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Integration of reduction and expansion processes in layout optimization*

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Abstract A two-stage layout optimization procedure, consisting of reduction and expansion processes, is presented. The object in developing this procedure is to use the advantages of both processes. In the reduction process, a reduced structure with a limited number of members and joints is established by solving large scale idealized problems. An expansion process is then employed, where members and joints are added to the initial reduced structure. At this stage, relatively small problems are solved, considering general variables, all relevant constraints and the real objective function.

1 Introduction

Most of the work that has been done on optimum structural design is related to optimization of cross-sections. Much less effort has been devoted to optimization of the layout (geometry and topology). It is recognized, however, that optimization of the structural layout can greatly improve the design (Bendsøe and Mota Soares 1992; Kirsch 1989, 1993; Rozvany *et al.* 1995; Topping 1983). Because of the complexity in simultaneous optimization of the geometry, the topology and the cross-sections, two classes of problems are often considered in this type of optimization (Kirsch 1990).

(a) *Topological optimization*, where the spatial sequence of members and joints is optimized.

(b) *Geometrical optimization*, where joint coordinates and cross-sectional sizes are optimized.

The solution of each problem affects indirectly the other one. That is, the geometry is affected by elimination of members during topological optimization whereas the tolopogy might be changed due to zero cross-sections or the coalescence of joints during geometrical optimization.

In this paper layout optimization is viewed as a two-stage procedure, consisting of reduction and expansion processes (Table 1). The object in developing this procedure is to use the advantages of both processes. Topological optimization is usually based on a reduction process, where members and joints are eliminated from an initial highly-connected ground structure. In a typical reduction process, the solution of a large scale idealized problem is achieved by assuming various simplifications. In the approach presented in this study, the object at this stage is to establish an initial reduced structure (IRS) with a limited number of members and joints, using available analytical and numerical methods. An expansion process, where members and joints are added to an initial structure, is uncommon due to the lack of effective systematic procedures. In the procedure presented, relatively small problems are solved during the expansion process, considering general variables, all relevant constraints and the real objective function. The object at this stage is to find the final optimum by adding successively members and joints to the IRS. For each candidate topology the geometry is optimized and all intermediate solutions are feasible. It is shown that a near optimal design can be achieved by a simplified expansion procedure.

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Table 1. Reduction and expansion processes

Process	Reduction	Expansion
Problem size	Large	Small
Formulation	Idealized-simplified	Real-practical
Solution stages	Exact LB (analytical) Approximate LB (discretized) Establishing an IRS	Selection of geometrical variables Geometrical optimization Adding members and joints
Difficulty	Computational effort	Selection of variables
Advantage	Automated process	Improved feasible solutions

2 The reduction process

2.1 General considerations

A reduction process is characterized by elimination of members and joints from an initial structural topology. The number of members in the latter topology might be indefinitely large (in exact-analytical formulations), very large (in approximate-discretized formulations) or a reduced one. In a typical reduction process, a large scale structure with numerous members is often solved. Although most of the work on layout optimization is related to reduction processes, the problem solved is usually highly idealized. That is, various simplifications are often assumed in the problem formulation due to some basic difficulties involved in the solution process.

One problem is that, unlike common optimization problems, the structural model is itself allowed to vary during the design process. Another difficulty is that the number of possible element-joint connectivities in the initial structure is very large. In addition, the problem can have singular global optima that cannot be reached by assuming a continuous set of variables. These and other difficulties make the topological optimizaton problem perhaps the most challenging of the structural optimization tasks. The various simplifications assumed in the problem formulation include consideration of simplified sizing variables (e.g. a single cross-sectional variable per member), only certain constraints (e.g. stress constraints), simplified objective function (e.g. weight or compliance), simple structural systems (e.g. trusses), approximate analysis models (e.g. rigid plastic) and a limited number of loading conditions.

In the approach presented, the object of the reduction process is to establish an initial reduced structure (IRS), consisting of a limited number of members and joints. Selection of the IRS is based on available analytical and numerical solutions of simplified problems. Since an idealized problem is solved, the solution can be viewed as a lower bound on the optimum. To obtain a practical topology for the IRS, the lower bound solution can be modified by eliminating or adding members and joints.

2.2 Exact and approximate lower bounds

Some analytical and numerical solutions used to establish the IRS are briefly described subsequently. Similar to most studies on layout optimization, truss structures will be considered; however, reduction concepts are applicable also in other types of structures. Exact-analytical solutions provide a theoretical lower bound on the weight of the structure. The general concept of the structural layout can often be established by the classical Michell theory. Michell structures can also be used as reference solutions for assessment of the efficiency of practical configurations. However, they are seldom suitable for direct use in practical design for several reasons.

- A fundamental limitation of this approach is that it is not general. General variables, constraints, objective function and loading conditions are not considered.
- The solutions usually consist of an indefinitely large number of infinitesimal members, and the resulting structures are potentially unstable if alternative loads are applied.
- Michell layouts have only been determined for a few simple loading conditions, and there is no systematic procedure to construct a structure for an arbitrary set of loads.

Michell's early work was further developed by others (see reviews by Kirsch 1989; Rozvany *et al.* 1995; Topping 1983). Techniques for assessing the efficiency of near optimal trusses have been presented, some practical aspects have been studied, and solutions for several alternative load conditions have been demonstrated. Prager and Rozvany (1977) developed a layout theory as a generalization of Michell's effort. It deals with the layout of low density grid-type structures, called gridlike continua. Recently, layout theory for high density structures has been developed (Bendsøe and Kikuchi 1988).

Approximate-discretized solutions are usually based on the ground structure approach. Member areas are allowed to reach zero and hence can be deleted automatically from the structure. While the displacement method is the prevalent structural analysis tool in current computational practice, the force method formulation is adopted in many topological optimization problems. The main reason is that linear programming (LP) formulation is obtained under certain assumptions (Kirsch 1989, 1993; Rozvany *et al.* 1995; Topping 1983). The main advantages of the LP formulation is that the global optimum is reached in a finite number of steps and large structures can efficiently be solved. The LP solution satisfies the equilibrium and stress constraints, but it might not satisfy the compatibility conditions or may represent unstable configuration under a general loading. For structures subjected to a single loading condition the optimum represents a statically determinate or an unstable structure. In cases where the optimal LP solution represents a statically indeterminate structure the compatibility conditions might not be satisfied, but a certain deviation from elastic force distribution is often allowed on account of the inelastic behaviour. Another approach to achieve an approximate lower bound, based on optimality criteria, has been used successfully in recent years (Rozvany 1989; Zhou and Rozvany 1990).

2.3 Establishing an IRS

Optimal layouts obtained by solution of idealized problems are often impractical. They might represent an unstable structure (a mechanism) or consist of too many members and joints. To achieve a feasible practical design for the IRS, the lower bound solutions are modified by eliminating or adding members and joints. Denoting the exact lower bound on the optimum achieved by analytical solutions as Z_L, the approximate lower bound achieved by discretized solutions as Z_A, and the solution corresponding to the selected IRS modified topology as Z_{IRS}, then

$$Z_L \leq Z_A \leq Z_{\mathrm{IRS}} \, .$$

It will be shown subsequently that the difference between the above solutions achieved during the reduction process is often insignificant. Moreover, the effect of geometrical optimization on the optimum might be larger than that of topological changes. In the following three typical examples of reduction (the results are summarized in Table 2), the objective function is the volume of material, the constraints are related only to stresses, and the allowable stress is 1.0 (arbitrary units have been assumed).

(a) Simply-supported structure, shown in Fig. 1a. The lower bound on the optimum is a Michell structure shown in Fig. 2b, an approximate lower bound achieved by LP is shown in Fig. 1c and a three-bar truss selected as an IRS is shown in Fig. 1d.

(b) Fixed-supported structure, shown in Fig. 2a. The lower bound on the optimum is a Michell structure shown in Fig. 2b, an approximate lower bound achieved by an optimality criteria method is shown in Fig. 2c (Zhou and Rozvany 1990), and a five-bar truss selected as an IRS is shown in Fig. 2d. It should be noted that all the topologies shown in Fig. 2 are unstable.

(c) Cantilever structure, shown in Fig. 3a. The lower bound is a Michell structure shown in Fig. 3b, an approximate lower bound is shown in Fig. 3c and the selected six bar IRS is shown in Fig. 3d.

It can be seen that the difference in weight between highly idealized lower bound solutions (with an indefinitely large number of members) and simplified structures consisting of 5-6 members might be only 3-4%.

Table 2. Results, reduction process

Structure	Fig.		Z_L	Z_A	Z_{IRS}
Simply-supported	1	Number of members (n)	∞	11	3
		Minimum weight	28.2	$1.06Z_L$	$1.35Z_L$
Fixed-supported	2	Number of members (n)	∞	13	5
		Minimum weight	2.57	$1.01Z_L$	$1.03Z_L$
Cantilever	3	Number of members (n)	∞	8	6
		Minimum weight	4.50	$1.02Z_L$	$1.04Z_L$

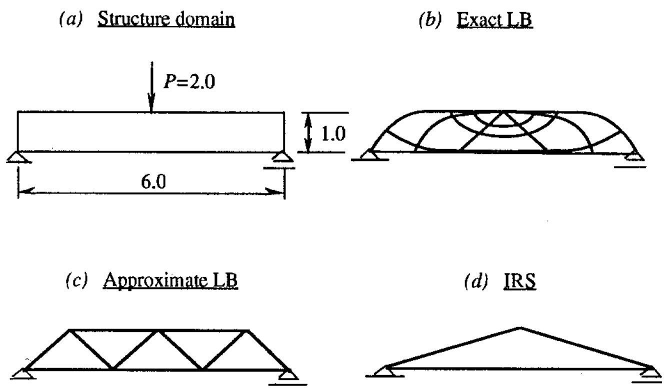

Fig. 1. Reduction process – simply-supported structure

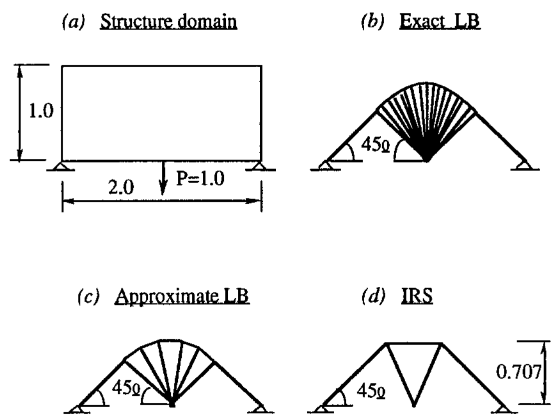

Fig. 2. Reducton process – fixed-supported structure

3 The expansion process

3.1 General considerations

Following the reduction process, an expansion process is employed, characterized by the addition of members and joints to an initial structural topology. The object at this stage is to find the final optimum by adding successively members and joints to the IRS and optimizing the real problem for each candidate topology. The expansion process consists of the following main stages.

a. Selecting geometrical variables for the given topology.
b. Optimizing the geometry and cross-sections.
c. Modifying the structural topology by adding members and joints.

To obtain an upper bound on the optimum Z_U, the IRS is first optimized considering geometrical and sizing variables, all relevant constraints and the real objective function. Introducing successively improved feasible designs by adding

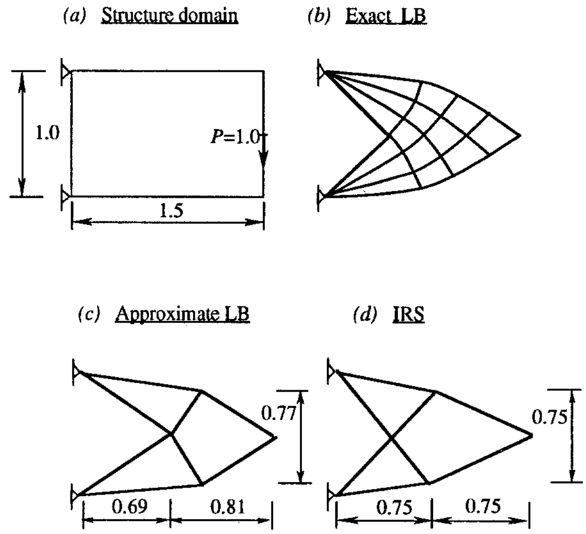

Fig. 3. Reduction process – cantilever structure

members and joints and optimizing the geometry of the resulting topologies, Z_U is improved and the final optimal design Z_{OPT} is approached from the interior side of the feasible region. The optimum is in between Z_U and the theoretical lower bound on the optimum Z_L,

$$Z_L \leq Z_{\mathrm{OPT}} \leq Z_U$$

The expansion process is characterized by the following features.

(a) The structures optimized are simple, consisting of a limited number of members and joints. Therefore, the effect of the number of members on the objective function can be considered directly, and the computational effort involved in the solution process is significantly reduced.

(b) Since at each iteration the real problem is solved, all intermediate solutions are feasible designs satisfying all the constraints.

(c) Problems of singular optima that might be encountered in the common reduction process are eliminated since we start with reduced structures having a small number of members.

3.2 Selecting the geometrical variables

In the stage of geometrical optimization the design variables are assumed to be continuous. Effective selection of these variables is most important for the following reasons.

a. Poor selection of the variables might lead to nonoptimal or singular solutions.
b. A large number of variables increases significantly the computational effort, whereas a small number of effective variables might be adequate to achieve a near optimal solution.

Modification of the topology during geometrical optimization might occur due to deletion of zero size members, obtained for certain geometries, or deletion of *nonzero size members* due to the coalescence of joints. Elimination of zero length or parallel members, in cases where some joints tend to coalesce during geometrical optimization, will change the topology and the resulting structures might represent singular optima (Kirsch 1995). To illustrate this phenomenon, assume the eleven-bar truss shown in Fig. 4, the optimal depth $Y = 4.24$ and only single geometrical variable X. Variation of Z with X is shown in Fig. 5. It can be seen that the global optimum is at point GS ($X = 3.0, Z = 17.0$) which is a singular point in the design space, representing a three-bar truss. However, the solution process converges to point $L(X = 1.0, Z = 31.1)$, which is a local optimum. That is, the solution reached by optimizing X is heavier than the global optimum by 83%. The singular optimum is the result of changes in the topology of the structure. Specifically, as X approaches 3.0, the joints A-B, C-D, E-F-G, and the members 6-7-8, 9-10-11 tend to coalesce. Just before that, the forces in members 6, 8, 9, 11 are identical, while the forces in members 7, 10 have the same magnitude but with opposite direction. At the limit ($X = 3.0$) the three members 6, 7, 8 (and 9, 10, 11) become a single member. The result is a reduction of 2/3 of the weight of the diagonal members, leading to the singular global optimum at GS.

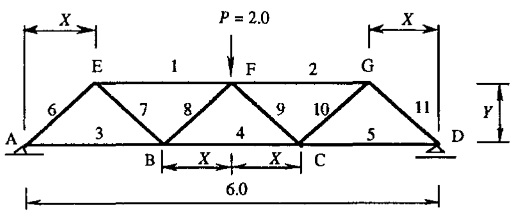

Fig. 4. Eleven-bar truss

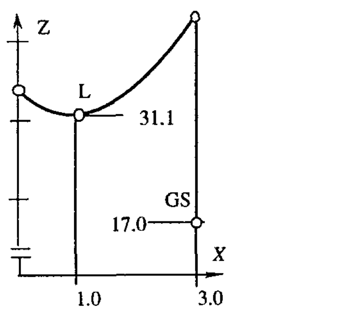

Fig. 5. Z versus X, eleven-bar truss

To illustrate the effect of alternative selections of variables, consider the following cases for the eleven-bar truss shown in Fig. 4.

(a) A uniform depth with a single vertical variable Y.
(b) A uniform depth with a vertical variable Y and a horizontal variable X.
(c) A nonuniform depth with two vertical variables: Y_F (joint F) and Y_E (joints E and G).

The optimal solutions are summarized in Table 3. It can be seen that in case a the global optimum is obtained, but better solutions can be achieved if more vertical variables are assumed (case c below). In case b a local opimum is obtained, since the true singular optimum could not be achieved by the optimization process. In case c the global optimum is achieved (the topology is changed into the three-bar truss). In conclusion, poor selection of geometrical variables might lead to nonoptimal or singular solutions. In the example presented, selection of two vertical variables (Y_F and Y_E) is most effective, since the global optimum is reached even in cases of changes in the topology during optimization.

As noted earlier, some important constraints are often neglected in the reduction process in order to simplify the solution. These constraints can readily be considered in the expansion process, since the structures optimized are relatively small and simple. To illustrate the effect of Euler buckling constraints, assume tubular members with a predetermined diameter-to-thickness ratio. The allowable buckling stress can be expressed as $\sigma_E = -cX/L^2$, where c is a constant depending on the modulus of elasticity, L is the member length, and X is the cross-sectional area. Assuming $c = 4.0$, then the variation of the optimal objective function value with the depth Y for the three-bar simply-supported truss (IRS) of Fig. 1d is shown in Fig. 6. It can be seen that the buckling constraints affect significantly both the optimal geometry and the objective function value. Specifically, if buckling is not considered the optimum is $Y = 4.24$, $Z_L = 17.0$. If buckling is considered for this geometry, then $Z_U = 34.2$. Optimizing the geometry with buckling constraints gives the near optimal solution $Y = 2.0, Z = 26.4$.

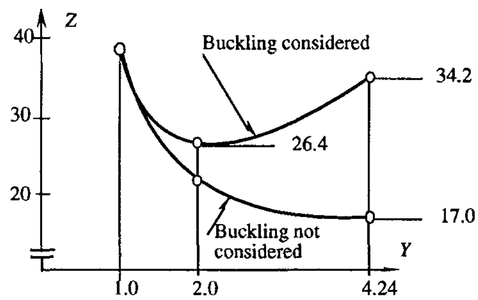

Fig. 6. Effect of buckling, three-bar truss

3.3 Adding members and joints

During the expansion process, modified structures are introduced successively by adding members and joints. Although several expansion processes have been proposed in the past, it is usually difficult to carry out this stage automatically and there is no single best method in terms of efficiency, reliability, and ease-of-implementation. The following approaches have been considered in this study.

(a) Adding a single joint and members connecting it with existing joints. The initial position of the joint is arbitrary, whereas its final location is determined by optimizing the geometry. The advantage is that this procedure can readily be automated, but in cases of multiple local optima the solution might be affected by the initial joint position.
(b) Adding multiple joints and members. This possibility is more general and also might be suitable for automated implementation. However, experience has shown that it is not very effective since some of the new members might not be needed and the final solution could be a local optimum.
(c) Adding a limited number of joints and members. This approach usually involves interactive decisions, but the combination of automated optimization and interactive design might prove useful.

To illustrate the latter approach, consider the three bar truss shown in Fig. 7a as an IRS. Assuming only some regular layouts, the solution process involves the following steps (Fig. 7).

(a) The geometry of the IRS (Fig. 7a) is optimized assuming a single geometrical variable Y_1.
(b) Adding four internal members and defining the new geometrical variables Y_1 and Y_2 shown in Fig. 7b, the resulting eleven-bar truss is optimized.
(c) Adding four internal members and defining the new geometrical variables Y_1, Y_2 and Y_3 shown in Fig. 7c, the resulting nineteen-bar truss is optimized.

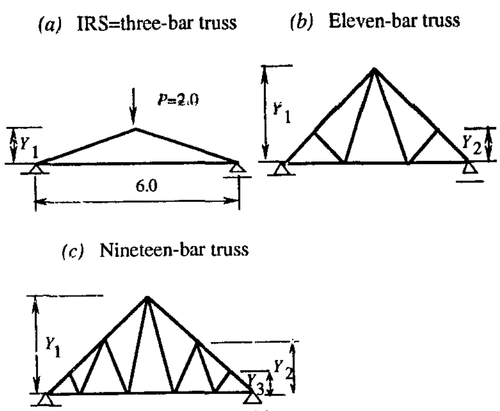

Fig. 7. A typical expansion process

From the results shown in Fig. 8 and Table 4 it can be seen that the optimal layout for all cases of upper limit on the depth (Y_U) is the eleven-bar truss. In addition, the effect of the depth is more significant than the effect of the number of members. In particular, the difference in weight between the eleven-bar and the nineteen-bar topologies is small.

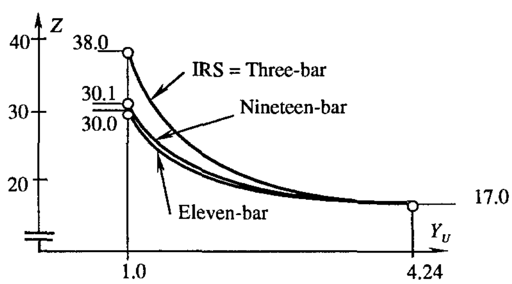

Fig. 8. Results, expansion process

Table 3. Eleven-bar truss, various geometrical variables

Case	Variables	Optimal geometry	Z	Type of optimum
a	Y	$Y = 2.0$	24.0	Global
b	Y, X	$Y = 2.0, X = 1.0$	24.0	Local
c	Y_F, Y_E	$Y_F = 4.24, Y_E = 1.41$	10.7	Global

4 Concluding remarks

The common approach in topological optimization is based on a reduction process, where members and joints are eliminated from an initial highly-connected structure. Since the problem solved is usually large, idealized simplified formulations are often assumed. An expansion process, where members and joints are added to an initial reduced structure, is not common due to the lack of automated systematic procedures. The approach presented in this paper is based on the integration of the two processes into a general design procedure. In the reduction process, several methods are used to establish the IRS. It has been shown that the difference in weight between highly idealized lower bounds on the optimum (with an indefinitely large number of infinitesimal members) and simplified structures consisting of 5-6 members might be only 3-4%. In addition, the effect of geometrical optimization on the optimum might be larger than that of topological changes.

Table 4. Results, expansion process

Three-bar			Eleven-bar			Nineteen-bar			
Y_U	Z	Y_1	Z	Y_1	Y_2	Z	Y_1	Y_2	Y_3
1.0	38.0	1.0	30.0	1.0	1.0	30.1	1.0	0.8	0.6
2.0	22.0	2.0	20.7	2.0	1.0	21.6	2.0	1.4	0.8
4.24	17.0	4.24	17.0	4.24	1.21	17.0	4.24	2.55	0.85

During the expansion process, problems with general variables, constraints and objective function are solved successively. The main advantages at this stage are as follows.

(a) The structures optimized are simple and small, therefore the computational effort involved in the solution process is considerably reduced.
(b) At each iteration the optimum is a feasible design satisfying all practical constraints.
(c) Problems of singular optima that might be encountered in the reduction process are eliminated.

It has been shown that poor selection of the geometrical variables might lead to local or singular optima that cannot be reached by numerical optimization. The procedure presented in this paper can find such optima at early stages while optimizing reduced structures with a small number of members. Finally, development of a systematic expansion approach for adding members and joints is still a challenge.

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