 0 & 1 & 0\\ & 0 & & -1 & 0 & 0\\ & & & 0 & 0 & 0\\ \hline 0 & -1 & 0 & & & \\ 1 & 0 & 0 & & 0 & \\ 0 & 0 & 0 & & & \end{array}\right].$$

With the spatially harmonic variation of $\mathbf{e}(\mathbf{r})$ assumed in (2), Eq. (4) reduces to an eigenvalue equation for q and $\mathbf{e}$:

$$(\underset{\sim}{K}-S_1\underset{\sim}{U}_1-S_2\underset{\sim}{U}_2-q_\alpha\underset{\sim}{U}_3)\mathbf{e}_\alpha=0. \tag{5}$$

The equation

$$\det[\underset{\sim}{K}-S_1\underset{\sim}{U}_1-S_2\underset{\sim}{U}_2-q_\alpha\underset{\sim}{U}_3]=0 \tag{6}$$

is a quartic in q_α (since $\underset{\sim}{U}_3$ has two empty rows), and the corresponding eigenvectors $\mathbf{e}_\alpha$ ($\alpha=\pm1,\pm2$) describe plane waves of which two are positive-going ($\alpha>0$) and two negative-going ($\alpha<0$) with respect to the z-axis.
The adjoint eigenmodes $\bar{\mathbf{e}}_\beta$ are obtained by rewriting (5) for the transposed constitutive tensor $\tilde{\underset{\sim}{K}}$:

$$[\tilde{\underset{\sim}{K}}-S_1\underset{\sim}{U}_1-S_2\underset{\sim}{U}_2-\bar{q}_\beta\underset{\sim}{U}_3]\bar{\mathbf{e}}_\beta=0. \tag{7}$$

Since the 6×6 matrix multiplying $\bar{\mathbf{e}}_\beta$ is the transpose of that multiplying $\mathbf{e}_\alpha$ in (6), ($\underset{\sim}{U}_1$, $\underset{\sim}{U}_2$, and $\underset{\sim}{U}_3$ are symmetric), the eigenvalues are the same

$$\bar{q}_\alpha=q_\alpha. \tag{8}$$

We now premultiply (6) by $\tilde{\bar{\mathbf{e}}}_\beta$, transpose (7) and postimultiply by $\mathbf{e}_\alpha$, and then substract the two to give the biorthogonality relation between the given and adjoint eigenmodes

$$(\bar{q}_\beta-q_\alpha)\tilde{\bar{\mathbf{e}}}_\beta\underset{\sim}{U}_3\mathbf{e}_\alpha=0 \tag{9}$$

which, with suitable normalization, gives

$$\tilde{\bar{\mathbf{e}}}_\beta\underset{\sim}{U}_3\mathbf{e}_\alpha=\operatorname{sign}(\alpha)\delta_{\alpha\beta}. \tag{10}$$

Because the third and sixth rows and columns of $\underset{\sim}{U}_3$ are vacant, (10) is a relation between the four tangential, x and y, components of $\bar{\mathbf{e}}_\beta$ and $\mathbf{e}_\alpha$. If we collect these tangential components to form the vectors $\mathbf{g}$ (or $\bar{\mathbf{g}}$), where typically

$$\tilde{\mathbf{g}}=[E_x,-E_y,\mathscr{H}_x,\mathscr{H}_y] \tag{11}$$

then (10) becomes

$$\tilde{g}_\beta U g_\alpha = \mathrm{sign}(\alpha)\delta_{\alpha\beta},$$

$$U = U^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}. \quad (12)$$

We now collect the four 4-component eigenvectors g_α (or $\bar{g}_\alpha$) to form the 4×4 modal matrices G and the 2×4 modal matrices $G_\pm$:

$$\begin{aligned} G &\equiv [g_1 \quad g_2 \quad g_{-1} \quad g_{-2}] \equiv [G_+ \quad G_-] \\ \bar{G} &\equiv [\bar{g}_1 \quad \bar{g}_2 \quad \bar{g}_{-1} \quad \bar{g}_{-2}] \equiv [\bar{G}_+ \quad \bar{G}_-] \end{aligned} \quad (13)$$

and (12) becomes

$$\tilde{\bar{G}} U G = J, \quad J = J^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}. \quad (14)$$

Hence

$$(\tilde{\bar{G}} U)(GJ) = (GJ)(\tilde{\bar{G}} U) = I,$$

where I is the 4×4 unit matrix, and finally, with $U = U^{-1}$,

$$GJ\tilde{\bar{G}} \equiv [G_+ \quad G_-]\begin{bmatrix} \tilde{\bar{G}}_+ \\ \tilde{\bar{G}}_- \end{bmatrix} = U. \quad (15)$$

This biorthogonality relationship, (14) or (15), is the same as that derived in [1] for a plane-stratified gyrotropic system, and is here seen to apply generally to all plane-stratified media, whatever be the symmetry of their 6×6 constitutive tensors.

2. Transformation of the Constitutive Tensors

We shall seek 6×6 matrices, $P = P^{-1}$, which will provide adjoint transformations of the constitutive tensor K, i.e. will transform K into its transpose $\tilde{K}$

$$\tilde{K} = PKP \quad (16)$$

with the constraint that the symmetric matrices U_1, U_2, and U_3 will be unchanged by the transformation, aside from a possible change in sign:

$$PU_iP = \pm U_i, \quad i = 1, 2, 3.$$

This will ensure that the transformed eigenmode Eq. (5) will lead to transformed eigenvectors that will represent physical propagating modes, i.e. that satisfy Maxwell's equations for the plane stratified medium. These requirements are satisfied by diagonal matrices of the type

$$P \equiv P_{n,\pm n} \equiv \begin{bmatrix} P_n & 0 \\ 0 & \pm P_n \end{bmatrix}, \quad n = 0, 1, 2, 3, \quad (17)$$

$$P_0 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \quad (18)$$

Transformation matrices of the type $P_{-n,\pm n}$ are not considered separately. We note that $P_{-n,\mp n} = -P_{n,\pm n}$, and since

$$PKP = (-P)K(-P), \quad PU_iP = (-P)U_i(-P)$$

no essentially new physical result is given by $P_{-n,\mp n}$, except that subsequently the transformed eigenvectors of the type $e'_\alpha = P\bar{e}_\alpha$ will be reversed in phase, together with their component wave-fields, E'_α and $\mathscr{H}'_\alpha$, but not their Poynting products $E'_\alpha \times H'_\alpha$.

Of the eight matrices considered, $P_{0,0}$ gives the trivial identity transformation, and will not be further discussed. We consider first the transformations $P_{n,\pm n}U_iP_{n,\pm n}$ of U_i, which are listed in Table 1.

Table 1. Transformations of U_1, U_2, and U_3

$P_{n,\pm n}$	$P_{n,\pm n}[U_1, U_2, U_3]P_{n,\pm n}$
$P_{0,-0}$	$-U_1, -U_2, -U_3$
$P_{1,\pm 1}$	$\pm U_1, \mp U_2, \mp U_3$
$P_{2,\pm 2}$	$\mp U_1, \pm U_2, \mp U_3$
$P_{3,\pm 3}$	$\mp U_1, \mp U_2, \pm U_3$

Next we consider transformations of the constitutive tensor K, satisfying (16), for various uniaxial media. For a gyrotropic medium, such as a magnetoplasma, the single symmetry axis will be directed along the external magnetic field. For a moving isotropic medium, which becomes bianisotropic by virtue of the Lorentz field transformations [3], the symmetry axis will be in the direction of motion. We discuss also uniaxial anisotropic and magneto-electric bianisotropic media [3, 4] which have a unique symmetry axis. We choose the x-axis, which is tangential to the stratification, to be in such a direction that the symmetry, or principal, axis of the constitutive tensor K lies in the $x-z$ plane. Hence, if the direction cosines of the symmetry axis are (l, m, n), then in all cases $m = 0$. We consider four types of media.

2.1. Gyrotropic Media. Suppose that either ε or μ (or both) are gyrotropic, such as in a magnetized plasma (gyroelectric) or a ferrite sheet (gyromagnetic). In principal axes, with the axis of symmetry in the $\hat{\mathbf{x}}$-direction, $\underset{\sim}{\varepsilon}$ in general has the form

$$\underset{\sim}{\varepsilon}(m,n=0)=\begin{bmatrix} a-c & 0 & 0 \\ 0 & a & -ib \\ 0 & ib & a \end{bmatrix}, \tag{19}$$

where b is proportional to the external magnetic field, and changes sign if the direction of the field is reversed. If the principal axis (the external magnetic field direction) has an arbitrary orientation in the $x-z$ plane, then $\underset{\sim}{\varepsilon}$ becomes

$$\underset{\sim}{\varepsilon}(m=0)=\begin{bmatrix} a-cl^2 & -ibn & -cln \\ ibn & a & -ibl \\ -cln & ibl & a-cn^2 \end{bmatrix}. \tag{20}$$

If $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\mu}$ have the same symmetry axis, or if either $\underset{\sim}{\varepsilon}$ or $\underset{\sim}{\mu}$ are isotropic, then $\underset{\sim}{\mathbf{P}}_{2,\pm 2}$ are the only two matrices which give an adjoint transformation of $\underset{\sim}{\mathbf{K}}$, i.e.

$$\tilde{\underset{\sim}{\mathbf{K}}}=\underset{\sim}{\mathbf{P}}_{2,\pm 2}\underset{\sim}{\mathbf{K}}\underset{\sim}{\mathbf{P}}_{2,\pm 2}\,. \tag{21}$$

When the gyrotropic axis is tangential to the stratification [along the x-axis, as in (19)], such as in an equatorial ionosphere or a magnetized ferrite sheet, then $\underset{\sim}{\mathbf{P}}_{3,\pm 3}$ also gives an adjoint transformation.

2.2. Uniaxial Anisotropic Media. Tetragonal, hexagonal and rhombohedral crystals are of this type. $\underset{\sim}{\varepsilon}$ is given by (19) or (20) with $b=0$. Both $\underset{\sim}{\mathbf{P}}_{2,\pm 2}$ and $\underset{\sim}{\mathbf{P}}_{0,-0}$ give adjoint transforms of $\underset{\sim}{\mathbf{K}}$. If the optic axis is tangential to the stratification, (in the $\hat{\mathbf{x}}$-direction), then $\underset{\sim}{\mathbf{K}}$ is transformed into $\tilde{\underset{\sim}{\mathbf{K}}}$ by all the matrices $\underset{\sim}{\mathbf{P}}_{n,\pm n}$ $(n=0,1,2,3)$.

2.3. Magneto-Electric Uniaxial Bianisotropic Media. When placed in an electric or a magnetic field the medium becomes both polarized and magnetized. Such a medium was conceived by Tellegen [4] as the basis of a new network element, the gyrator, and the constitutive relation (1), with $\underset{\sim}{\xi}=\underset{\sim}{\eta}$, was proposed by Dzyaloshinskii [5] on theoretical grounds for substances such as antiferromagnetic chromium oxide, in which the 3×3 tensors $\underset{\sim}{\varepsilon}$, $\underset{\sim}{\mu}$, and $\underset{\sim}{\xi}$, all have the same symmetry axis. The constitutive tensor, with the symmetry axis in the $\hat{\mathbf{x}}$ direction, is of the form

$$\underset{\sim}{\mathbf{K}}(m,n=0)=\left[\begin{array}{ccc|ccc} \varepsilon_x & 0 & 0 & \xi_x & 0 & 0 \\ 0 & \varepsilon & 0 & 0 & \xi & 0 \\ 0 & 0 & \varepsilon & 0 & 0 & \xi \\ \hline \xi_x & 0 & 0 & \mu_x & 0 & 0 \\ 0 & \xi & 0 & 0 & \mu & 0 \\ 0 & 0 & \xi & 0 & 0 & \mu \end{array}\right] \tag{22}$$

so that $\underset{\sim}{\mathbf{K}}$ transforms just as the uniaxial anisotropic medium, with $\tilde{\underset{\sim}{\mathbf{K}}}=\underset{\sim}{\mathbf{P}}_{2,2}\underset{\sim}{\mathbf{K}}\underset{\sim}{\mathbf{P}}_{2,2}$ in the general case $(m=0)$, and $\tilde{\underset{\sim}{\mathbf{K}}}=\underset{\sim}{\mathbf{P}}_{n,n}\underset{\sim}{\mathbf{K}}\underset{\sim}{\mathbf{P}}_{n,n}$ $(n=1,2,3)$ when the symmetry axis is parallel to the x-axis, as in (22). The transformation matrices $\underset{\sim}{\mathbf{P}}_{n,-n}$, including $\underset{\sim}{\mathbf{P}}_{0,-0}$, are here unacceptable since they change the sign of $\underset{\sim}{\xi}$.

2.4. Lorentzian Bianisotropy (Moving Media). Suppose that in the rest frame of the moving medium the permittivity and permeability tensors are isotropic

$$\underset{\sim}{\varepsilon}_{\text{r.f.}}=\begin{bmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}, \quad \underset{\sim}{\mu}_{\text{r.f.}}=\begin{bmatrix} \mu & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu \end{bmatrix}. \tag{23}$$

In the laboratory frame, in which the medium appears to be moving with velocity $\mathbf{v}$ in the $\hat{\mathbf{x}}$-direction, the constitutive matrix [Ref. 3, pp. 42–43] becomes

$$\underset{\sim}{\mathbf{K}}(m,n=0)=\left[\begin{array}{ccc|ccc} \varepsilon & 0 & 0 & 0 & 0 & 0 \\ 0 & \varepsilon' & 0 & 0 & 0 & \xi \\ 0 & 0 & \varepsilon' & 0 & -\xi & 0 \\ \hline 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & -\xi & 0 & \mu' & 0 \\ 0 & \xi & 0 & 0 & 0 & \mu' \end{array}\right], \tag{24}$$

where

$$\varepsilon'=\varepsilon(1-\beta^2)/(1-\mathrm{n}^2\beta^2)$$
$$\mu'=\mu(1-\beta^2)/(1-\mathrm{n}^2\beta^2)$$
$$\xi=-(\beta/c)(\mathrm{n}^2-1)/(1-\mathrm{n}^2\beta^2)$$
$$\beta=v/c, \quad \mathrm{n}=c(\varepsilon\mu)^{1/2}.$$

If $\mathbf{v}$ has a component along $\hat{\mathbf{z}}$ too, then $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\mu}$ transform as in (20), with $b=0$. The off-diagonal matrices become

$$\underset{\sim}{\xi}=\begin{bmatrix} 0 & \xi n & 0 \\ -\xi n & 0 & \xi l \\ 0 & -\xi l & 0 \end{bmatrix}$$

and $\underset{\sim}{\mathbf{K}}(m=0)$ cannot be transformed into its transpose by any of the matrices $\underset{\sim}{\mathbf{P}}_{n,\pm n}$. On the other hand, with $\mathbf{v}$ parallel to the x-axis then $\underset{\sim}{\mathbf{K}}$, given by (24), is transformed to $\tilde{\underset{\sim}{\mathbf{K}}}$ by $\underset{\sim}{\mathbf{P}}_{1,\pm 1}$.

3. Transformation of the Adjoint Eigenmode Equation and Formulation of the Conjugate Problem

We substitute (16) into the adjoint eigenmode Eq. (7). To be concrete, suppose $\underset{\sim}{\mathbf{K}}$ transforms to $\tilde{\underset{\sim}{\mathbf{K}}}$ as in (21). With $\underset{\sim}{\mathbf{P}}=\underset{\sim}{\mathbf{P}}_{2,\pm 2}$, and with the aid of Table 1, (7) becomes

$$\mathbf{P}[\mathbf{K} \pm S_1 \mathbf{U}_1 \mp S_2 \mathbf{U}_2 \pm \bar{q}_\alpha \mathbf{U}_3]\,\mathbf{P}\mathbf{e}_\alpha = 0. \qquad (25)$$

Let us choose new propagation constants, S_1' and S_2', such that the coefficients multiplying $\mathbf{U}_1$ and $\mathbf{U}_2$ regain their original values. This requires that

$$S_1' = \mp S_1, \quad S_2' = \pm S_2 \qquad (26)$$

leading to

$$[\mathbf{K} - S_1'\mathbf{U}_1 - S_2'\mathbf{U}_2 \pm \bar{q}_\alpha \mathbf{U}_3]\,\mathbf{P}\bar{\mathbf{e}}_\alpha = 0 \qquad (27)$$

in which we have premultiplied (25) by $\mathbf{P}$ $(=\mathbf{P}^{-1})$. This leads immediately to a conjugate problem in which the tangential propagation constants (the x and y components of the vector $\mathbf{k}/k_0$) are S_1' and S_2' given by (26)

$$[\mathbf{K} - S_1'\mathbf{U}_1 - S_2'\mathbf{U}_2 - q_\beta' \mathbf{U}_3]\,\mathbf{e}_\beta' = 0. \qquad (28)$$

Comparison of (27) and (28) gives a relationship between the conjugate and adjoint eigenmodes. With $\mathbf{P} = \mathbf{P}_{2,2}$ [upper sign in (26) and (27)], we have with the aid of (8)

$$q_\alpha' = -\bar{q}_{-\alpha} = -q_{-\alpha}, \quad \mathbf{e}_\alpha' = \mathbf{P}_{2,2}\bar{\mathbf{e}}_{-\alpha} \quad (\alpha = \pm 1, \pm 2) \qquad (29)$$

(where the sign of the subscript α gives the direction of propagation, positive or negative, with respect to the z-axis). With $\mathbf{P} = \mathbf{P}_{2,-2}$, [lower sign in (27)], we have

$$q_\alpha' = \bar{q}_\alpha = q_\alpha, \quad \mathbf{e}_\alpha' = \mathbf{P}_{2,-2}\bar{\mathbf{e}}_\alpha. \qquad (30)$$

All matrices $\mathbf{P}_{n,\pm n}$ in (17) and (18), except only $\mathbf{P}_{3,-3}$, change the signs of $\mathbf{U}_1$ or $\mathbf{U}_2$ or both (Table 1), and in each case this leads to a conjugate problem in which the new propagation constants, S_1' and S_2', are so chosen as to restore the coefficients of $\mathbf{U}_1$ and $\mathbf{U}_2$ to their initial values. The crucial question then is what happens to the sign of $\mathbf{U}_3$ under this transformation. If it changes sign, then an upgoing mode in the conjugate problem is related to a downgoing adjoint mode, as in (29), and hence to a downgoing mode in the original (given) problem, via (10), leading to a reciprocity type scattering relation. If the sign of $\mathbf{U}_3$ does not change, then upgoing modes in a given and conjugate problem are related as in (30), but the results, mentioned later are of no special interest.

We note finally that, except for the gyrotropic media, all the constitutive tensors discussed are symmetric (self-adjoint), and are therefore unchanged by the adjoint transformation

$$\mathbf{K} = \mathbf{P}_{n,\pm n}\mathbf{K}\mathbf{P}_{n,\pm n}, \quad \mathbf{K} = \tilde{\mathbf{K}}. \qquad (31)$$

In these cases, transformations which leave the sign of $\mathbf{U}_3$ unchanged are of special interest, leading to an equivalence theorem in which the scattering matrices in the given and conjugate problems are identical.

4. Derivation of the Scattering Theorems

We shall here set down in brief and follow a method elaborated in an earlier paper [1]. Consider a plane interface separating media v and $v+1$. Eigenmodes incident on the interface will have amplitudes a_1^v and a_2^v in medium v, and a_{-1}^{v+1} and a_{-2}^{v+1} in medium $v+1$. Waves departing from the interface will similarly have amplitudes a_{-1}^v, a_{-2}^v, a_1^{v+1}, and a_2^{v+1} respectively. We adopt the following notation:

$$\mathbf{a}_+ = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}, \quad \mathbf{a}_- = \begin{bmatrix} a_{-1} \\ a_{-2} \end{bmatrix}, \quad \mathbf{a}_{\text{in}} = \begin{bmatrix} \mathbf{a}_+^v \\ \mathbf{a}_-^{v+1} \end{bmatrix}, \quad \mathbf{a}_{\text{out}} = \begin{bmatrix} \mathbf{a}_-^v \\ \mathbf{a}_+^{v+1} \end{bmatrix} \qquad (33)$$

and define the 4×4 scattering matrix $\mathbf{S}$ by means of

$$\mathbf{a}_{\text{out}} = \mathbf{S}\mathbf{a}_{\text{in}} \qquad (34)$$

or, in full

$$\begin{bmatrix} \mathbf{a}_-^v \\ \mathbf{a}_+^{v+1} \end{bmatrix} = \begin{bmatrix} \mathbf{r}^+ & \mathbf{t}^- \\ \mathbf{t}^+ & \mathbf{r}^- \end{bmatrix} \begin{bmatrix} \mathbf{a}_+^v \\ \mathbf{a}_-^{v+1} \end{bmatrix}, \qquad (35)$$

where $\mathbf{r}^\pm$ and $\mathbf{t}^\pm$ are the 2×2 reflection and transmission matrices comprising the interface scattering matrix $\mathbf{S}$.

In terms of the 2×4 modal matrices $\mathbf{G}_\pm$, introduced in (13), we define the 4×4 modal matrices

$$\mathbf{G}_{\text{in}} = [\mathbf{G}_+^v \quad -\mathbf{G}_-^{v+1}], \quad \mathbf{G}_{\text{out}} = [-\mathbf{G}_-^v \quad \mathbf{G}_+^{v+1}]. \qquad (37)$$

It has been shown in [1] that the scattering matrix $\mathbf{S}$, defined in (34) in terms of modal *amplitudes*, is also related to the modal matrices in (37) through

$$\mathbf{G}_{\text{out}}\mathbf{S} = \mathbf{G}_{\text{in}}, \quad \mathbf{S} = (\mathbf{G}_{\text{out}})^{-1}\mathbf{G}_{\text{in}} \qquad (38)$$

and this result will be essential in the calculations following. We consider two basic types of transformation.

4.1. Type 1 transformations, in which $\mathbf{P}\mathbf{K}\mathbf{P} = \tilde{\mathbf{K}}$, $\mathbf{P}\mathbf{U}_3\mathbf{P} = -\mathbf{U}_3$. The matrices $\mathbf{P}_{0,-0}$, $\mathbf{P}_{1,1}$, $\mathbf{P}_{2,2}$ and $\mathbf{P}_{3,-3}$ belong to this class (Table 1), and the corresponding conjugating transformations are given in Table 2.

We apply (15) to each side, v and $v+1$ of an interface to obtain

$$[\mathbf{G}_+^v \quad \mathbf{G}_-^v]\begin{bmatrix} \tilde{\bar{\mathbf{G}}}_+^v \\ -\tilde{\bar{\mathbf{G}}}_-^v \end{bmatrix} - [\mathbf{G}_+^{v+1} \quad \mathbf{G}_-^{v+1}]\begin{bmatrix} \tilde{\bar{\mathbf{G}}}_+^{v+1} \\ -\tilde{\bar{\mathbf{G}}}_-^{v+1} \end{bmatrix} = \mathbf{U} - \mathbf{U} = 0. \qquad (39)$$

Now for this class of transformation we have, as in (29), $\mathbf{e}_\alpha' = \mathbf{P}\bar{\mathbf{e}}_{-\alpha}$, and in terms of the 4-component eigenvectors $\mathbf{g}_\alpha$ this becomes

$$\mathbf{g}_\alpha' = \mathbf{L}\bar{\mathbf{g}}_{-\alpha}, \quad \bar{\mathbf{g}}_\alpha = \mathbf{L}\mathbf{g}_{-\alpha}' \qquad (40)$$

in which $\mathbf{L}$ $(=\mathbf{L}^{-1})$ is the 4×4 diagonal matrix derived from $\mathbf{P}$ by deleting the third and sixth rows and

Table 2. Type 1 transformations $\mathbf{PKP}=\tilde{\mathbf{K}}$, $\mathbf{PU}_3\mathbf{P}=-\mathbf{U}_3$, and the corresponding conjugate systems

$\mathbf{P}\equiv\mathbf{P}_{n,\pm n}$	Conjugating transformation	Applicable to following unaxial media ($m=0$ in all cases)
$\mathbf{P}_{0,-0}$	$S_1'=-S_1, S_2'=-S_2$	Anisotropic (leading to Lorentz reciprocity)
$\mathbf{P}_{1,1}$	$S_1'=S_1, \quad S_2'=-S_1$	Anisotropic and bianisotropic (magneto-electric) with $l=0$ or $n=0$. Gyrotropic with $l=0$ and bianisotropic (moving media) with $n=0$
$\mathbf{P}_{2,2}$	$S_1'=-S_1, S_2'=S_2$	Gyrotropic, anisotropic and bianisotropic (magneto-electric)
$\mathbf{P}_{3,-3}$	$S_1'=S_1, \quad S_2'=S_2$	Anisotropic with $l=0$ or $n=0$. Gyrotropic with $n=0$. This problem is selfconjugate and leads to: $\mathbf{S}=\tilde{\mathbf{S}}$ ($\mathbf{R}=\tilde{\mathbf{R}}$, $\mathbf{T}^+=\tilde{\mathbf{T}}^-$)

Table 3. Type 2 transformations $\mathbf{PKP}=\mathbf{K}$, $\mathbf{PU}_3\mathbf{P}=\mathbf{U}_3$, and the corresponding conjugate systems

$\mathbf{P}\equiv\mathbf{P}_{n,\pm n}$	Conjugating transformation	Applicable to following uniaxial media ($m=0$ in all cases)
$\mathbf{P}_{1,-1}$	$S_1'=-S_1, S_2'=S_2$	Anisotropic with $l=0$ or $n=0$. Gyrotropic with $n=0$
$\mathbf{P}_{2,-2}$	$S_1'=S_1, \quad S_2'=-S_2$	Anisotropic
$\mathbf{P}_{3,3}$	$S_1'=-S_1, S_2'=-S_2$	All media with $l=0$. Anisotropic and bianisotropic (magneto-electric) with $n=0$

columns. In terms of the 2×4 modal matrices $\mathbf{G}_\pm$ in (13), this becomes

$$\tilde{\mathbf{G}}_\pm=\mathbf{L}\mathbf{G}'_\mp\,. \tag{41}$$

Substituting (41) in (39) and regrouping we obtain

$$\left\{[\mathbf{G}_+^\nu \quad -\mathbf{G}_-^{\nu+1}]\begin{bmatrix}-\tilde{\mathbf{G}}_-'^\nu\\ \tilde{\mathbf{G}}_+'^{\nu+1}\end{bmatrix} -[-\mathbf{G}_-^\nu \quad \mathbf{G}_+^{\nu+1}]\begin{bmatrix}\tilde{\mathbf{G}}_+'^\nu\\ -\tilde{\mathbf{G}}_-'^{\nu+1}\end{bmatrix}\right\}\mathbf{L}=\mathbf{0}$$

or, postmultiplying by $\mathbf{L}^{-1}=\mathbf{L}$,

$$\begin{aligned}&\mathbf{G}_{\text{in}}\tilde{\mathbf{G}}'_{\text{out}}=\mathbf{G}_{\text{out}}\tilde{\mathbf{G}}'_{\text{in}}\\ &\mathbf{G}_{\text{out}}^{-1}\mathbf{G}_{\text{in}}=\text{trans}[(\mathbf{G}'_{\text{out}})^{-1}\mathbf{G}'_{\text{in}}]\end{aligned} \tag{42}$$

which, with (38), gives the scattering relation for a plane interface

$$\mathbf{S}=\tilde{\mathbf{S}}' \tag{43}$$

and with the iteration procedure described in [1, 6], can be extended and applied to any plane-stratified multi-layer medium, in each layer of which the constitutive tensor $\mathbf{K}$ can be transformed by the same $\mathbf{P}$ matrices. Then (43) becomes

$$\mathbf{R}^\pm=\tilde{\mathbf{R}}'^\pm, \quad \mathbf{T}^\pm=\tilde{\mathbf{T}}'^\mp \tag{44}$$

where $\mathbf{R}^\pm$ and $\mathbf{R}'^\pm$, $\mathbf{T}^\pm$ and $\mathbf{T}'^\pm$ are the reflection and transmission matrices for a multilayer slab, for positive- or negative-going incidence, in a given and in a conjugate problem. This result was obtained in [1, 2] for a gyrotropic medium, and is here shown to apply, under suitable conditions, to all the media considered hereto, with the appropriate conjugating transformations in each case given in Table 2.

4.2. Type 2 transformations, in which $\mathbf{PKP}=\mathbf{K}$ and $\mathbf{PU}_3\mathbf{P}=\mathbf{U}_3$.

The matrices $\mathbf{P}_{1,-1}$, $\mathbf{P}_{2,-2}$, and $\mathbf{P}_{3,3}$ belong to this class (Table 1), and the corresponding conjugating transformations are shown in Table 3. With $\mathbf{K}=\tilde{\mathbf{K}}$, i.e. if gyrotropic media are excluded, the eigenmode equation is self-adjoint, and equations such as (30), appropriate for this class of transformation, become in general

$$\mathbf{e}'_\alpha=\mathbf{P}\bar{\mathbf{e}}_\alpha=\mathbf{P}\mathbf{e}_\alpha$$

with the 4-component analogs

$$\mathbf{g}'_\alpha=\mathbf{L}\mathbf{g}_\alpha, \quad \mathbf{L}\mathbf{G}'_\pm=\mathbf{G}_\pm, \quad \mathbf{L}\mathbf{G}'_{\text{in}}=\mathbf{G}_{\text{in}}, \quad \mathbf{L}\mathbf{G}'_{\text{out}}=\mathbf{G}_{\text{out}}\,. \tag{45}$$

Application of (38) gives directly

$$\mathbf{S}=\mathbf{S}' \tag{46}$$

which by iteration [1, 6], may be extended to the multilayer system as a whole to give

$$\mathbf{R}^\pm=\mathbf{R}'^\pm, \quad \mathbf{T}^\pm=\mathbf{T}'^\pm \tag{47}$$

a result which equates the scattering matrices in the given and conjugate problems, as defined by the conjugating transformations in Table 3.

4.3. Scattering Relations that Apply to an Interface Only. Consider first a Type 1 transformation applied to a restricted class of media in which the constitutive tensors are symmetric:

$$\underset{\sim}{P}\underset{\sim}{K}\underset{\sim}{P}=\tilde{\underset{\sim}{K}}=\underset{\sim}{K}, \quad \underset{\sim}{P}\underset{\sim}{U}_3\underset{\sim}{P}=-\underset{\sim}{U}_3.$$

The system is self-adjoint, so that $\bar{\underset{\sim}{G}}_{in}=\underset{\sim}{G}_{in}$, $\bar{\underset{\sim}{G}}_{out}=\underset{\sim}{G}_{out}$, and application of (41) to (37) gives, with $\underset{\sim}{L}=\underset{\sim}{L}^{-1}$,

$$\underset{\sim}{G}_{in}=-\underset{\sim}{L}\underset{\sim}{G}'_{out}, \quad \underset{\sim}{G}_{out}=-\underset{\sim}{L}\underset{\sim}{G}'_{in} \tag{48}$$

and hence, using (38), we have

$$\underset{\sim}{S}=\underset{\sim}{G}_{out}^{-1}\underset{\sim}{G}_{in}=(\underset{\sim}{S}')^{-1}$$

$$\underset{\sim}{S}\underset{\sim}{S}'=\underset{\sim}{I}, \tag{49}$$

where $\underset{\sim}{I}$ is the 4×4 unit matrix. But this is a Type 1 transformation for which (43) applies, $\underset{\sim}{S}=\tilde{\underset{\sim}{S}}'$, giving with (49)

$$\underset{\sim}{S}\tilde{\underset{\sim}{S}}=\tilde{\underset{\sim}{S}}\underset{\sim}{S}=\underset{\sim}{I}. \tag{50}$$

These two results, (49) and (50), apply however to an interface only, and cannot be extended to a multilayer system.

Consider next a Type 2 transformation, $\underset{\sim}{P}\underset{\sim}{U}_3\underset{\sim}{P}=\underset{\sim}{U}_3$ and $\underset{\sim}{P}\underset{\sim}{K}\underset{\sim}{P}=\tilde{\underset{\sim}{K}}=\underset{\sim}{K}$, for which equations such as (30) apply, so that, in general

$$\underset{\sim}{G}'_{\pm}=\underset{\sim}{L}\bar{\underset{\sim}{G}}_{\pm}. \tag{51}$$

Substitution in (39), with rearrangement, yields

$$\underset{\sim}{G}_{out}\underset{\sim}{G}'_{out}=\underset{\sim}{G}_{in}\underset{\sim}{G}'_{in}$$

or

$$\underset{\sim}{S}\tilde{\underset{\sim}{S}}'=\underset{\sim}{I}. \tag{52}$$

Furthermore, we have seen, in (46), that for a Type 2 transformation $\underset{\sim}{S}=\underset{\sim}{S}'$, and hence (52) yields

$$\underset{\sim}{S}\tilde{\underset{\sim}{S}}=\tilde{\underset{\sim}{S}}\underset{\sim}{S}=\mathbf{I}$$

which is the same interface orthogonality relation found in (50) for the Type 1 transformation when $\mathbf{K}=\tilde{\underset{\sim}{K}}$. The conclusion then, is that for all the symmetric constitutive tensors considered (the gyrotropic medium being excluded), the interface scattering matrix $\underset{\sim}{S}$ is orthonormal. For a gyrotropic medium in which $\underset{\sim}{P}\underset{\sim}{U}_3\underset{\sim}{P}=\underset{\sim}{U}_3$, $\underset{\sim}{P}\underset{\sim}{K}\underset{\sim}{P}=\tilde{\underset{\sim}{K}}\neq\underset{\sim}{K}$, then (52) still applies, but not (50), since $\underset{\sim}{S}\neq\tilde{\underset{\sim}{S}}'$.

5. Discussion

In the ionospheric literature scattering relations of the type: $\underset{\sim}{R}^{\pm}=\tilde{\underset{\sim}{R}}'^{\pm}$, $\underset{\sim}{T}^{+}=\tilde{\underset{\sim}{T}}'^{-}$, have often been called "reciprocity" relations. Communication workers, on the other hand, usually understand "reciprocity" to mean Lorentz reciprocity relating currents and fields, or specifically, transmitting and receiving antennas. It could perhaps be expected that this type of scattering relation would lead to a form of Lorentz reciprocity, if one applied a scattering theorem to the angular spectrum of eigenmodes, generated by a transmitting antenna and received by a second antenna, in the presence of, or within, a plane stratified medium. It can be shown [7] that this is indeed so. It is possible to ascribe a simple physical model to the conjugating transformations derived in this paper. Type 1 transformations (in which $\underset{\sim}{P}\underset{\sim}{U}_3\underset{\sim}{P}=-\underset{\sim}{U}_3$), lead to an interchange of transmitter and receiver. For a specific conjugating transformation in which $S'_1=-S_1$, $S'_2=-S_2$, "classical" Lorentz reciprocity is achieved, with the roles of transmitting and receiving antennas interchanged, without a change in their locations or orientations. For other conjugating transformations of this class the roles of receiver and transmitter are also interchanged, but the "conjugate problem" then involves a mirroring of the two antennas with respect to some plane, or rotated with respect to some axis.

In the case of Type 2 transformations which yield an equivalence relation, ($\underset{\sim}{R}^{\pm}=\underset{\sim}{R}'^{\pm}$, $\underset{\sim}{T}^{\pm}=\underset{\sim}{T}'^{\pm}$), these again lead to a relationship between currents and fields, or transmitting and receiving antennas, but these are no longer interchanged in the conjugate problem, but merely mirrored with respect to a plane, or rotated with respect to an axis.

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