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CONSTRUCTIVE APPROXIMATION

Univariate Multiquadric Approximation: Quasi-Interpolation to Scattered Data

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Abstract. The univariate multiquadric function with center $x_j \in \mathbf{R}$ has the form $\{\varphi_j(x) = [(x - x_j)^2 + c^2]^{1/2}, x \in \mathbf{R}\}$ where c is a positive constant. We consider three approximations, namely, $\mathscr{L}_{\mathscr{A}} f$, $\mathscr{L}_{\mathscr{B}} f$, and $\mathscr{L}_{\mathscr{C}} f$, to a function $\{f(x), x_0 \leq x \leq x_N\}$ from the space that is spanned by the multiquadrics $\{\varphi_j : j = 0, 1, \ldots, N\}$ and by linear polynomials, the centers $\{x_j : j = 0, 1, \ldots, N\}$ being given distinct points of the interval $[x_0, x_N]$. The coefficients of $\mathscr{L}_{\mathscr{A}} f$ and $\mathscr{L}_{\mathscr{B}} f$ depend just on the function values $\{f(x_j) : j = 0, 1, \ldots, N\}$. while $\mathscr{L}_{\mathscr{C}} f$ also depends on the extreme derivatives $f'(x_0)$ and $f'(x_N)$. These approximations are defined by quasi-interpolation formulas that are shown to give good accuracy even if the distribution of the centers in $[x_0, x_N]$ is very irregular. When f is smooth and $c = \mathcal{O}(h)$, where h is the maximum distance between adjacent centers, we find that the error of each quasi-interpolant is $\mathcal{O}(h^2 |\log h|)$ away from the ends of the range $x_0 \leq x \leq x_N$. Near the ends of the range, however, the accuracy of $\mathscr{L}_{\mathscr{A}} f$ and $\mathscr{L}_{\mathscr{B}} f$ is only $\mathcal{O}(h)$, because the polynomial terms of these approximations are zero and a constant, respectively. Thus, some of the known accuracy properties of quasi-interpolation when there is an infinite regular grid of centers $\{x_j = jh : j \in \mathscr{Z}\}$, given by Buhmann (1988), are preserved in the case of a finite range $x_0 \leq x \leq x_N$, and there is no need for the centers $\{x_j : j = 0, 1, \ldots, N\}$ to be equally spaced.

1. Introduction

A multiquadric approximating function of d variables has the form

$$s(x) = \sum_{j=0}^{N} \lambda_j [\|x - x_j\|^2 + c^2]^{1/2}, \qquad x \in \mathbf{R}^d, \tag{1.1}$$

where $\{\lambda_j : j = 0, 1, \ldots, N\}$ and $\{x_j : j = 0, 1, \ldots, N\}$ are real coefficients and fixed points in $\mathbf{R}^d$, respectively, where the vector norm is Euclidean, and where c is a positive constant. Thus $\{s(x), x \in \mathbf{R}^d\}$ is infinitely differentiable. The use of such functions was proposed by Hardy (1971), and they perform well in many calculations including the numerical experiments that are reported by Franke (1982). An important property is that, for any choice of distinct points $\{x_j : j = 0, 1, \ldots, N\}$,

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the $(N+1) \times (N+1)$ matrix that has the elements

$$A_{ij} = [\|x_i - x_j\|^2 + c^2]^{1/2}, \qquad i, j = 0, 1, \ldots, N, \tag{1.2}$$

is nonsingular (Micchelli, 1986). Thus the interpolation equations $\{s(x_i) = f_i, i = 0, 1, \ldots, N\}$ define the coefficients $\{\lambda_j : j = 0, 1, \ldots, N\}$ uniquely for any given right-hand sides $\{f_i : i = 0, 1, \ldots, N\}$. Further, interpolation on an infinite regular square grid in $\mathbf{R}^d$ reproduces all polynomials of degree d (Buhmann, 1990), so we can achieve $\mathcal{O}(h^{d+1})$ accuracy when interpolating smooth functions, h being the grid size. There is no uniform bound on the norm of the interpolation operator, however, when the centers $\{x_j : j = 0, 1, \ldots, N\}$ are in general position.

In the univariate case ($d = 1$), the ability to reproduce linear polynomials does not require the centers $\{x_j\}$ to be equally spaced when there are infinitely many of them, $\{x_j : j \in \mathscr{Z}\}$ say, that extend to both ends of the real line (Powell, 1990). Further, letting the centers be in strictly ascending order and letting φ_j denote the function $\{\varphi_j(x) = [(x - x_j)^2 + c^2]^{1/2}, x \in \mathbf{R}\}$, that paper defines the normalized second divided difference

$$\psi_j(x) = \frac{\varphi_{j+1}(x) - \varphi_j(x)}{2(x_{j+1} - x_j)} - \frac{\varphi_j(x) - \varphi_{j-1}(x)}{2(x_j - x_{j-1})}, \qquad x \in \mathbf{R}, \tag{1.3}$$

for every integer j, and then proves that the quasi-interpolation scheme

$$s(x) = \sum_{j \in \mathscr{Z}} f(x_j)\psi_j(x), \qquad x \in \mathbf{R}, \tag{1.4}$$

gives the identity $\{s(x) = f(x), x \in \mathbf{R}\}$ whenever f is a linear polynomial. We are going to draw some conclusions from this result in the usual case when the approximating function $\{s(x), x \in \mathbf{R}\}$ is derived from a finite number of function values, $\{f(x_i) : i = 0, 1, \ldots, N\}$ say.

Three quasi-interpolation schemes that define an approximation s are considered, and for each one we establish bounds on the error

$$\|f - s\|_\infty = \max_{x_0 \le x \le x_N} |f(x) - s(x)|. \tag{1.5}$$

We believe that these error bounds are the first that have been published for univariate multiquadric approximation in the case when the number of centers is finite, and our method of analysis imposes no conditions on the positions of the centers, except that we assume the strict ordering

$$x_0 < x_1 < x_2 < \cdots < x_N. \tag{1.6}$$

The case when there are equally spaced centers $\{x_j = jh : j \in \mathscr{Z}\}$ throughout the real line $-\infty \le x \le \infty$ has been studied by Buhmann (1988), and he provides a bound on the error $\|f - s\|_\infty$ of the approximation (1.4) that is similar to our results. We extend the notation (1.6) by letting $\{x_j : j = -1, -2, \ldots\}$ and $\{x_j : j = N+1, N+2, \ldots\}$ be any prescribed, infinite, strictly monotonic sequences that diverge to $-\infty$ and $+\infty$, respectively. We retain the definition (1.3) for all $j \in \mathscr{Z}$, and we recall from Powell (1990) that these functions can be expressed in the form

$$\psi_j(x) = \tfrac{1}{2}c^2 \int_{x_{j-1}}^{x_{j+1}} \frac{B_j(\theta)}{[(x - \theta)^2 + c^2]^{3/2}}\, d\theta, \qquad x \in \mathbf{R}, \tag{1.7}$$

where $\{B_j(\theta), \theta \in \mathbf{R}\}$ is the piecewise linear hat function that has the knots $\{x_{j-1}, x_j, x_{j+1}\}$ and the normalization $B_j(x_j) = 1$.

Section 2 considers the approximation $\mathscr{L}_{\mathscr{B}} f = s$ that is defined by the formula

(1.8)
$$s(x) = \sum_{j=-\infty}^{-1} f(x_0)\psi_j(x) + \sum_{j=0}^{N} f(x_j)\psi_j(x) + \sum_{j=N+1}^{\infty} f(x_N)\psi_j(x), \qquad x_0 \le x \le x_N.$$

We find that this approximating function is independent of the particular choice of the extra centers $\{x_j: j < 0 \text{ or } j > N\}$, and that it is in the $(N + 2)$-dimensional linear space $\mathscr{B}$ that is spanned by the multiquadric functions $\{\varphi_j: j = 0, 1, \ldots, N\}$ and constants. Further, we derive a bound on the error (1.5) that is expressed in terms of the modulus of continuity of f. All our error bounds depend on the maximum spacing between adjacent data points, which is denoted by the symbol

$$h = \max_{1 \le j \le N} (x_j - x_{j-1}). \tag{1.9}$$

Thus we avoid the details of the positions of the interior data points.

Section 3 addresses the question of confining s to the $(N + 1)$-dimensional space $\mathscr{A}$ that is spanned by the multiquadrics $\{\varphi_j: j = 0, 1, \ldots, N\}$. Specifically, we form $\mathscr{L}_{\mathscr{A}} f$ by replacing the constant term that occurs in the approximation (1.8) by the constant times the function

$$\{[(x - x_0)^2 + c^2]^{1/2} + [(x - x_N)^2 + c^2]^{1/2}\}/(x_N - x_0), \qquad x_0 \le x \le x_N, \tag{1.10}$$

because the value of this function is quite close to one for all x in the interval $[x_0, x_N]$ provided that c is small. We note the resultant change to the error $\{f(x) - s(x), x_0 \le x \le x_N\}$, the variable x occurring explicitly in order to demonstrate that the change is smaller when x is well inside the range $[x_0, x_N]$.

When f is differentiable, we let $\mathscr{L}_{\mathscr{C}} f$ be the approximation

$$\begin{aligned} s(x) = &\sum_{j=-\infty}^{-1} [f(x_0) + (x_j - x_0)f'(x_0)]\psi_j(x) + \sum_{j=0}^{N} f(x_j)\psi_j(x) \\ &+ \sum_{j=N+1}^{\infty} [f(x_N) + (x_j - x_N)f'(x_N)]\psi_j(x), \qquad x_0 \le x \le x_N, \end{aligned} \tag{1.11}$$

which is usually more accurate than expression (1.8). Indeed, it follows from the polynomial reproduction properties of formula (1.4) that we now have $\{s(x) = f(x), x \in \mathbf{R}\}$ whenever f is a linear polynomial. This scheme is the subject of Section 4. Again we find that the actual positions of the centers $\{x_j: j < 0 \text{ or } j > N\}$ are irrelevant, but now s is in the $(N + 3)$-dimensional linear space $\mathscr{C}$ that includes $\mathscr{A}$ and all linear polynomials. Assuming that the derivative $\{f'(x), x_0 \le x \le x_N\}$ is Lipschitz continuous, we bound the error (1.5) by an expression that is proportional to the Lipschitz constant.

The given error bounds are explicit. We find, for instance, that $\|f - \mathscr{L}_{\mathscr{B}} f\|_\infty$ is at most $(1 + c/h)\omega(f, h)$, where $\{\omega(f, \delta), \delta > 0\}$ is the modulus of continuity of f. Further, examples are presented that show that the bounds are optimal or nearly optimal. A referee has pointed out that the uniform convergence of $\mathscr{L}_{\mathscr{B}} f$ to f as $h \to 0$ can also be deduced from the fact that equations (1.7) and (1.8) allow $\mathscr{L}_{\mathscr{B}} f$

to be expressed as a convolution of a linear interpolating spline with a positive kernel. We employed yet another technique initially which makes explicit use of the asymptotic properties of the functions $\{\psi_j\}$. The theory that is presented in this paper, however, includes exact analytic evaluations of several integrals. Thus it gives tighter error bounds than the other two techniques.

The conclusions of Sections 2–4 are compared in Section 5. We note that the differences between the approximations $\{(\mathscr{L}_{\mathscr{A}} f)(x), x_0 \le x \le x_N\}$, $\{(\mathscr{L}_{\mathscr{B}} f)(x), x_0 \le x \le x_N\}$, and $\{(\mathscr{L}_C f)(x), x_0 \le x \le x_N\}$ are greatest when x is near to an end of the interval $[x_0, x_N]$. Further, when f is smooth and c is bounded above by a constant multiple of h, we find that typically the errors $\|\mathscr{L}_{\mathscr{A}} f - f\|_\infty$, $\|\mathscr{L}_{\mathscr{B}} f - f\|_\infty$, and $\|\mathscr{L}_{\mathscr{C}} f - f\|_\infty$ are of magnitudes $\mathscr{O}(h)$, $\mathscr{O}(h)$, and $\mathscr{O}(h^2|\log h|)$, respectively, but the differences $|(\mathscr{L}_{\mathscr{A}} f)(x) - (\mathscr{L}_{\mathscr{C}} f)(x)|$ and $|(\mathscr{L}_{\mathscr{B}} f)(x) - (\mathscr{L}_{\mathscr{C}} f)(x)|$ are only $\mathscr{O}(h^2)$ when x is well inside $[x_0, x_N]$. Such bounds do not hold, however, if c is independent of h. Therefore, in order to obtain good accuracy from the quasi-interpolation schemes when h is small, it is necessary to ensure that c is not much larger than h. We can also reach this conclusion by studying the decay of $\psi_j(x)$ to zero as $|x - x_j|$ becomes large, using expression (1.3) or (1.7). On the other hand, some recent work of Buhmann and Dyn (1991) shows that it may be advantageous to keep c fixed as $h \to 0$ when the approximation s is defined by interpolation.

2. Approximation from the Space $\mathscr{B}$

It is straightforward to deduce from (1.7) that the functions $\{\psi_j : j \in \mathscr{Z}\}$ are positive and a partition of unity, which means that the equation

$$\sum_{j\in\mathscr{Z}} \psi_j(x) = 1, \qquad x \in \mathbf{R}, \tag{2.1}$$

is obtained. Therefore, the infinite sums of the definition (1.8) are absolutely convergent for every x. We write this definition in the form

$$(\mathscr{L}_{\mathscr{B}} f)(x) = f(x_0)\beta_0(x) + \sum_{j=1}^{N-1} f(x_j)\psi_j(x) + f(x_N)\beta_N(x), \qquad x \in \mathbf{R}, \tag{2.2}$$

where β_0 and β_N are the functions $\sum_{j\le 0} \psi_j$ and $\sum_{j\ge N} \psi_j$, respectively, the notation β being employed because of its affinity to the name of the linear space $\mathscr{B}$. Further, remembering that the numerator of the integrand of expression (1.7) is a normalized hat function, we have the identity

$$\begin{aligned}\beta_0(x) &= \tfrac{1}{2}c^2 \int_{-\infty}^{x_0} \frac{1}{[(x-\theta)^2 + c^2]^{3/2}}\, d\theta + \tfrac{1}{2}c^2 \int_{x_0}^{x_1} \frac{(x_1-\theta)/(x_1-x_0)}{[(x-\theta)^2+c^2]^{3/2}}\, d\theta \\ &= \tfrac{1}{2} + \frac{[(x-x_1)^2+c^2]^{1/2} - [(x-x_0)^2+c^2]^{1/2}}{2(x_1-x_0)}, \qquad x \in \mathbf{R},\end{aligned} \tag{2.3}$$

and a similar argument provides the equation

$$\beta_N(x) = \tfrac{1}{2} - \frac{[(x-x_N)^2+c^2]^{1/2} - [(x-x_{N-1})^2+c^2]^{1/2}}{2(x_N-x_{N-1})}, \qquad x \in \mathbf{R}. \tag{2.4}$$

It follows that $\mathscr{L}_{\mathscr{B}} f$ is in the space $\mathscr{B}$ as claimed. Further, the relations (1.3) and (2.2)–(2.4) imply that the approximation $\mathscr{L}_{\mathscr{B}} f$ is the same as f when f is a constant function, which is a restatement of condition (2.1).

Next we seek bounds on the error function $f - \mathscr{L}_{\mathscr{B}} f$ when $\{f(x), x_0 \le x \le x_N\}$ is continuous. Using the form (1.8) of $\mathscr{L}_{\mathscr{B}} f$ and condition (2.1), we deduce the equation

$$(2.5)\quad f(x) - (\mathscr{L}_{\mathscr{B}} f)(x) = \sum_{j=-\infty}^{-1} [f(x) - f(x_0)]\psi_j(x) + \sum_{j=0}^{N} [f(x) - f(x_j)]\psi_j(x) + \sum_{j=N+1}^{\infty} [f(x) - f(x_N)]\psi_j(x), \qquad x_0 \le x \le x_N.$$

In order to bound the differences in f that occur in the square brackets, we let $\{\omega(f, \delta), \delta > 0\}$ be the modulus of continuity of f. Therefore, if x is in the interval $[x_k, x_{k+1}]$, the definition (1.9) implies the inequalities

$$(2.6)\qquad |f(x) - f(x_k)| \le \omega(f, h) \qquad \text{and} \qquad |f(x) - f(x_{k+1})| \le \omega(f, h).$$

Further, the conditions

$$(2.7)\qquad |f(x) - f(x_j)| \le (1 + |x - x_j|/h)\omega(f, h), \qquad j = 0, 1, \ldots, N,$$

are also satisfied.

Let $\{\sigma(\theta), \theta \in \mathbf{R}\}$ be the linear spline with the knots $\{x_j : j \in \mathscr{Z}\}$ that takes the values

$$(2.8)\qquad \sigma(x_j) = \begin{cases} f(x) - f(x_0), & j \le 0, \\ f(x) - f(x_j), & 0 \le j \le N, \\ f(x) - f(x_N), & j \ge N. \end{cases}$$

Then (2.5) and (1.7) give the identity

$$(2.9)\qquad f(x) - (\mathscr{L}_{\mathscr{B}} f)(x) = \tfrac{1}{2}c^2 \int_{-\infty}^{\infty} \frac{\sigma(\theta)}{[(x-\theta)^2 + c^2]^{3/2}}\, d\theta.$$

It follows from conditions (2.6)–(2.8) that we have the inequality

$$(2.10)\qquad |f(x) - (\mathscr{L}_{\mathscr{B}} f)(x)| \le \tfrac{1}{2}c^2 \int_{-\infty}^{\infty} \frac{(1 + |x-\theta|/h)\omega(f, h)}{[(x-\theta)^2 + c^2]^{3/2}}\, d\theta = (1 + c/h)\omega(f, h), \qquad x_0 \le x \le x_N,$$

which is the main result of this section. We state it as a theorem.

Theorem 1. *Let $\mathscr{L}_{\mathscr{B}} f$ be defined by the quasi-interpolation formula* (2.2), *where β_0, $\{\psi_j : j = 1, 2, \ldots, N-1\}$, and β_N are the functions* (2.3), (1.3), *and* (2.4), *respectively. Then the maximum value of the error function $\{f(x) - (\mathscr{L}_{\mathscr{B}} f)(x), x_0 \le x \le x_N\}$ satisfies the bound*

$$(2.11)\qquad \|f - \mathscr{L}_{\mathscr{B}} f\|_\infty \le (1 + c/h)\omega(f, h),$$

where h is the maximum spacing between adjacent data points.

The factor $(1 + c/h)$ that multiplies $\omega(f, h)$ in the statement of the theorem cannot be replaced by a smaller number whose only dependence on the data points $\{x_j: j = 0, 1, \ldots, N\}$ is the value of h. We prove this assertion by considering the following example.

Let η be any number from the open interval $(0, h)$, and let $\{f(x), x \in \mathbf{R}\}$ be a linear spline whose knots are all the integer multiples of h and the points $\{\pm(kh + \eta): k = 0, 1, 2, \ldots\}$. Further, we let the values of f at its knots be the numbers

$$\left.\begin{aligned} f(\pm kh) &= k\omega, \\ f(\pm(kh + \eta)) &= (k + 1)\omega, \end{aligned}\right\} \quad k = 0, 1, 2, \ldots, \tag{2.12}$$

where ω is a positive constant, this notation being deliberate because we have $\omega = \omega(f, h)$. We consider the error $|f(x) - (\mathscr{L}_{\mathscr{B}} f)(x)|$ at $x = 0$ when the data points $\{x_j: j = 0, 1, \ldots, N\}$ are all the elements of the set $\{\pm(kh + \eta): k = 0, 1, 2, \ldots\}$ that are in the interval $[-Mh - \eta, Mh + \eta]$ for some large integer M. In this case, we have $N = 2M + 1$ and $f(x) = 0$. Further, expression (2.8) becomes the equation

$$\sigma(x_j) = \begin{cases} -(M + 1)\omega, & j \le 0, \\ -[1 + (|x_j| - \eta)/h]\omega, & 0 \le j \le N, \\ -(M + 1)\omega, & j \ge N. \end{cases} \tag{2.13}$$

Therefore the identity (2.9) gives the bound

$$|f(0) - (\mathscr{L}_{\mathscr{B}} f)(0)| > \tfrac{1}{2}c^2 \int_{-Mh}^{Mh} \frac{[1 + (|\theta| - \eta)/h]\omega}{(\theta^2 + c^2)^{3/2}}\, d\theta. \tag{2.14}$$

We let η and M tend to zero and infinity, respectively, for fixed h. Thus in the limit the first line of inequality (2.10) when $x = 0$ is satisfied as an equation, which implies the required optimality of condition (2.11).

3. Approximation from the Space $\mathscr{A}$

Equations (2.2)–(2.4) show that the approximation $\mathscr{L}_{\mathscr{B}} f$ is a linear combination of the multiquadric functions $\{\varphi_j: j = 0, 1, \ldots, N\}$ plus the constant term $\tfrac{1}{2}[f(x_0) + f(x_N)]$, but now we require an approximation that has no constant term. Therefore, recalling from Section 1 that the expression (1.10) can be a good substitute for one, we define our approximation from $\mathscr{A}$ by the quasi-interpolation formula

$$\begin{aligned} (\mathscr{L}_{\mathscr{A}} f)(x) &= (\mathscr{L}_{\mathscr{B}} f)(x) + \tfrac{1}{2}[f(x_0) + f(x_N)] \\ &\quad \times \left\{ \frac{[(x - x_0)^2 + c^2]^{1/2} + [(x - x_N)^2 + c^2]^{1/2}}{x_N - x_0} - 1 \right\} \\ &= f(x_0)\alpha_0(x) + \sum_{j=1}^{N-1} f(x_j)\psi_j(x) + f(x_N)\alpha_N(x), \qquad x \in \mathbf{R}, \end{aligned} \tag{3.1}$$

where α_0 and α_N are the functions

$$\left.\begin{aligned} \alpha_0(x) &= \frac{\varphi_1(x) - \varphi_0(x)}{2(x_1 - x_0)} + \frac{\varphi_0(x) + \varphi_N(x)}{2(x_N - x_0)}, \\ \alpha_N(x) &= -\frac{\varphi_N(x) - \varphi_{N-1}(x)}{2(x_N - x_{N-1})} + \frac{\varphi_0(x) + \varphi_N(x)}{2(x_N - x_0)}, \end{aligned}\right\} \quad x \in \mathbf{R}. \tag{3.2}$$

Because expression (1.10) is bounded below by one, the strict positivity of β_0 and β_N is inherited by α_0 and α_N, but formula (3.1) does not reproduce nonzero constant functions.

Equation (3.1) implies the difference

$$(\mathscr{L}_{\mathscr{A}} f)(x) - (\mathscr{L}_{\mathscr{B}} f)(x) = \frac{f(x_0) + f(x_N)}{2(x_N - x_0)} \{[(x - x_0)^2 + c^2]^{1/2} - (x - x_0) + [(x_N - x)^2 + c^2]^{1/2} - (x_N - x)\}, \quad x_0 \le x \le x_N. \tag{3.3}$$

The term in braces has the value

$$\frac{c^2}{(x - x_0) + [(x - x_0)^2 + c^2]^{1/2}} + \frac{c^2}{(x_N - x) + [(x_N - x)^2 + c^2]^{1/2}}, \tag{3.4}$$

which for x in $[x_0, x_N]$ is bounded below and above by the numbers

$$\frac{\frac{1}{2}c^2}{\min(x - x_0, x_N - x) + c} \quad \text{and} \quad \frac{2c^2}{\min(x - x_0, x_N - x) + c}, \tag{3.5}$$

respectively. Thus we deduce the relation

$$|(\mathscr{L}_{\mathscr{A}} f)(x) - (\mathscr{L}_{\mathscr{B}} f)(x)| \le \frac{c^2 |f(x_0) + f(x_N)|}{(x_N - x_0)[\min(x - x_0, x_N - x) + c]}, \quad x_0 \le x \le x_N, \tag{3.6}$$

the right-hand side being at most four times the left-hand side. We see that, if c is small, then the difference $|(\mathscr{L}_{\mathscr{A}} f)(x) - (\mathscr{L}_{\mathscr{B}} f)(x)|$ is relatively large when x is near the ends of the interval $[x_0, x_N]$. Further, because the maximum value of expression (3.6) occurs at $x = x_0$, it follows from Theorem 1 and the triangle inequality that we have the following result.

Theorem 2. *The maximum error of the quasi-interpolant $\mathscr{L}_{\mathscr{A}} f$ on the interval $x_0 \le x \le x_N$ satisfies the bound*

$$\|f - \mathscr{L}_{\mathscr{A}} f\|_\infty \le \frac{c|f(x_0) + f(x_N)|}{x_N - x_0} + (1 + c/h)\omega(f, h). \tag{3.7}$$

We complete this section by proving that, for any values of the two numbers $|f(x_0) + f(x_N)|$ and $\omega(f, h)$, there exists a function $\{f(x), x_0 \le x \le x_N\}$ such that the left-hand side of inequality (3.7) is at least one-quarter of the right-hand side.

Specifically, we let f satisfy the condition

$$\|f - \mathscr{L}_{\mathscr{B}} f\|_\infty \geq \tfrac{1}{2}(1 + c/h)\omega(f, h), \tag{3.8}$$

which is shown to be possible in the last paragraph of Section 2, and then we add a constant to f, if necessary, to achieve the given value of $|f(x_0) + f(x_N)|$, which preserves inequality (3.8). Further, we construct $\hat{f}$ by altering f by the constant term that reverses the sign of $f(x_0) + f(x_N)$, so the expressions $\hat{f} - \mathscr{L}_{\mathscr{B}} \hat{f}$ and $\omega(\hat{f}, h)$ are equal to $f - \mathscr{L}_{\mathscr{B}} f$ and $\omega(f, h)$, respectively, but the difference $\mathscr{L}_{\mathscr{A}} \hat{f} - \mathscr{L}_{\mathscr{B}} \hat{f}$ is minus the difference $\mathscr{L}_{\mathscr{A}} f - \mathscr{L}_{\mathscr{B}} f$.

If the condition

$$(1 + c/h)\omega(f, h) \geq c|f(x_0) + f(x_N)|/(x_N - x_0) \tag{3.9}$$

holds, we let ξ be a point of $[x_0, x_N]$ that satisfies the equation

$$|f(\xi) - (\mathscr{L}_{\mathscr{B}} f)(\xi)| = \|f - \mathscr{L}_{\mathscr{B}} f\|_\infty, \tag{3.10}$$

and we consider the elementary bound

$$\|f - \mathscr{L}_{\mathscr{A}} f\|_\infty \geq |[f(\xi) - (\mathscr{L}_{\mathscr{B}} f)(\xi)] + [(\mathscr{L}_{\mathscr{B}} f)(\xi) - (\mathscr{L}_{\mathscr{A}} f)(\xi)]|. \tag{3.11}$$

The option of replacing f by $\hat{f}$ allows us to assume that the two terms in square brackets do not have opposite signs. Therefore it follows from expressions (3.11), (3.10), and (3.8) that we have the relation

$$\|f - \mathscr{L}_{\mathscr{A}} f\|_\infty \geq \tfrac{1}{2}(1 + c/h)\omega(f, h), \tag{3.12}$$

and then condition (3.9) implies that $\|f - \mathscr{L}_{\mathscr{A}} f\|_\infty$ is at least one-fourth of the right-hand side of inequality (3.7) as required.

Alternatively, when condition (3.9) fails, we employ the bound

$$\begin{aligned} \|f - \mathscr{L}_{\mathscr{A}} f\|_\infty &\geq |[f(x_0) - (\mathscr{L}_{\mathscr{B}} f)(x_0)] - [(\mathscr{L}_{\mathscr{A}} f)(x_0) - (\mathscr{L}_{\mathscr{B}} f)(x_0)]| \\ &\geq |(\mathscr{L}_{\mathscr{A}} f)(x_0) - (\mathscr{L}_{\mathscr{B}} f)(x_0)|, \end{aligned} \tag{3.13}$$

where, as before, the last line is obtained by replacing f by $\hat{f}$ if necessary. Because the term (3.4) is bounded below by c when $x = x_0$, we deduce from expressions (3.3) and (3.13) that the inequality

$$\|f - \mathscr{L}_{\mathscr{A}} f\|_\infty \geq \tfrac{1}{2}c|f(x_0) + f(x_N)|/(x_N - x_0) \tag{3.14}$$

holds. Therefore the required result follows from the failure of condition (3.9).

4. Approximation from the Space $\mathscr{C}$

We assume throughout this section that $\{f(x),\ x_0 \leq x \leq x_N\}$ has a Lipschitz continuous first derivative, we recall from Section 1 that $\mathscr{L}_{\mathscr{C}} f$ is the approximation (1.11), and we seek bounds on the error $f - \mathscr{L}_{\mathscr{C}} f$. The definition (1.8) of $\mathscr{L}_{\mathscr{B}} f$ allows us to express $\mathscr{L}_{\mathscr{C}} f$ in the form

$$(\mathscr{L}_{\mathscr{C}} f)(x) = (\mathscr{L}_{\mathscr{B}} f)(x) + f'(x_0)\gamma_0(x) + f'(x_N)\gamma_N(x), \qquad x \in \mathbf{R}, \tag{4.1}$$

where γ_0 and γ_N are independent of f. Specifically, in view of (1.7), we see that γ_0 is the function

$$\gamma_0(x) = \sum_{j=-\infty}^{0} (x_j - x_0)\psi_j(x) = \tfrac{1}{2}c^2 \int_{-\infty}^{x_0} \frac{\theta - x_0}{[(x-\theta)^2 + c^2]^{3/2}}\, d\theta \qquad (4.2)$$

$$= \tfrac{1}{2}(x - x_0) - \tfrac{1}{2}[(x - x_0)^2 + c^2]^{1/2}, \qquad x \in \mathbf{R},$$

and a similar argument gives the identity

$$\gamma_N(x) = \tfrac{1}{2}[(x_N - x)^2 + c^2]^{1/2} - \tfrac{1}{2}(x_N - x), \qquad x \in \mathbf{R}. \qquad (4.3)$$

Hence, as claimed in Section 1, $\mathscr{L}_{\mathscr{C}} f$ is in the space $\mathscr{C}$ and is independent of the positions of the additional centers $\{x_j : j < 0 \text{ or } j > N\}$. Further, we can deduce directly from (1.3), (2.2)–(2.4), and (4.1)–(4.3) that $\mathscr{L}_{\mathscr{C}} f \equiv f$ when f is any linear polynomial. We omit this task, however, because it has been noted already that, due to the equivalence of formulas (1.4) and (1.11) when $f \in \Pi_1$, this property is a consequence of a theorem of Powell (1990). Because it implies the relation $\{\sum_{j \in \mathscr{X}} x_j \psi_j(x) = x,\ x \in \mathbf{R}\}$ in addition to condition (2.1), we have the equation

$$f(x) = \sum_{j \in \mathscr{X}} [f(x) + (x_j - x) f'(x)]\psi_j(x), \qquad x_0 \le x \le x_N, \qquad (4.4)$$

which is important to the error bounds that are going to be derived.

Indeed, this equation and the definition (1.11) of $\mathscr{L}_{\mathscr{C}} f$ provide the identity

(4.5)

$$\begin{aligned} f(x) - (\mathscr{L}_{\mathscr{C}} f)(x) = &\sum_{j=-\infty}^{-1} [f(x) + (x_j - x) f'(x) - f(x_0) - (x_j - x_0) f'(x_0)]\psi_j(x) \\ &+ \sum_{j=N+1}^{\infty} [f(x) + (x_j - x) f'(x) - f(x_N) - (x_j - x_N) f'(x_N)]\psi_j(x) \\ &+ \sum_{j=0}^{N} [f(x) + (x_j - x) f'(x) - f(x_j)]\psi_j(x), \qquad x_0 \le x \le x_N. \end{aligned}$$

In order to bound the terms in square brackets, we require the derivative $\{f'(x), x_0 \le x \le x_N\}$ to be Lipschitz continuous, and we use the notation

$$\Omega = \operatorname*{ess\,sup}_{x_0 \le x \le x_N} |f''(x)| \qquad (4.6)$$

for the Lipschitz constant. For every x in the interval $[x_0, x_N]$, this assumption yields the inequalities

$$|f(x) + (x_j - x) f'(x) - f(x_j)| \le \tfrac{1}{2}(x - x_j)^2 \Omega, \qquad j = 0, 1, \ldots, N, \qquad (4.7)$$

$$\begin{aligned} &|f(x) + (x_j - x) f'(x) - f(x_0) - (x_j - x_0) f'(x_0)| \qquad (4.8) \\ &\quad \le \tfrac{1}{2}(x - x_0)^2 \Omega + |x_j - x_0| \, |f'(x) - f'(x_0)| \\ &\quad \le (x - x_0)[\tfrac{1}{2}(x - x_0) + (x_0 - x_j)]\Omega, \qquad j \le 0, \end{aligned}$$

and

(4.9) $$|f(x)+(x_j-x)f'(x)-f(x_N)-(x_j-x_N)f'(x_N)| \le (x_N-x)[\tfrac{1}{2}(x_N-x)+(x_j-x_N)]\Omega, \qquad j \ge N.$$

Instead of expression (2.8), we now let the numbers $\{\sigma(x_j): j \in \mathscr{Z}\}$ be the right-hand sides of inequalities (4.7)–(4.9) for the values of j that are displayed. Therefore (4.5) implies the bound

(4.10) $$|f(x)-(\mathscr{L}_{\mathscr{C}}f)(x)| \le \sum_{j=-\infty}^{\infty} \sigma(x_j)\psi_j(x), \qquad x_0 \le x \le x_N.$$

It follows from the identity (1.7) that the right-hand side of this bound has the value

(4.11) $$\tfrac{1}{2}c^2 \int_{-\infty}^{\infty} \frac{\sum_j \sigma(x_j)B_j(\theta)}{[(x-\theta)^2+c^2]^{3/2}}\,d\theta = \tfrac{1}{2}c^2 \int_{-\infty}^{\infty} \frac{\sigma(\theta)}{[(x-\theta)^2+c^2]^{3/2}}\,d\theta,$$

where $\{\sigma(\theta), \theta \in \mathbf{R}\}$ is the linear spline with the knots $\{x_j: j \in \mathscr{Z}\}$ that interpolates the numbers $\{\sigma(x_j): j \in \mathscr{Z}\}$ that have been defined already. We see that the functions $\{\sigma(\theta), \theta \le x_0\}$ and $\{\sigma(\theta), \theta \ge x_N\}$ are straight lines, but $\{\sigma(\theta), x_0 \le \theta \le x_N\}$ is a piecewise linear interpolant to the quadratic function $\{\tfrac{1}{2}(x-\theta)^2\Omega, x_0 \le \theta \le x_N\}$. Therefore, the usual bound for the error of linear interpolation gives the condition

(4.12) $$\sigma(\theta) \le \tfrac{1}{2}(x-\theta)^2\Omega + \tfrac{1}{8}h^2\Omega, \qquad x_0 \le \theta \le x_N.$$

Hence expressions (4.10) and (4.11) imply the inequality

(4.13) $$\begin{aligned}|f(x)-(\mathscr{L}_{\mathscr{C}}f)(x)| \le{}& \tfrac{1}{2}c^2\Omega \int_{-\infty}^{x_0} \frac{(x-x_0)[\tfrac{1}{2}(x+x_0)-\theta]}{[(x-\theta)^2+c^2]^{3/2}}\,d\theta \\ &+ \tfrac{1}{2}c^2\Omega \int_{x_0}^{x_N} \frac{\tfrac{1}{2}(x-\theta)^2+\tfrac{1}{8}h^2}{[(x-\theta)^2+c^2]^{3/2}}\,d\theta \\ &+ \tfrac{1}{2}c^2\Omega \int_{x_N}^{\infty} \frac{(x_N-x)[\theta-\tfrac{1}{2}(x+x_N)]}{[(x-\theta)^2+c^2]^{3/2}}\,d\theta.\end{aligned}$$

Analytic integration shows that the three terms on the right-hand side have the values

(4.14) $$\tfrac{1}{2}c^2\Omega\left[\frac{x-x_0}{[(x-\theta)^2+c^2]^{1/2}} + \frac{\tfrac{1}{2}(x-x_0)^2(x-\theta)}{c^2[(x-\theta)^2+c^2]^{1/2}}\right]_{-\infty}^{x_0},$$

(4.15) $$\tfrac{1}{2}c^2\Omega\left[\tfrac{1}{2}\sinh^{-1}\!\left(\frac{\theta-x}{c}\right) + \frac{(\tfrac{1}{2}c^2-\tfrac{1}{8}h^2)(x-\theta)}{c^2[(x-\theta)^2+c^2]^{1/2}}\right]_{x_0}^{x_N},$$

and

(4.16) $$\tfrac{1}{2}c^2\Omega\left[\frac{x-x_N}{[(x-\theta)^2+c^2]^{1/2}} + \frac{\tfrac{1}{2}(x-x_N)^2(x-\theta)}{c^2[(x-\theta)^2+c^2]^{1/2}}\right]_{x_N}^{\infty},$$

respectively. Thus we find the relation

$$
\begin{aligned}
(4.17)\quad |f(x) - (\mathscr{L}_{\mathscr{C}} f)(x)| \le\ & \tfrac{1}{4}\Omega(x - x_0)\{[(x - x_0)^2 + c^2]^{1/2} - (x - x_0)\} \\
&+ \tfrac{1}{4}\Omega(x_N - x)\{[(x_N - x)^2 + c^2]^{1/2} - (x_N - x)\} \\
&+ \tfrac{1}{4}c^2\Omega\left[\sinh^{-1}\left(\frac{x - x_0}{c}\right) + \sinh^{-1}\left(\frac{x_N - x}{c}\right)\right] \\
&+ \tfrac{1}{16}h^2\Omega\left[\frac{x - x_0}{[(x - x_0)^2 + c^2]^{1/2}} + \frac{x_N - x}{[(x_N - x)^2 + c^2]^{1/2}}\right], \\
&\qquad x_0 \le x \le x_N.
\end{aligned}
$$

Now the expressions in the first, second, and fourth lines of the right-hand side are at most $\frac{1}{8}c^2\Omega$, $\frac{1}{8}c^2\Omega$, and $\frac{1}{8}h^2\Omega$, respectively, while the third line contains a concave function of x whose maximum value occurs at $x = \frac{1}{2}(x_0 + x_N)$. Therefore, we have the bound

$$
(4.18)\qquad \|f - \mathscr{L}_{\mathscr{C}} f\|_\infty \le \tfrac{1}{4}c^2\Omega\left[1 + 2\sinh^{-1}\left(\frac{x_N - x_0}{2c}\right)\right] + \tfrac{1}{8}h^2\Omega.
$$

It is more usual, however, to employ logarithms instead of inverse hyperbolic functions, so we invoke the elementary inequality $\{\sinh^{-1} t \le \log(2t + 1),\ t \ge 0\}$. Thus expression (4.18) implies the following theorem.

Theorem 3. *If f has a Lipschitz continuous first derivative, then the maximum error of the quasi-interpolant $\mathscr{L}_{\mathscr{C}} f$ on the interval $x_0 \le x \le x_N$ satisfies the bound*

$$
(4.19)\qquad \|f - \mathscr{L}_{\mathscr{C}} f\|_\infty \le \tfrac{1}{4}c^2\Omega\left[1 + 2\log\left(1 + \frac{x_N - x_0}{c}\right)\right] + \tfrac{1}{8}h^2\Omega,
$$

where Ω is defined by (4.6).

In the limiting case when $c = 0$ this theorem gives the usual bound on the error of piecewise linear interpolation, but, when c is not much less than h, the $\frac{1}{8}h^2\Omega$ term is unimportant. Therefore, it is worthwhile to test the slackness in inequality (4.19) without giving careful attention to the value of h. It is appropriate to let f be the quadratic function $\{f(x) = \frac{1}{2}\Omega x^2,\ x_0 \le x \le x_N\}$, because in this case the inequalities (4.7)–(4.9) are all satisfied as equations even if the modulus signs are replaced by the factor -1 on each of the left-hand sides. Thus condition (4.10) also holds as an equation. Deleting the h^2 term from the relation (4.12), however, provides the lower bound $\{\sigma(\theta) \ge \frac{1}{2}\Omega(x - \theta)^2,\ x_0 \le \theta \le x_N\}$. Therefore we set h to zero in expressions (4.13) and (4.17) and we reverse these inequalities. Thus, because each of the functions of x in the first three lines of the right-hand side of expression (4.17) is concave, we find the bound

$$
\begin{aligned}
(4.20)\quad |f(x) - (\mathscr{L}_{\mathscr{C}} f)(x)| \ge\ & \tfrac{1}{4}\Omega(x_N - x_0)\{[(x_N - x_0)^2 + c^2]^{1/2} - (x_N - x_0)\} \\
&+ \tfrac{1}{4}c^2\Omega\sinh^{-1}\left(\frac{x_N - x_0}{c}\right), \qquad x_0 \le x \le x_N.
\end{aligned}
$$

We can delete the first term on the right-hand side because it is nonnegative. It follows from the elementary relation $\{\sinh^{-1} t \geq \log(2t), t > 0\}$ that we have the inequality

$$(4.21) \quad |f(x) - (\mathscr{L}_{\mathscr{C}} f)(x)| \geq \tfrac{1}{4}c^2\Omega\left[\log 2 + \log\left(\frac{x_N - x_0}{c}\right)\right], \qquad x_0 \leq x \leq x_N,$$

when f is the function $\{\frac{1}{2}\Omega x^2, x_0 \leq x \leq x_N\}$. It follows that Theorem 3 usually provides a good indication of the magnitude of $\|f - \mathscr{L}_{\mathscr{C}} f\|_\infty$. Further, because condition (4.21) is valid for every x in the interval $[x_0, x_N]$, we can deduce that in many cases the ratio of the maximum to the minimum value of the error function $\{|f(x) - (\mathscr{L}_{\mathscr{C}} f)(x)|, x_0 \leq x \leq x_N\}$ is at most two, due to the dominance of the $\sinh^{-1}$ terms of expressions (4.18) and (4.20) when $h \leq c \ll (x_N - x_0)$.

5. Discussion

Theorems 1, 2, and 3 all suggest that the given quasi-interpolation schemes fail to provide good accuracy as $h \to 0$ unless $c \to 0$ too. This suggestion is confirmed in the last paragraph of Section 4. Therefore in this section we assume that c satisfies the bound

$$(5.1) \qquad c \leq Dh,$$

where D is a positive constant. Thus our theorems provide the inequalities

$$(5.2) \qquad \|f - \mathscr{L}_{\mathscr{B}} f\|_\infty \leq (1 + D)\omega(f, h),$$

$$(5.3) \qquad \|f - \mathscr{L}_{\mathscr{A}} f\|_\infty \leq \frac{|f(x_0) + f(x_N)|Dh}{x_N - x_0} + (1 + D)\omega(f, h),$$

and

$$(5.4) \qquad \|f - \mathscr{L}_{\mathscr{C}} f\|_\infty \leq \left[\tfrac{1}{8} + \tfrac{1}{4}D^2 + \tfrac{1}{2}D^2 \log\left(1 + \frac{x_N - x_0}{Dh}\right)\right]\Omega h^2.$$

It is usual to replace $\omega(f, h)$ by $h\|f'\|_\infty$ when f is smooth, and in this case we expect $\mathscr{L}_{\mathscr{C}} f$ to be a substantially more accurate approximation to $\{f(x), x_0 \leq x \leq x_N\}$ than $\mathscr{L}_{\mathscr{A}} f$ or $\mathscr{L}_{\mathscr{B}} f$. Indeed, inequalities (5.2)–(5.4) provide the $\mathcal{O}(h)$, $\mathcal{O}(h)$, and $\mathcal{O}(h^2|\log h|)$ error bounds that are mentioned in Section 1.

The example at the end of Section 4 shows that the $\mathcal{O}(h^2|\log h|)$ term of expression (5.4) cannot be removed if our only condition on f is that it has a Lipschitz continuous first derivative. Inequality (4.21) also demonstrates that the dependence of the relation (5.4) on $x_N - x_0$ is unavoidable too, and that the error of the approximation $\mathscr{L}_{\mathscr{C}} f \approx f$ becomes unbounded if $x_N - x_0$ tends to infinity for any fixed value of c. When there is a constant upper bound on $\|f'\|_\infty$, however, in addition to the Lipschitz condition (4.6), then inequality (5.4) can be strengthened to a form that has no $x_N - x_0$ term. For example, Theorem 6 of Buhmann (1988) establishes that, if the function $\{f(x), x \in \mathbf{R}\}$ has finite first and second derivatives and if there are an infinite number of equally spaced centers $\{x_j =$

$jh\colon j \in \mathscr{Z}\}$, then the error $\|f - s\|_\infty$ of the quasi-interpolation scheme (1.4) is $\mathcal{O}(h^2 + c^2|\log h|)$. A method of proof is to bound the square bracket terms of equation (4.5) by the least numbers that can be derived from the values of $\|f'\|_\infty$ and Ω.

Inequality (4.21), when f is a quadratic function, suggests that in general expression (5.4) provides a useful bound on $|f(x) - (\mathscr{L}_{\mathscr{C}} f)(x)|$ for every x in $[x_0, x_N]$. On the other hand, the following remarks show that the bounds (5.2) and (5.3) on $|f(x) - (\mathscr{L}_{\mathscr{B}} f)(x)|$ and $|f(x) - (\mathscr{L}_{\mathscr{A}} f)(x)|$, respectively, are pessimistic when f is smooth and x is well inside the interval $[x_0, x_N]$. Equations (4.1)–(4.3) imply the relation

$$(5.5)\quad \begin{aligned} |(\mathscr{L}_{\mathscr{B}} f)(x) - (\mathscr{L}_{\mathscr{C}} f)(x)| &\le [|\gamma_0(x)| + |\gamma_N(x)|]\|f'\|_\infty \\ &\le \left[\frac{\frac{1}{2}c^2}{(x - x_0) + [(x - x_0)^2 + c^2]^{1/2}} \right. \\ &\qquad \left. + \frac{\frac{1}{2}c^2}{(x_N - x) + [(x_N - x)^2 + c^2]^{1/2}}\right]\|f'\|_\infty \\ &\le \frac{D^2\|f'\|_\infty h^2}{\min(x - x_0, x_N - x) + Dh}, \qquad x_0 \le x \le x_N, \end{aligned}$$

where the last line depends on condition (5.1) and the upper bound (3.5) on expression (3.4). Therefore, if x satisfies $x_0 + \Delta \le x \le x_N - \Delta$, where Δ is a positive number, then $|(\mathscr{L}_{\mathscr{B}} f)(x) - (\mathscr{L}_{\mathscr{C}} f)(x)|$ is at most a constant multiple of $\Delta^{-1}h^2$. Further, inequality (3.6) shows that $|(\mathscr{L}_{\mathscr{A}} f)(x) - (\mathscr{L}_{\mathscr{C}} f)(x)|$ also enjoys this property. Thus we deduce the conditions

$$(5.6)\quad \left.\begin{aligned} &|f(x) - (\mathscr{L}_{\mathscr{B}} f)(x)| \le D^2\|f'\|_\infty \Delta^{-1}h^2 + \|f - \mathscr{L}_{\mathscr{C}} f\|_\infty, \\ &|f(x) - (\mathscr{L}_{\mathscr{A}} f)(x)| \le \left[\|f'\|_\infty + \frac{|f(x_0) + f(x_N)|}{x_N - x_0}\right] D^2\Delta^{-1}h^2 + \|f - \mathscr{L}_{\mathscr{C}} f\|_\infty, \end{aligned}\right\}$$

when $x \in [x_0 + \Delta,\ x_N - \Delta]$. Remembering the $h^2|\log h|$ term that occurs in $f(x) - (\mathscr{L}_{\mathscr{C}} f)(x)$, it follows usually that the errors $f(x) - (\mathscr{L}_{\mathscr{A}} f)(x)$, $f(x) - (\mathscr{L}_{\mathscr{B}} f)(x)$, and $f(x) - (\mathscr{L}_{\mathscr{C}} f)(x)$ are approximately equal when x is well inside the interval $x_0 \le x \le x_N$.

The authors are continuing to study multiquadric radial basis function approximations that have a finite number of centers, giving particular attention to interpolation on the regular grid $\{x_j = jh\colon j = 0, 1, \ldots, N\}$ when $d = 1$ (Beatson and Powell, 1991). Interpolation tends to be more accurate than quasi-interpolation because its error vanishes at the centers $\{x_j\}$ instead of often having a constant sign. Further, if x is any fixed interior point of $[x_0, x_N]$ and if h tends to zero, then in the case of a regular grid one can bound the interpolation error at x by a multiple of h^2 (Powell, 1991), which is better than the $\mathcal{O}(h^2|\log h|)$ results that we have derived. On the other hand, the effort of calculating the coefficients of quasi-interpolants is less than the work of solving the linear systems of equations that occur in interpolation and, more importantly, the accuracy properties of the given quasi-interpolation shcemes are valid when the spacings $\{x_j - x_{j-1}: j = 1, 2, \ldots, N\}$ between adjacent data points are highly irregular.

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