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Dynamic Optimization in Non-Convex Models with Irreversible Investment: Monotonicity and Turnpike Results

By

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1. Introduction

In this paper we give a virtually complete characterization of the structure and asymptotic behavior of optimal programs in a one good model of intertemporal optimization distinguished from the "classical" model by two important features:

(i) The technology admits nonconvexity (increasing returns to scale): in fact, we consider arbitrary increasing production functions;

(ii) we allow for irreversibility of investment.

Since actual economic systems exhibit both features, it is of interest to see how the traditional models of economic growth are affected if these factors are taken into account[1]. Independently of this, the generality of our framework is likely to have applications to intertemporal optimization problems that have no direct bearing on economic growth: one may mention here the question of optimal man-

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[1] For the importance of increasing returns cf. e. g. Hicks (1960): "I find it hard to believe that increasing returns and growth by capital accumulation are not tied very closely together. I could quote authority (Adam Smith or Allyn Young) for this belief". The (partial) irreversibility of investment has also often been stressed in the literature. For example, in a different context, it motivated the well known "putty-clay" models in the theory of economic growth.

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agement of renewable resources in which non-convexitites have been recognized (e. g. the literature on optimal fishery-management, cf. Clark, 1971).

We give now a brief overview of the paper. In *Sec. 2*, we consider finite-horizon problems in which the technology and tastes are allowed to vary arbitrarily over time. The main result is that optimal investment must go up (or remain the same) *in every period* if the initial stock and/or the final stock is increased (T. 2.1). This is true even if the optimal program is not unique. Example 2.1 illustrates some qualitative differences between our model and the "classical" one with a convex technology and reversible investment.

In *Sec. 3*, we study optimal infinite-horizon programs (in a "stationary environment") where the future is discounted. The major results are: Every optimal program is necessarily *monotonic* (i. e. there are no "planning cycles", T. 3.1), and converges either to zero ("extinction") or to a "local golden rule" (T. 3.2). In sharp contrast to the classical model, the asymptotic behavior of optimal programs depends crucially on the initial stock. Moreover, for very small discount factors (i. e. high interest rates) all optimal programs are extinction programs, whereas for sufficiently large discount factors all optimal programs converge to the "global golden rule" (T. 3.3, T. 3.4).

In *Sec. 4* we consider the undiscounted case, where optimal programs are defined by the Ramsey-Weizsäcker "overtaking" criterion. The results are analogous to those of Sec. 3, for very low interest rates. I. e. all optimal programs are monotonic (T. 4.1) and converge to the "global golden rule" (T. 4.2).

All proofs are collected in *Sec. 5*. One may note here that the traditional analysis of qualitative properties of optimal programs relies heavily on exploiting (a) the duality theory for optimization problems with convex structures and (b) the "Euler" equation characterizing interior optimal programs. New techniques are needed in our framework since the absence of convexity in the technology precludes a routine application of (a), and (b) is precluded by the irreversibility constraint. Further complications arise out of the non-uniqueness of optimal programs. This accounts for the length and difficulty of our proofs. It should also be poinetd out that while we impose no convexity restrictions on the technology, we do require the utility functions to be strictly concave. This assumption is essential for our results, as the reader may easily convince himself by simple counterexamples.

Turning to the literature, the results of Brock (1971) on sensitivity of optimal programs with respect to changes in the *target* stocks in the traditional convex model *without* irreversibility have been extended to a convex model *with* irreversible investment in a recent note by Mitra (1983). Mitra did observe the difficulties caused by the fact that optimal programs are not necessarily Euler programs when one introduces irreversibility. Mitra (1979) also obtained sensitivity results with respect to changes in *initial* stocks in the convex model *without* irreversibility. Our counterexample and sensitivity results can thus be viewed as a continuation of these and other works cited in these papers.

Majumdar and Mitra (1982) considered the problem of intertemporal allocation with an *S*-shaped technology without irreversibility. In a recent paper, Dechert and Nishimura (1981) re-examined the same model and developed some useful arguments for qualitative analysis. An extended list of references to the literature on non-convexity is given in Majumdar and Mitra (1982).

The problem of optimal growth with irreversibility of investment was treated in a continuous time infinite horizon convex model by Cass (1965), and later by Arrow and Kurz (1970), in another convex model. To be sure, these papers did not deal with sensitivity or non-convexity at all.

The systematic account that follows is of some interest as a synthesis of earlier results that can now be viewed as special cases.

2. Finite-Horizon Programs

In this Section we consider a dynamic optimization model with changing technology and tastes and with irreversibility of investment, in the sense that the capital stock in any period must be at least as large as the capital stock in the previous period, multiplied by a "depreciation factor".

The *technology* is given by a sequence of production functions, g_t from $\mathbb{R}_+$ into itself, and depreciation factors δ_t. We assume, for $t=0, 1, 2, \ldots$:

(A.1) *g_t is differentiable and strictly increasing, with $g_t(0)=0$.*

(A.2) $0 \leqq \delta_t \leqq 1$.

Note that (A.1) allows for arbitrary non-convexities, not just the "*S*-shaped" production functions studied previously.

Given a non-negative input ξ in period t, it is possible to produce a "current" output $g_t(\xi)$ in period $t+1$. The total output (including the depreciated capital stock) is denoted by

$$f_t(\xi) := g_t(\xi) + \delta_t \cdot \xi.$$

The planner's *preferences* are represented by a sequence of real-valued utility functions u_t, defined either on $\mathbb{R}_+$ or $\mathbb{R}_{++}$. We assume, for $t=1, 2, \ldots$:

(A.3) *u_t is differentiable, strictly increasing and strictly concave; and when $u_t(0)$ is undefined,* $\lim_{\gamma \to 0} u_t(\gamma) = -\infty$.

The planning period is given by two integers S, T, with $0 \leqq S < T$, the initial capital stock $a > 0$ is a positive real number, and the final or target stock $b \geqq 0$ is a non-negative real number. We write $\zeta = (S, T, a, b)$ for these parameters, and define

$$b_{S,T}(a) := f_{T-1}(f_{T-2}(\ldots f_S(a))\ldots)$$

as the maximum capital stock attainable at time T starting from initial stock a at time S (by following the "pure accumulation" plan). We make the following consistency assumption on ζ:

(A.4) $b < b_{S,T}(a)$. (I. e. b is attainable from a).

Now consider the dynamic optimization problem:

$$\max U(c) = \sum_{t=S+1}^{T} u_t(c_t) \tag{2.1.a}$$

s. t.

$$\left.\begin{array}{ll} c_t + x_t = f_{t-1}(x_{t-1}) & t = S+1, \ldots, T \\ x_t \geqq \delta_{t-1} x_{t-1} & t = S+1, \ldots, T \\ c_t \geqq 0, \; x_t \geqq 0 & t = S+1, \ldots, T \\ x_S = a, \; x_T \geqq b & \end{array}\right\} \tag{2.1.b}$$

The constraints in (2.1. b) are referred to as the production, irreversibility, non-negativity, and initial and final stock constraints, respectively. Note that $x_t \geqq 0$ is already implied by irreversibility.

c_t (resp. x_t) is consumption (resp. investment) in period t. We write $c = (c_t)_{t=S+1}^{T}$, $x = (x_t)_{t=S}^{T}$, and say that (x, c) is a *feasible* program if (2.1. b) is satisfied. A solution (x, c) to the problem (2.1) is referred to as an *optimal* program. x (resp. c) is called feasible (optimal) if (x, c) is feasible (optimal) for some c (resp. x). For a feasible x, we define the *associated consumption program* $c = (c_t)_{t=S+1}^{T}$ by $c_t := f_{t-1}(x_{t-1}) - x_t$, $S+1 \leqq t \leqq T$, and the associated total utility by

$$U(x) := U(c) = \sum_{t=S+1}^{T} u_t(c_t).$$

Both feasibility and optimality depend on the parameters $\zeta = (S, T, a, b)$. If we want to make this dependence explicit, we say that a program is feasible for ζ (optimal for ζ) or simply ζ-feasible (ζ-optimal). If c^* (ζ) is an optimal consumption program for ζ, we can define the value function at ζ (for the problem (2.1)) by

$$V(\zeta) := U(c^*(\zeta)). \tag{2.2}$$

Under assumptions (A.1)—(A.4) it is easy to prove the following results:

R.2.1: (Existence): *For any ζ, an optimal program exists.*

R.2.2: *V is strictly increasing in a and non-increasing in b,* i. e., *if $a < a'$, $b < b'$, then $V(S, T, a, b') \leqq V(S, T, a, b) < V(S, T, a', b)$.*

R.2.3: (Principle of Optimality): *Let (x, c) be optimal for $\zeta = (S, T, a, b)$. Then*

(i) $V(S, T, a, b) = V(S, s, a, x_s) + V(s, T, x_s, b)$ *for* $S \leqq s \leqq T$.

(ii) *The program* $(x_t)_{t=\alpha}^{\beta}$ *is optimal for* $\xi = (\alpha, \beta, x_\alpha, x_\beta)$ *for* $S \leqq \alpha < \beta \leqq T$.

In what follows, we assume $S=0$ unless otherwise stated, and abbreviate $\zeta = (0, T, a, b)$ to $\zeta = (T, a, b)$.

We say that two programs $x = (x_0, x_1, \ldots, x_T)$, $y = (y_0, y_1, \ldots, y_T)$ *cross* if there are periods r, s, $0 \leqq r, s \leqq T$, such that $x_r < y_r$, $x_s > y_s$.

Two parameters $\zeta = (T, a, b)$, $\zeta' = (T, a', b')$ are called *equivalent* if there exists a program that is optimal for both. Clearly this is possible only if $a = a'$ (cf. R.2.2) and the lower one of the two final stock constraints b, b' is not binding.

We can now state the main result of this section.

T.2.1. (Monotonicity of investment levels). Under (A.1)—(A.4), *let x and x' be optimal for $\zeta = (T, a, b)$ and $\zeta' = (T, a', b')$ respectively, where $a \leqq a'$, $b \leqq b'$. Then x and x' do not cross, and if ζ and ζ' are not equivalent, then $x_t \leqq x_t'$ for $t = 0, 1, \ldots, T$.*

The proof of Theorem 2.1 is based on the following Lemma:

L.2.1. ("Crossing Lemma"): *Under* (A.1)—(A.4) *let* $x = (x_0, x_1, \ldots, x_T)$, $y = (y_0, y_1, \ldots, y_T)$ *be feasible and assume that* $x_0 < y_0$, $x_t = y_t$ *for* $1 \leqq t \leqq T-1$, $x_T > y_T$. *Then there exist feasible programs* $\tilde{x} = (y_0, \tilde{x}_1, \ldots, \tilde{x}_{T-1}, x_T)$, $\tilde{y} = (x_0, \tilde{y}_1, \ldots, \tilde{y}_{T-1}, y_T)$ *such that*

$$A = A(x, y, \tilde{x}, \tilde{y}) := U(\tilde{x}) + U(\tilde{y}) - U(x) - U(y) > 0.$$

If ζ and ζ' are not equivalent, then T.2.1 implies that *every* ζ'-optimal program lies "above" *every* ζ-optimal program, even if the optimal programs are not unique. This is illustrated schematically in Diagram 2.1.

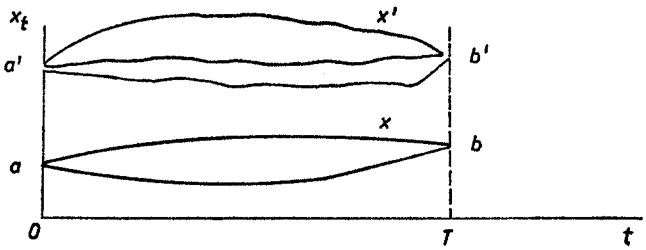

Diagram 2.1

An interesting application of Theorem 2.1 concerns changes not in the target stock, but in the length of the planning horizon. Assume that there is no final stock requirement, i. e. $b=0$. Then, if the planning horizon is increased, investment must go up in every period. Formally:

Cor. 2.1 (Change in Planning Horizon): Let x and x' be optimal for $\zeta=(T, a, 0)$ and $\zeta'=(T', a, 0)$ respectively, where $T \leqq T'$, and assume that ζ and (T, a, x_T') are not equivalent. Then $x_t \leqq x_t'$ for $t=0, 1, \ldots, T$.

This Corollary implies the existence of a "limit path" as the planning horizon T goes to infinity. Among other things, it seems likely that this can be used to extend the work of Hammond, 1975, and Hammond and Kennan, 1979, on "agreeable" resp. "uniformly optimal" plans to the irreversible case; but we do not pursue this question here.

In order to prove a (partial) analog of T.2.1 for consumption levels, we make the further assumption, for $t=0, 1, \ldots$:

(A.5) *g_t is concave.*

R.2.4. (Uniqueness): *Under* (A.1)—(A.5), *the optimal program is unique.*

We denote the optimal program for $\zeta=(T, a, b)$ by $(x^*(\zeta), c^*(\zeta))$. If z, z' are any two scalars or vectors of the same dimension, we denote convex combinations by

$$z^\lambda = \lambda z + (1-\lambda) z', \text{ for } 0 \leqq \lambda \leqq 1.$$

R.2.5. *Under* (A.1)—(A.5) *$V(T, a, b)$ is a concave function of (a, b), i. e., if $\zeta=(T, a, b)$, $\zeta'=(T, a', b')$, $\zeta^\lambda=(T, a^\lambda, b^\lambda)$, then $V(\zeta^\lambda) \geqq \lambda V(\zeta) + (1-\lambda) V(\zeta')$ for $0 \leqq \lambda \leqq 1$.*

T.2.2. (Monotonicity of consumption levels). *Under* (A.1)—(A.5), *assume that* $\zeta=(T, a, b)$, $\zeta'=(T, a', b)$, *with* $a \leqq a'$. *Then* $c_t^*(\zeta) \leqq c_t^*(\zeta')$ *for all* $t=1, 2, \ldots, T$.

For a convex model *without* irreversibility of investment, Brock (1971) showed that optimal consumption is also monotonic with respect to changes in the target stock b, viz. if b is raised, then consumption goes down in every period. Moreover, he and Mitra (1979) obtained *strict* monotonicity of optimal inputs and consumptions with respect to changes in initial or final stocks, whereas our theorems are in the form of weak monotonicity (the relevant inequalities in T.2.1 and T.2.2 are not strict inequalities). It is also known (for a convex model without irreversibility) that the value function V is differentiable (see, e. g., Benveniste and Scheinkman (1979), Mirman and Zilcha (1976)) and, trivially, that the final stock contraint is always binding.

The following example shows that all these properties need not hold in models with irreversible investment (even if the production functions are concave).

Example 2.1. Let $T=2$, $a>0$, $u_t(c_t)=\log c_t$, $f_{t-1}(x_{t-1})=k x_{t-1}$, $\delta_t=\delta$ for $t=1, 2$, and assume that $0<\delta \leqq 1$, $k>\delta$. Define $\underline{b}:=\max\{\delta^2 a, 1/2\, k\, \delta a\}$, $\bar{b}:=k^2 \delta a/(2k-\delta)$. It is easy to verify that $0<\underline{b}<\bar{b}<b_{0,2}(a)=k^2 a$, and (A.1)—(A.5) are satisfied for $\zeta=(2, a, b)$ if and only if $0 \leqq b<k^2 a$. For such ζ, the optimal program $x=(x_1, x_2)$, $c=(c_1, c_2)$ is given by (the tedious computations, based on the Kuhn-Tucker Theorem, are omitted):

(i) If $0 \leqq b \leqq \underline{b}=\delta^2 a$:

$$x=(\delta a, \delta^2 a),\; c=(a(k-\delta), \delta a(k-\delta)).$$

The irreversibility constraint is binding in both periods ("path of pure decumulation").

(ii) If $0 \leqq b \leqq \underline{b}=\frac{1}{2} k \delta a$:

$$x=\left(\frac{1}{2} k a,\, \delta \cdot \frac{1}{2} k a\right),\; c=\left(\frac{1}{2} k a,\, \frac{1}{2} k a(k-\delta)\right).$$

Only the irreversibility constraint in the second period is binding.

(iii) If $\underline{b} \leqq b \leqq \bar{b}$:

$$x=\left(\frac{b}{\delta},\, b\right),\; c=\left(k a-\frac{b}{\delta},\, \frac{b}{\delta}(k-\delta)\right).$$

The irreversibility constraint in the second period and the final stock constraint are binding.

(iv) If $\bar{b} \leqq b < k^2 a$:

$$x = \left(\frac{1}{2k}(k^2 a + b),\ b\right),\ c = \left(\frac{1}{2k}(k^2 a - b),\ \frac{1}{2}(k^2 a - b)\right).$$

Only the final stock constraint is binding.

Perhaps the most striking effect of the irreversibility of investment is that optimal consumption in certain periods may be an *increasing* function of the final stock requirement b (cf. $c_2 = b \cdot \frac{k-\delta}{\delta}$ in Case (iii)). Generally, this will be the case whenever the irreversibility constraint in the last period *and* the final stock constraint are both binding, for then the optimal consumption in the last period is given by

$$c_T = g_{T-1}(x_{T-1}) = g_{T-1}\left[\frac{1}{\delta_{T-1}} \cdot b\right],$$

which is an increasing function of b (where $\delta_{T-1} \neq 0$, of course). The point of Example 2.1 is that it shows that such a situation is not degenerate.

The example also shows that T.2.1 and T.2.2 cannot be strengthened to assert strict monotonicity in any direction (in Case (i), the optimal plan is independent of b; in Case (iii), optimal levels in the second period are independent of a).

Moreover, it is easy to check that the value function $V(T, a, b)$ has a kink both as a function of a and as a function of b at the point $b = \delta^2 a$, provided $\underline{b} = \delta^2 a$ (consider the transition from Case (i) to Case (iii)), i. e., V is not differentiable.

Finally, Example 2.1 shows also that the final stock constraint need not be binding (if $b < \underline{b}$), again in contrast to the model without irreversibility of investment. More generally, one can show:

P.2.1. *Under* (A.1)—(A.5), *for every* (T, a) *there exists some* $\underline{b} = \underline{b}(T, a)$, $0 \leqq \underline{b} < b_{0,T}(a)$ *such that the constraint* $x_T \geqq b$ *is binding for* $\zeta = (T, a, b)$ *if and only if* $b \geqq \underline{b}$.

3. Infinite-Horizon Programs: The Discounted Case

We now give the basic comparative dynamic and turnpike results for the infinite horizon case (in a "stationary environment"). The problem (2.1) is reconsidered with $T = \infty$. The following assumptions are made in this Section, for $t = 0, 1, 2, \ldots$:

(A.1′) $u_{t+1} = \varrho^t \cdot u$, $0 < \varrho < 1$, *where* $u: \mathbb{R}_+ \to \mathbb{R}_+$ *is twice continuously differentiable and satisfies* $u(0) = 0$, $u' > 0$, $u'' < 0$, $\lim_{\gamma \to 0} u'(\gamma) = \infty$.

(A.2′) $\delta_t = \delta$, $0 \leqq \delta < 1$.

(A.3′) *$f_t = f$, where f: $\mathbb{R}_+ \to \mathbb{R}_+$ is twice continuously differentiable and satisfies $f(0) = 0$, $f'(\xi) > \delta$, and there is a maximum sustainable capital stock $\xi_{\max} > 0$ such that $f(\xi_{\max}) = \xi_{\max}$, $f'(\xi) < 1$ for $\xi \geqq \xi_{\max}$. Moreover, the equation $f''(\xi) = 0$ has only finitely many roots.*

(A.4′) $0 < a < \xi_{\max}$.

A capital stock $\hat{x}$ is called a (modified) *local golden rule* if the function $\varrho f(\xi) - \xi$ achieves a local maximum at $\xi = \hat{x}$ and if the associated consumption $\hat{c} = f(\hat{x}) - \hat{x}$ is positive.

A capital stock ξ is called a *steady state* if $\varrho \cdot f'(\xi) = 1$. The second part of assumption (A.3′) means that f consists of finitely many strictly concave (resp. strictly convex) segments.

Our problem now takes the form (by stationarity, we choose the initial time $S = 0$ without loss of generality):

$$\max U(c) = \sum_{t=1}^{\infty} \varrho^{t-1} u(c_t) \tag{3.1.a}$$

s. t.

$$\left.\begin{array}{ll} c_t + x_t = f(x_{t-1}) & \text{for } t = 1, 2, \ldots \\ x_t \geqq \delta \cdot x_{t-1} & \text{for } t = 1, 2, \ldots \\ c_t \geqq 0,\ x_t \geqq 0 & \text{for } t = 1, 2, \ldots \\ x_0 = a & \end{array}\right\} \tag{3.1.b}$$

We use the same notation and terminology as before. In particular, a program (x, c), where $x = (x_t)_{t=0}^{\infty}$, $c = (c_t)_{t=1}^{\infty}$, is called feasible if it satisfies (3.1.b); and *optimal* if it is a solution of problem (3.1). If we want to emphasize the dependence on the initial stock a (the only parameter), we say that (x, c) is *feasible from a* (optimal from a). If $c^* = c^*(a)$ is optimal from a, we define the value of problem (3.1) for initial stock a by $V(a) := U(c^*(a))$.

Under (A.1′)—(A.4′), it is easy to prove[2]:

R.3.1. For any a, an optimal program exists.

[2] R. 3.1., for example, is true because the set of feasible consumption programs defined by (3.1. b) is compact in the product topology, and the target function (3.1. a) is continuous over this set. Formally, R. 3.1. follows from Cor. 1 in Nermuth (1978), p. 293, with minor modifications: simply put $l = 1$, add the irreversibility constraint in (3b) on p. 292, and restrict y to the closed interval $[0, \xi_{\max}]$ in Ass. (A.0. i) on p. 293. Then the assumptions (A.1), (A.2′), (A.3′), (A.4) on pp. 297—298 are satisfied, and hence also the Main Ass. (A.0) on p. 293.

R.3.2: *If* $a < a'$ *then* $V(a) < V(a')$.

R.3.3: *Let* $x = (x_t)_{t=0}^{\infty}$ *be optimal from* a. *Then, for* $s \geqq 1$:

(i) $V(a) = V(s, a, x_s) + \varrho^s \cdot V(x_s)$.

(ii) *The subprogram* $(x_{s+t})_{t=0}^{\infty}$ *is optimal from* x_s.

The basic monotonicity property of finite-horizon programs remains true for infinite-horizon programs:

L.3.1. *Under* (A.1′)—(A.4′), *let* x *and* x' *be optimal from* a *and* a' *respectively, where* $a < a'$. *Then* $x_t \leqq x_t'$ *for all* $t \geqq 0$.

We say that a program $x = (x_t)$ is *monotonic* if either $x_t \leqq x_{t+1}$ for $t = 0, 1, 2, \ldots$ or $x_t \geqq x_{t+1}$ for $t = 0, 1, 2, \ldots$.

T.3.1. (Monotonicity). *Under* (A.1′)—(A.4′), *every optimal program is monotonic.*

The theorem implies that there can be no "planning cycles". Moreover, the irreversibility constraint can be binding in at most finitely many initial periods (except possibly for "extinction programs"). This should be compared with Arrow and Kurz's statement (in a continuous time framework): "... it is possible to have — as an optimal policy — practically any structure of alternating intervals in which the inequality (i. e. irreversibility] constraint is effective or ineffective." (Arrow and Kurz, 1970, p. 332).

We note also that the monotonicity properties L.3.1 and T.3.1 do not really depend on (A.3′), but are true for any production function satisfying $f(0) = 0$, $f'(\xi) > \delta$. The second part of (A.3′) is needed only for the turnpike results below. As a preliminary step, we have

L.3.2. *Every optimal program converges either to zero or to a steady state* $\bar{x}$ *with* $\bar{c} = f(\bar{x}) - \bar{x} > 0$.

We now state the main result of this Section. It says, loosely speaking, that every optimal program converges monotonically to a local golden rule or to zero.

T.3.2. (Turnpike). *Assume* (A.1′)—(A.4′) *and that* $f''(\bar{\xi}) = 0$ *implies* $\varrho f'(\bar{\xi}) \neq 1$ (i. e., *inflection points of* f *do not occur at steady states). Then there exists an index* $n \geqq 0$ *and numbers* $0 = \xi_0 = \hat{x}_0 \leqq \xi_1 \leqq \hat{x}_1 \leqq \xi_2 \leqq \ldots \hat{x}_n \leqq \xi_{n+1} = \xi_{\max}$, *where* $\hat{x}_i$ *is a local golden rule for* $i \geqq 1$, *such that every optimal program from an initial stock* $a \in (\xi_i, \xi_{i+1})$ *converges monotonically to* $\hat{x}_i$, *for* $i = 0, 1, \ldots, n$.

The ξ_i's are called *critical levels*. At initial stocks $a=\xi_i$ the asymptotic behavior of optimal programs changes. When ξ_i is not a steady state, then the proof of T.3.2 shows that all optimal programs starting from $a=\xi_i$ converge either to $\hat{x}_{i-1}$ or $\hat{x}_i$. When ξ_i is a steady state, we cannot rule out the possibility that the constant program $(a, a, \ldots)$ is also optimal from $a=\xi_i$.

We conclude this Section with two theorems characterizing the asymptotic behavior of optimal programs for very small or very large discount factors.

Denote the maximum marginal product by

$$M: = \max \{f'(\xi)/0 \leqq \xi \leqq \xi_{\max}\}.$$

By (A.3′) M is well-defined, positive, and finite.

T.3.3. *If* $\varrho < \frac{1}{M}$, *then all optimal programs converge to zero.*

Note that the condition of T.3.3 is independent of the initial stock a and of the utility function u.

Next we want to show that if ϱ is sufficiently large, then all optimal programs converge to the "best" golden rule, i. e., to the steady state which gives the largest consumption among all steady states. In order to ensure that this "best" golden rule is unique and can be reached from all initial stocks, we assume, in addition to (A.1′)—(A.4′):

(A.5′) *The function* $r_1(\xi) = f(\xi) - \xi$ *is positive in* $(0, \xi_{\max})$ *and achieves a global maximum* $\hat{c} = f(\hat{x}) - \hat{x}$ *at a unique point* $\hat{x} \in (0, \xi_{\max})$. *Moreover,* $r_1'(0) > 0$.

(A.5)′ implies, by continuity, that for all ϱ sufficiently near to one the function $r_\varrho(\xi) := \varrho \cdot f(\xi) - \xi$ is positive in $(0, \xi_{\max})$ and achieves a global maximum $\hat{c}(\varrho) := r_\varrho[\hat{x}(\varrho)]$ at a unique point $\hat{x}(\varrho) \in (0, \xi_{\max})$, and that moreover $r_\varrho'(0) > 0$.

$\hat{x}(\varrho)$ (with associated consumption $\hat{c}(\varrho)$) will be referred to as the *global golden rule* for discount factor ϱ.

T.3.4. *Under* (A.1′)—(A.5′) *there exists a* $\bar{\varrho} < 1$ *such that for* $\varrho \geqq \bar{\varrho}$ *all optimal programs converge to the global golden rule* $\hat{x}(\varrho)$, *independently of the initial stock.*

Summing up the turnpike results of this section, if the future is discounted very heavily (resp. very little), then all optimal programs are extinction programs (resp. converge to the global golden rule), independently of the initial stock. In the intermediate case, the asymptotic behavior of optimal programs depends on the initial stock; in general, they converge to some local golden rule (l. g. r). This distinguishes our model from the "classical" one. Moreover, although the irreversibility constraint cannot be binding in the long run if the program approaches a l. g. r., it still plays a role for its asymptotic behaviour, because it may affect the decision at the initial time between accumulating (and reaching a "higher" l. g. r.) and decumulating (and reaching a "lower" l. g. r.).

4. Infinite Horizon Programs: The Undiscounted Case

In this section we consider the infinite-horizon dynamic optimization problem when future utilities are not discounted, i. e. the discount factor satisfies $\varrho=1$.

With this modification, we continue to assume (A.1′)—(A.5′). Using the same notation and terminology as in Sec. 3, $(\hat{x}, \hat{c})$ is the unique global golden rule, $\hat{u} := u(\hat{c})$ the corresponding one-period utility, and a program (x, c), where $x=(x_t)_{t=0}^{\infty}$, $c=(c_t)_{t=1}^{\infty}$ is called *feasible* (from initial stock a) if it satisfies (3.1.b). We define the total utility associated with (x, c) by

$$U(x)=U(c)=\liminf_{T\to\infty} \sum_{t=1}^{T} [u(c_t)-\hat{u}]. \tag{4.1}$$

A feasible program x is called *good* if $U(x) > -\infty$, and *optimal* (from a) if no other feasible program (from a) gives higher total utility. The following results are well known:

R.4.1. *For every feasible x, either $U(x)=-\infty$ or $U(x)$ is finite.*

R.4.2. *From every initial stock $a \in (0, \xi_{\max})$ there exists an optimal program which is a good program.*

R.4.3. *Let $x=(x_t)_{t=0}^{\infty}$ be optimal from a. Then, for $s \geqq 1$:*

(i) *$(x_{s+t})_{t=0}^{\infty}$ is optimal from x_s.*

(ii) *$(x_t)_{t=0}^{s}$ if optimal for $\zeta=(0, s; a, x_s)$ in the sense of* Sec. 2.

Again we have the monotonicity and turnpike properties:

L.4.1. *If x is optimal from a, x' is optimal from a', and $a<a'$,* then $x_t \leqq x_t'$ for $t=1, 0, 2, \ldots$

T.4.1. *Every optimal program is monotonic.*

T.4.2. *Every optimal program converges monotonically to the global golden rule $\hat{x}$, independently of the initial stock a.*

5. Proofs

Proof of L.2.1: The Lemma is proved by explicitly constructing the "comparison programs" $\tilde{x}, \tilde{y}$. For $T=1$ this is very simple, for the general case a little complicated. The assumption that the utility functions are strictly concave is essential.

The associated consumption programs for $x, y, \tilde{x}, \tilde{y}$ will be denoted by $c, d, \tilde{c}, \tilde{d}$ respectively. Assume first that $U(x) > -\infty$.

$T=1$:

Def. $\tilde{x}: = (y_0; x_1)$

$\tilde{y}: = (x_0; y_1)$

This is obviously feasible; moreover, $d_1 = \tilde{c}_1 + \varepsilon$, $\tilde{d}_1 = c_1 + \varepsilon$, $c_1 < \tilde{c}_1$ where $\varepsilon: = x_1 - y_1 > 0$. This leads to $A = u_1(c_1+\varepsilon) - u_1(c_1) - [u_1(\tilde{c}_1+\varepsilon) = u_1(\tilde{c}_1)] > 0$, by strict concavity of u_1.

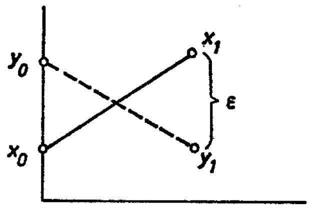

Diagram 5.1

$T \geqq 2$:

Choose $\varepsilon > 0$ and define

$$\tilde{x}: = (y_0; x_1+\varepsilon, H_2(x_1+\varepsilon), \ldots H_{T-1}(x_1+\varepsilon), x_T)$$
$$\tilde{y}: = (x_0; x_1-\varepsilon, H_2(x_1-\varepsilon), \ldots H_{T-1}(x_1-\varepsilon), y_T)$$

where the functions H_t are defined recursively as follows:

For a real variable $\alpha \geqq 0$, and for $t = 2, \ldots, T-1$ define

$$h_{t-1}(\alpha): = \begin{cases} \delta_{t-1} \cdot \alpha & \text{if} \quad x_t = \delta_{t-1} x_{t-1} \\ x_t & \text{if} \quad \delta_{t-1} x_{t-1} < x_t < f_{t-1}(x_{t-1}) \\ f_{t-1}(\alpha) & \text{if} \quad x_t = f_{t-1}(x_{t-1}). \end{cases}$$

Now put

$$H_1(\alpha): = \alpha, \quad H_t(\alpha): = h_{t-1}[H_{t-1}(\alpha)], \qquad t = 2, \ldots, T-1.$$

The functions h_{t-1} and hence H_t are differentiable in α, and satisfy $H_t(x_1)=x_t$ for $t=1,\dots,T-1$. Therefore, the differences $|\tilde{x}_t-x_t|$, $|\tilde{y}_t-x_t|$ can be made arbitrarily small for all $t=1,\dots,T-1$, by choosing ε sufficiently small. From the definition it is clear that

$$\tilde{x}_t \geqq x_t \geqq \tilde{y}_t \text{ for } t=1,\dots,T-1.$$

We will now argue that $\tilde{x}$ is feasible for $\varepsilon>0$ sufficiently small. The definition of $\tilde{x}$ means the following: in the first period, increase investment from x_1 to $x_1+\varepsilon$. This is feasible after y_0 because x_1 is feasible after x_0 and $x_0<y_0$. In periods $t=2,\dots,T-1$, if the original program x follows a path of pure decumulation ($x_t=\delta_{t-1}x_{t-1}$) or pure accumulation ($x_t=f_{t-1}(x_{t-1})$), do the same. Otherwise,

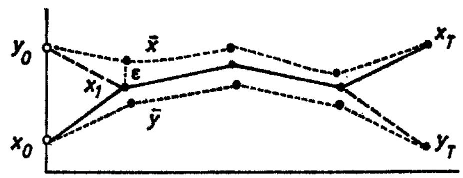

Diagram 5.2

revert to the original program. This is feasible because pure accumulation or decumulation is always feasible and because $\delta_{t-1}x_{t-1}<x_t<f_{t-1}(x_{t-1})$ implies $\delta_{t-1}\tilde{x}_{t-1}<x_t<f_{t-1}(\tilde{x}_{t-1})$, by the continuity remarked above. Finally, in period T, revert to the original program ($\tilde{x}_T=x_T$). This is feasible because $\tilde{x}_{T-1}$ is greater than, but close to x_{T-1}, and because both x_T and $y_T<x_T$ are feasible after x_{T-1}. Therefore, $\tilde{x}$ is feasible for ε sufficiently small.

Similarly, $\tilde{y}$ is feasible for ε sufficiently small (note in particular that $x_t>0$ for $t=1,\dots,T-1$, because $x_T>y_T\geqq 0$).

Given feasibility of $\tilde{x}$ and $\tilde{y}$, we can write, for $t=2,\dots,T-1$:

$$\tilde{c}_t=G_t(x_1+\varepsilon)$$
$$\tilde{d}_t=G_t(x_1-\varepsilon),$$

where $G_t(\alpha):=f_{t-1}[H_{t-1}(\alpha)]-H_t(\alpha)$ is a differentiable function, with $G_t(x_1)=c_t$.

Define $A_t:=u_t(\tilde{c}_t)+u_t(\tilde{d}_t)-u_t(c_t)-u_t(d_t)$ and write

$$A=U(\tilde{c})+U(\tilde{d})-U(c)-U(d)=\sum_{t=1}^{T} A_t.$$

We want to show that this expression is positive, for ε sufficiently small.

For $t=1$ we have, by definition:

$$d_1=\tilde{c}_1+\varepsilon,\ \tilde{d}_1=c_1+\varepsilon, \text{ and } c_1<\tilde{c}_1.$$

This implies

$$\begin{aligned} A_1 &= u_1\ (c_1+\varepsilon)-u_1\ (c_1)-[u_1\ (\tilde{c}_1+\varepsilon)-u_1\ (\tilde{c}_1)] \\ &= \varepsilon\cdot[u_1'\ (c_1)-u_1'\ (\tilde{c}_1)]+o\ (\varepsilon)=\varepsilon\cdot b_1+o\ (\varepsilon) \end{aligned}$$

where $b_1=u_1'\ (c_1)-u_1'\ (\tilde{c}_1)>0$ because u_1 is strictly concave.

For $2\leqq t\leqq T-1$ we have $c_t=d_t$ because $x_t=y_t$, $x_{t-1}=y_{t-1}$, hence

$$\begin{aligned} \mathrm{A}_t &= u_t\ [G_t\ (x_1+\varepsilon)]+u_t\ [G_t\ (x_1-\varepsilon)]-2\ u_t\ (c_t) \\ &= u_t\ (c_t)+\varepsilon\cdot u_t'\ (c_t)\cdot G_t'\ (x_1)+u_t\ (c_t)-\varepsilon\cdot u_t'\ (c_t)\ G_t'\ (x_1)\ +o\ (\varepsilon) \\ &\quad -2u_t\ (c_t)=o\ (\varepsilon). \end{aligned}$$

Finally, for $t=T$:

$$\begin{aligned} A_T &= u_T\ [f_{T-1}\ (H_{T-1}\ (x_1+\varepsilon))-x_T]+u_T\ [f_{T-1}\ (H_{T-1}\ (x_1-\varepsilon))-y_T] \\ &\quad -u_T\ (c_T)-u_T\ (d_T)=u_T\ (c_T)+\varepsilon\cdot u_T'\ (c_T)\cdot f'_{T-1}\ (x_{T-1})\cdot H'_{T-1}\ (x_1) \\ &\quad +u_T\ (d_T)-\varepsilon\cdot u_T'\ (d_T)\cdot f'_{T-1}\ (x_{T-1})\cdot H'_{T-1}\ (x_1)+o\ (\varepsilon)-u_T\ (c_T) \\ &\quad -u_T\ (d_T)=\varepsilon\cdot[u_T'\ (c_T)-u_T'\ (d_T)]\cdot f'_{T-1}\ (x_{T-1})\cdot H'_{T-1}\ (x_1)\ +o\ (\varepsilon) \\ &= \varepsilon\cdot b_T+o\ (\varepsilon) \end{aligned}$$

where $b_T=[u_T'\ (c_T)-u_T'\ (d_T)]\cdot f'_{T-1}\ (x_{T-1})\cdot H'_{T-1}\ (x_1)\geqq 0$ because $c_T<d_T$, and u_T is concave, and because f_{T-1} and H_{T-1} are both nondecreasing.

Collecting these results, we obtain:

$$A=\varepsilon\ (b_1+b_T)+o\ (\varepsilon)>0$$

for ε sufficiently small, because $b_1+b_T>0$. This proves the Lemma for the case $U\ (x)>-\infty$.

If $U\ (x)=-\infty$, it suffices to observe that there exist programs $\tilde{x}, \tilde{y}$, feasible for $\zeta=(T, y_0, x_T)$, $\zeta'=(T, x_0, y_T)$ respectively, which have strictly positive consumption in every period, so that

$$U\ (\tilde{x})+U\ (\tilde{y})>-\infty=U\ (x)+U\ (y).$$

This follows from (A.1) and the fact that x is feasible for (T, x_0, x_T), and $x_0<y_0$, $x_T>y_T$. Q. E. D.

Proof of T.2.1. Write $y_t := x_t'$ for $t=0, 1, \ldots, T$, and $y=(y_0; y_1, \ldots, y_T)$. Of course $x_0=a$, $y_0=a'$. If $y_t=x_t$ for $t=1, \ldots, T$, there is nothing to prove. Assume henceforth that $y_t \neq x_t$ for some t, $1 \leqq t \leqq T$.

$c, d, \tilde{c}, \tilde{d}$ will denote associated consumption programs for investment programs $x, y, \tilde{x}, \tilde{y}$.

The essence of the proof is to show that x and y cannot cross. This is done indirectly, by assuming that they do, and then using L.2.1 to construct two comparison programs $\tilde{x}, \tilde{y}$, feasible for ζ' resp. ζ, s. t. the *sum* of their total utilities is greater than the sum of the utilities of the original x and y. This implies that x and y cannot both be optimal, contrary to assumption. We proceed in four steps.

Case 1: $a < a'$, $b = b'$: We want to show that $x_t \leqq y_t$ for all t. Assume not; then there exist indices r, s, $0 \leqq r < s$ such that

$$x_r < y_r,\ x_t = y_t \text{ for } t = r+1, \ldots, s-1,\ x_s > y_s.$$

Now write

$$\tilde{x} = (y_0; y_1, \ldots, y_r, \tilde{x}_{r+1}, \ldots, \tilde{x}_{s-1}, x_s, x_{s+1}, \ldots, x_T)$$

$$\tilde{y} = (x_0; x_1, \ldots, x_r, \tilde{y}_{r+1}, \ldots, \tilde{y}_{s-1}, y_s, y_{s+1}, \ldots, y_T).$$

By L.2.1 resp. R.2.3 $\tilde{x}, \tilde{y}$ can be chosen so that

$$U(\tilde{x}) + U(\tilde{y}) > U(x) + U(y). \tag{5.1}$$

By def., $\tilde{x}$ is feasible for ζ' and $\tilde{y}$ is feasible for ζ. Therefore, (5.1) contradicts the optimality of x, y.

Case 2: $a = a'$: We want to show that x and y cannot cross. Assume indirectly that they do, and use a similar argument as in Case 1, relying on L.2.1 and R.2.3.

Case 3: $a < a'$, $b < b'$: We want to show that $x_t \leqq y_t$ for $t = 1, \ldots, T$. Let $z = (y_0; z_1, z_2, \ldots, z_T)$ be optimal for $\zeta'' = (T, y_0, b)$.

By Case 1, $z_t \geqq x_t$ for $t = 1, \ldots, T$.
By Case 2, z and y cannot cross.

If $z \leqq y$ (in obvious notation), we are done.

If $z \geqq y$, then z is also feasible for ζ', therefore $U(z) \leqq U(y)$. On the other hand, $U(z) \geqq U(y)$ by R.2.2, hence $U(z) = U(y)$. Therefore y must be optimal for ζ''. The assertion follows, by Case 1.

Case 4: $a = a'$, ζ *and* ζ' *not equivalent:* in this case, we must have $x_T < y_T$ (otherwise x would be optimal for both ζ and ζ'). The assertion follows, by Case 2.

Q. E. D.

Proof of Cor. 2.1: By the Principle of Optimality (R.2.3) the subprogram $(x_t')_{t=0}^{T}$ is optimal for $\zeta'' = (T, a, x'_T)$. Since $x'_T \geqq 0$, the Corollary follows by T.2.1.

Q. E. D.

Proof of T.2.2. We use induction on T. It is easy to verify that the result is true for $T = 1$ and we make the Induction Hypothesis that T.2.2 is true for all $\bar{\zeta} = (t, a, b)$ with $t \leqq T - 1$.

Now let (x, c) (resp. (y, d)) be optimal T-period programs for $\zeta = (T, a, b)$ (resp. $\zeta' = (T, a', b)$), where $a < a'$ (if $a = a'$, there is nothing to prove). We want to show that $c_t \leqq d_t$ for all $t = 1, \ldots, T$.

By R.2.3, the $(T-1)$-period subprograms $(x_t, c_t)_{t=2}^{T}$ and $(y_t, d_t)_{t=2}^{T}$ are optimal for $(T-1, x_1, b)$ and $(T-1, y_1, b)$, respectively. By Theorem 2.1, $x_1 \leqq y_1$, hence, by Induction Hypothesis, $c_t \leqq d_t$ for $t = 2, \ldots, T$. It remains to show that $c_1 \leqq d_1$.

Assume, to the contrary, that $c_1 > d_1$.

First note that this implies $x_1 < y_1$, for if $x_1 = y_1$, we should have $c_1 = f_0(a) - x_1 < f_0(a') - y_1 = d_1$, contradicting $c_1 > d_1$.

Secondly, this also implies $y_1 > \delta_0 a'$, for if $y_1 = \delta_0 a'$, we should have $d_1 = f_0(a') - \delta_0 a' = g_0(a') > g_0(a) \geqq c_1$, again contradicting $c_1 > d_1$.

Now choose ε, $0 < \varepsilon < (1/2) \cdot \min\{c_1 - d_1, y_1 - x_1\}$, and define T-period programs $(\tilde{x}, \tilde{c})$ resp. $(\tilde{y}, \tilde{d})$ as follows: The first components are given by

$$\tilde{x}_1 = x_1 + \varepsilon, \quad \tilde{c}_1 = c_1 - \varepsilon$$

$$\tilde{y}_1 = y_1 - \varepsilon, \quad d_1 = d_1 + \varepsilon$$

and the remaining $T-1$ components $(t = 2, \ldots, T)$ are the optimal programs for $(T-1, x_1 + \varepsilon, b)$ resp. $(T-1, y_1 - \varepsilon, b)$. If ε is sufficiently small, $(\tilde{x}, \tilde{c})$ resp. $(\tilde{y}, \tilde{d})$ are feasible for ζ resp. ζ', by the two observations made above and by feasibility of (x, c) resp. (y, d).

By R.2.2, R.2.5, the function $v(k) := V(1, T, k, b)$ is concave and increasing in k, and hence possesses non-negative left-hand and right-hand derivatives $v'_-(k)$, $v'_+(k)$ in the interior of its domain of definition. Moreover (cf. Nikaido, 1968, Th. 3.15), these derivatives are decreasing functions of k, and there exists a number $\mu \geqq 0$ such that

$$v'_+(x_1 + \varepsilon) \geqq \mu \geqq v'_-(y_1 - \varepsilon). \tag{5.2}$$

Similarly, by strict concavity of u_1 and because $d_1+\varepsilon<c_1-\varepsilon$ by construction, we obtain:

$$u'_{1-}(d_1+\varepsilon)>u'_{1+}(c_1-\varepsilon). \tag{5.3}$$

By R.2.3, we can write

$$U(c)=u_1(c_1)+v(x_1);\; U(d)=u_1(d_1)+v(y_1);$$
$$U(\tilde{c})=u_1(c_1-\varepsilon)+v(x_1+\varepsilon);\; U(\tilde{d})=u_1(d_1+\varepsilon)+v(y_1-\varepsilon).$$

Optimality of (x, c) for ζ implies $U(c)\geqq U(\tilde{c})$. This leads to

$$u_1(c_1)-u_1(c_1-\varepsilon)\geqq v(x_1+\varepsilon)-v(x_1).$$

By concavity of u_1 we have $u_1(c_1)-u_1(c_1-\varepsilon)\leqq\varepsilon\cdot u_1{}'_+(c_1-\varepsilon)$, and by (5.2) we have $v(x_1+\varepsilon)-v(x_1)\geqq\varepsilon\cdot v_+{}'(x_1+\varepsilon)\geqq\varepsilon\cdot\mu$. Hence, $u'_{1+}(c_1-\varepsilon)\geqq\mu$.

By a similar argument, the optimality of (y, d) for ζ' implies $u'_{1-}(d_1+\varepsilon)\leqq\mu$. Hence $u'_{1-}(d_1+\varepsilon)\leqq u'_{1+}(c_1-\varepsilon)$, contradicting (5.3). Therefore we must have $c_1\leqq d_1$, and the Theorem is proved.

Q. E. D.

Proof of L.3.1: The proof is exactly the same as for T.2.1, Case 1. We need only ignore the final stock constraint there and delete the symbols x_T resp. y_T in the definitions of $\tilde{x}$ resp. $\tilde{y}$ (i. e. "extend $\tilde{x}$, $\tilde{y}$ to infinity"). The assertion follows.

Q. E. D.

Proof of T.3.1: The proof is strikingly simple, and follows immediately from L.3.1, by the optimality principle R.3.3.

Let $x=(x_0, x_1, x_2, \ldots)$ be optimal, and assume that it is not constant. Let r be the first index s. t. $x_r\neq x_{r+1}$, for example $x_r<x_{r+1}$. By R.3.3, the subprograms $(x_r, x_{r+1}, \ldots)$ and $(x_{r+1}, x_{r+2}, \ldots)$ are optimal from x_r resp. x_{r+1}. By L.3.1, $x_{r+t}\leqq x_{r+1+t}$ for $t\geqq 1$.

Q. E. D.

Proof of L.3.2: The idea of the proof is first to deduce convergence from T.3.1 (i. e. monotonicity), and then use the Euler equation to show that the limit must be a steady state.

Let (x, c) be optimal from some $a>0$. By (A.3′) the sequence $x=(x_t)_{t=0}^{\infty}$ is bounded, and by T.3.1 it is monotonic, hence it must converge to some number $\bar{x}\in[0, \xi_{\max}]$. Either $\bar{x}=0$ or $\bar{x}>0$. If $\bar{x}>0$ then, for t sufficiently large, the irreversibility constraint $x_t\geqq\delta\cdot x_{t-1}$ cannot be binding, by (A.2′). Moreover, by (A.1′), we must have $x_t > 0$, $c_t > 0$ for all t. Hence, for t sufficiently large, (x_t, c_t) must satsify the Euler equation (cf. e. g. Takayama, 1974, Ch. 5):

$$u'(c_t) = \varrho f'(x_t) \cdot u'(c_{t+1}).$$

Taking limits for $t \to \infty$, this implies

$$u'(\bar{c}) = \varrho f'(\bar{x})\, u'(\bar{c}), \text{ where } \bar{c} = f(\bar{x}) - \bar{x}. \tag{5.4}$$

We claim that $\bar{c} > 0$. Assume indirectly that $\bar{c} = 0$.

Choose $\varepsilon > 0$ such that

$$u(\varepsilon) < \frac{1}{2}(1-\varrho)\, u[g(\bar{x})]. \tag{5.5}$$

There exists a $T \geqq 1$ such that for all $t \geqq T$:

$$u(c_t) < u(\varepsilon), \text{ and } u[g(x_t)] > \frac{1}{2} u[g(\bar{x})]. \tag{5.6}$$

Define an alternative path (x', c') by

$$x_t' := x_t \text{ for } t \leqq T, \text{ and } x_t' = \delta \cdot x_{t-1}' \text{ for } t \geqq T+1.$$

Then $c_t' = c_t$ for $t \leqq T$, and $c_T' = g(x_T)$.

This implies

$$U(x') - U(x) = \sum_{t=T+1}^{\infty} \varrho^{t-1} [u(c_t') - u(c_t)] >$$

$$\varrho^T \left[u[g(x_T)] - \frac{u(\varepsilon)}{1-\varrho} \right] > 0 \text{ by (5.5), (5.6),}$$

contradicting the optimality of x. Therefore $\bar{c} > 0$, and (5.4) implies $\varrho f'(\bar{x}) = 1$, i. e., $\bar{x}$ is a steady state. Q. E. D.

Proof of T.3.2: Essentially, we have to refine the result of L.3.2 by showing that the limit steady state of an optimal program cannot be a relative ***minimum*** of the function r_ρ. The rest then follows easily from (A.3′) and L.3.1.

Let (x, c) be optimal from some $a \in (0, \xi_{\max})$ and assume that (x_t) converges to a steady state $\bar{x} > 0$ with $\bar{c} = f(\bar{x}) - \bar{x} > 0$ (cf. L.3.2). By definition, $\bar{x}$ is a solution of the equation $r_\rho'(\xi) = 0$, where $r_\rho(\xi) := \varrho f(\xi) - \xi$. By the assumptions of the Theorem, $r_\rho''(\bar{x}) \neq 0$, i. e. the function $r_\rho(\xi)$ has either a strict relative maximum or a strict relative minimum at $\xi = \bar{x}$.

In the first case, $\bar{x}$ is a local golden rule.

In the second case, we claim that x must be the constant program $x=(\bar{x}, \bar{x}, \bar{x}, \ldots)$.

Assume the contrary. By R.3.3 resp. Lemma 3.1 one may assume w. l. o. g. that $x_0=a\neq\bar{x}$, $x_1\neq x_0$, and that a lies in a sufficiently small neighborhood of $\bar{x}$, such that $f(a)-a>0$, $\varrho f'(\xi)\neq 1$, $f''(\xi)>0$ for all ξ between a and $\bar{x}$.

Then the sequence $r_\rho(x_t)$ is monotonically decreasing by Theorem 3.1, and

$$\varrho f(x_0)-x_0>\varrho f(x_t)-x_t \text{ for } t=1,2,\ldots$$

This implies

$$\frac{1}{1-\varrho}\cdot[\varrho f(x_0)-x_0]>\sum_{t=1}^{\infty}\varrho^{t-1}[\varrho f(x_t)-x_t]=$$
$$\sum_{t=1}^{\infty}\varrho^{t-1}[f(x_{t-1})-x_t]-f(x_0)$$

or

$$\frac{1}{1-\varrho}\cdot[f(x_0)-x_0]>\sum_{t=1}^{\infty}\varrho^{t-1}c_t.$$

By Jensen's inequality, this implies:

$$U(x)=\frac{1}{1-\varrho}\cdot\sum_{t=1}^{\infty}(1-\varrho)\,\varrho^{t-1}u(c_t)\leqq\frac{1}{1-\varrho}u\,[\sum_{t=1}^{\infty}(1-\varrho)\varrho^{t-1}c_t]<$$
$$\frac{1}{1-\varrho}u\,[f(x_0)-x_0]=U(x'),$$

where $x'=(x_0, x_0, x_0, \ldots)$, contradicting the optimality of x.

Collecting our results so far, we have shown that an optimal program $x=(x_0, x_1, \ldots)$ either converges to zero, or to a local golden rule, or is of the form $x=(\bar{x}, \bar{x}, \bar{x}, \ldots)$, where $\bar{x}$ is a steady state.

By (A.3′) there exist only finitely many local golden rules. Denote those which are limits of nonconstant lpotima programs by $\hat{x}_1, \ldots, \hat{x}_n$ $(n\geqq 0)$, where w. l. o. g.

$$0<\hat{x}_1<\hat{x}_2<\ldots<\hat{x}_n<\xi_{\max}, \text{ and define } \hat{x}_0:=0.$$

By L.3.1 the set I_i of all initial stocks a from which there exists an optimal program converging to some given $\hat{x}_i$ $(0\leqq i\leqq n)$ must be an interval, and I_{i-1} and I_i must be adjacent, so that we can write

$\sup I_{i-1} = \inf I_i = \xi_i$ for $i = 1, \ldots, n$. By R.3.3 we have $\xi_i \leqq \hat{x}_i \leqq \xi_{i+1}$ for $i \geqq 0$. Moreover, from $a \in (\xi_i, \xi_{i+1})$ *all* optimal programs converge to $\hat{x}_i$, again by Lemma 3.1, and when ξ_i is not a steady state, then *all* optimal programs from $a = \xi_i$ converge either to $\hat{x}_{i-1}$ or to $\hat{x}_i$. When ξ_i is a steady state, we cannot rule out the possibility that the constant program from $a = \xi_i$ is also optimal. This completes the proof. Q. E. D.

Proof of T.3.3: If $\varrho \cdot M < 1$, then the equation $\varrho \cdot f'(\xi) = 1$ has no solution in $[0, \xi_{\max}]$, i. e. no steady state exists. The assertion follows, by L.3.2. Q. E. D.

Proof of T.3.4: The idea of the proof is to show that for ϱ sufficiently large, the utility loss due to staying away from the global golden rule will eventually outweigh any temporary gains. Incidentally, the same idea recurs in T.4.2.

If an optimal program does not converge to $\hat{x}(\varrho)$, then it converges to zero or to some other steady state, by L.3.2.

Step 1: First we show that it cannot converge to zero if ϱ is sufficiently near to one.

By (A.5′) there exist $\varrho_0 < 1$ and $\varepsilon_0 > 0$ such that $\varrho f'(\xi) > 1$ for

$$\varrho_0 < \varrho < 1, \quad 0 \leqq \xi \leqq \varepsilon_0. \tag{5.7}$$

We claim that if $\varrho \geqq \varrho_0$, then no optimal program converges to zero. By L.3.1 it sufficies to prove this for optimal programs starting from initial stocks $a \in (0, \varepsilon_0]$. Let (x, c) be such a program, and assume first that there is no irreversibility constraint. Then (x, c) must satisfy the Euler equation:

$$u'(c_t) = \varrho \cdot f'(x_t) \cdot u'(c_{t+1}) \quad \text{for} \quad t = 1, 2, \ldots, \tag{5.8}$$

If (x_t) converged to zero we would have, for all $t \geqq 1$: $\varrho f'(x_t) > 1$ by (5.7) resp. T.3.1, and this would imply $c_t < c_{t+1}$ by (5.8) and (A.1′), which is clearly incompatible with (x_t) converging to zero. Therefore, by L.3.2, (x_t) must converge to some steady state $\bar{x} > 0$.

Moreover by T.3.1, (x_t) is monotonically increasing, since (5.7) implies $x_0 \leqq \varepsilon_0 < \bar{x}$. Therefore the program (x, c) is feasible — and of course also optimal — even *with* an irreversibility constraint. This completes Step 1.

Step 2: Let ϱ_0, ε_0 be chosen as in Step 1.

By (A.3′), (A.5′), (5.7) there exist $\varrho_1, \varrho_0 \leqq \varrho_1 < 1$, $\varepsilon > 0$, $\eta_0 > 0$, and an integer $L \geqq 1$ such that for all ϱ with $\varrho_1 \leqq \varrho < 1$ the following is true:

(i) From every initial stock $a \in [\varepsilon_0, \xi_{\max}]$ it is possible to reach the global golden rule $\hat{x}(\varrho)$ in at most L steps (by following a path of pure accumulation or decumulation).

(ii) All steady states for ϱ lie in $(\varepsilon_0, \xi_{\max})$.

(iii) $r_\rho(\xi) \leqq \hat{c}(\varrho) - 2\eta_0$ for $\xi \notin U_\varepsilon[\hat{x}(\varrho)]$.

(iv) No other steady state except $\hat{x}(\varrho)$ lies in $U_\varepsilon[\hat{x}(\varrho)]$, where $U_\varepsilon[\hat{x}(\varrho)] := (\hat{x}(\varrho) - \varepsilon, \hat{x}(\varrho) + \varepsilon)$ is an ε-neighborhood of $\hat{x}(\varrho)$.

Define further

$$\eta := \frac{1}{2}\eta_0 \cdot u'(\xi_{\max}) > 0$$

$$\hat{u} := u(\hat{c}) \quad (\hat{c} \text{ was defined in (A.5′)})$$

$$\bar{\varrho} := \max\left\{\varrho_1, 1 - \frac{\eta}{L \cdot u}\right\} < 1.$$

Step 3: We claim that for $\varrho \geqq \bar{\varrho}$ every optimal pgroram converges to $\hat{x}(\varrho)$, independently of the initial stock.

Let (x, c) be an optimal program for some $\varrho \geqq \bar{\varrho}$ and assume, indirectly, that (x_t) does not converge to $\hat{x}(\varrho)$. Then, by L.3.2 and Step 1, (x_t) must converge to some steady state $\bar{x} \neq \hat{x}(\varrho)$. By Step 2, there exists a $T \geqq 1$ such that for all $t \geqq T$:

$$x_t \geqq \varepsilon_0, \quad c_t \leqq \hat{c}(\varrho) - \eta_0. \tag{5.9}$$

By R.3.3 we may assume w. l. o. g. that $T = 1$. Now consider an alternative program $(\tilde{x}, \tilde{c})$ from the same initial stock defined as follows: go to the global golden rule $\hat{x}(\varrho)$ in the first L periods and stay there ever after (by Step 2 (i), (5.9) this is possible).

We have

$$U(c) = \sum_{t=1}^{\infty} \varrho^{t-1} u(c_t) \leqq \sum_{t=1}^{\infty} \varrho^{t-1} u[\hat{c}(\varrho) - \eta_0] < \sum_{t=1}^{\infty} \varrho^{t-1} \cdot [u(\hat{c}(\varrho)) - \eta]$$

$$U(\tilde{c}) = \sum_{t=1}^{\infty} \varrho^{t-1} u(\tilde{c}_t) \geqq \sum_{t=L+1}^{\infty} \varrho^{t-1} u[\hat{c}(\varrho)],$$

and, thus,

$$U(\tilde{c}) - U(c) > -\sum_{t=1}^{L} \varrho^{t-1} u[\hat{c}(\varrho)] + \frac{\eta}{1-\varrho} \geqq -L \cdot \hat{u} + \frac{\eta}{\eta / L\hat{u}} = 0,$$

contradicting the optimality of (x, c). Q. E. D.

Proof of L.4.1 and T.4.1: Analogous to L.3.1 and T.3.1.

Proof of T.4.2: Let (x, c) be optimal. By Th. 4.1, (x_t) must converge to some $\bar{x} \in [0, \xi_{\max}]$, and hence (c_t) converges to $\bar{c} = f(\bar{x}) - \bar{x}$. If $\bar{x} \neq \hat{x}$, then by (A.5′), $\bar{c} \leqq \hat{c} - \varepsilon$ for some $\varepsilon > 0$. Then there exists a $t_0 \geqq 1$ such that for all $t \geqq t_0$

$$u(c_t) \leqq \hat{u} - \eta \quad \text{for some } \eta > 0.$$

This implies immediately that $U(x) = -\infty$, contradicting the optimality of x (cf. R.4.2). Therefore $\bar{x} = \hat{x}$.

Q. E. D.

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