HELEN A. KHOURY AND RINA ZAZKIS

ON FRACTIONS AND NON-STANDARD REPRESENTATIONS: PRE-SERVICE TEACHERS' CONCEPTS

ABSTRACT. The present study examined the reasoning strategies and arguments given by pre-service school teachers as they solved two problems regarding fractions in different symbolic representations. In the first problem, the pre-service school teachers were asked to compare between two different fractions having the same numerical representation, and in the second problem, they were asked to compare between different notational representations of the same fraction. Numeration systems in bases other than ten were used to generate various representations of fractions. All students were asked to provide justifications to their responses. Strategies and arguments relative to pre-service teachers' concepts of fractions and place value were identified and analyzed based on results of 38 individual clinical interviews, and written responses of 124 students. It was found that the majority of students believe that fractions change their numerical value under different symbolic representations.

1.

What is a fraction? There are different ways to introduce the concept and to think about it. Initially, Freudenthal (1983) sees fractional number knowledge as organizing lived situations which involve fractioning. Later on, such situations need to be varied, and need to be experienced at different levels of abstraction for a more integrated rational number knowledge building. Most researchers agree that the part-whole, the quotient and the ratio concepts are central to understanding fractions. However, Behr et al. (1992) pointed out the need to examine students' concepts of fractions as mathematical entities and not only as processes of partitioning or division.

According to Piaget et al. (1960), fractions take on a dual character, "... they are parts of the original whole – within a nested system –, and they are also wholes in their own right, and as such they too can be subdivided further... Invariance of a whole is an essential condition of operational subdivision, and applies with equal force to qualitative and quantitative subdivision..." (pp. 310–311). Thus, it is not only essential to regard a fraction as a mathematical entity, it is equally necessary to realize that this entity remains invariant under different transformations including notational or symbolic representations. Further, the invariance

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of fractions is also considered necessary for the conceptual understanding of fraction operations. Are pre-service school teachers, who seem to operate successfully with fractions at the symbolic level, aware that fractions do not change their values as a function of symbolic representation?

In his book, "Why Johnny can't add?", Morris Kline wrote: "Symbolism can serve three purposes. It can communicate ideas effectively, it can conceal ideas, and it can conceal the absence of ideas" (p. 86). Markovits and Sowder (1991) observed that "using different symbols to represent the same idea and similar-looking symbols to represent different ideas" (p. 5) can be quite complex to some learners. Do pre-service school teachers experience such a complexity?

The goal of the present study was to examine the reasoning strategies and arguments given by pre-service school teachers, as they solved problems on fractions under non-standard and unfamiliar symbolic representations. Numeration systems in bases other than ten were used to generate various representations of fractions.

2. METHOD

2.1. *Subjects*

A group of 124 college students in their junior and senior years, that consisted of 100 pre-service elementary school teachers majoring in elementary education and 24 pre-service secondary school teachers majoring in mathematical sciences, participated in the study. The students had previous experience with whole number representations in different bases, however, the idea of non-integer rational number representation in bases other than ten was unfamiliar to them prior to the administration of the assessment instrument.

2.2. *Assessment of Students' Concepts*

The assessment was conducted in two parts: written assessment, and clinical interviews. The following are the two items which were administered to investigate the pre-service school teachers' concepts of fractions under different notation representations:

Item 1: Is $(0.2)_{\text{three}}$ equal to $(0.2)_{\text{five}}$?

Item 2: Is the number "one-half" in base three equal to the number "one-half" in base five?

For each of the above items, students were asked to explain their decision, and in case of inequality, to choose the larger number. In Item 1, the symbolic representation, that is 0.2, was similar in both cases, although the numerical values were different because the different bases of representation assigned different place values to the digit symbols. Whereas in Item 2, the numerical value (one-half) was the same in both cases, but the referents were representations in different bases. In order to avoid pre-assigning symbols to the fraction one-half, the word "one-half" was used, and not any of the possible common symbolic representations, like .5, 0.5, 1/2 or $\frac{1}{2}$, to this number in the statement of Item 2. Although, this question may be interpreted in more than one way, we kept the ambiguity to find out what interpretation to "the number one half" would be chosen by the students.

An unfamiliar mathematical domain was chosen for the assessment problems to avoid the possible direct application of any specific pre-existing algorithmic knowledge students may have. Numeration systems in bases other than ten were specifically used as a research tool and not as a topic for instruction.

For the first part of the study, 124 pre-service teachers were asked to respond in writing to the above two problems and to show their computational work, if any, as well as to provide written explanations for their reasoning. The students' computational work and justifications were analyzed, arguments and reasoning strategies were identified.

For the second part of the study, a subset of 38 elementary pre-service teachers, that represented roughly equally various strategies identified in the first part, were asked to discuss the assessment items in an individual clinical interview setting. All interviews were audiotaped, and later transcribed. The students' protocols were then analyzed in order to validate the reasoning strategies identified in the first part of the study, and to describe the pre-service teachers' conceptual understanding and explanations behind their use of the specific reasoning strategies.

3. RESULTS AND INTERPRETATIONS

For the purposes of this study, the analysis focused on the identification of the most common explanation arguments or reasoning strategies used. As presented in Table I, the frequencies of correct performance on the first item were: 63 out of 100 elementary education majors, and 24 out of 24 mathematics education majors, while on the second item, only 26 out of 100 elementary education majors, and only four out of 24 mathematics education majors performed correctly.

TABLE I

Frequencies and percentages of correct/incorrect arguments used by pre-service teachers on items 1 and 2

School level of pre-service teacher	n	Item 1				Item 2			
		Correct		Incorrect		Correct		Incorrect	
		Frequency	%	Frequency	%	Frequency	%	Frequency	%
Elementary	100	63	63	37	37	26	26	74	74
Secondary	24	24	100	–	–	4	17	20	83
Total	124	87	70	37	30	30	24	94	76

3.1. *Strategies on Item 1: "Is* 0.2_{three} *Equal to* 0.2_{five}*?"*

Most of the strategies used on this item were of computational nature. The students converted each of the number representations to a decimal fraction or to a common fraction and then compared both numbers. No student claimed that the numbers 0.2_{three} and 0.2_{five} were equal. Some students made their initial decisions about the inequality of these numbers based on the observation that "the bases are different, so the numbers can't be equal" and then proceeded to perform conversion. Although the performed conversion revealed at times students' incomplete understanding of base-systems and place value.

3.2. *Conventional Algorithmic Conversion*

All 24 secondary education majors and 63 out of 100 elementary education majors applied conventional algorithmic conversion. As an example of conventional conversion, the student may have written the following:

$$(0.2)_{\text{three}} = 2 \times 1/3 = 2/3; \quad (0.2)_{\text{five}} = 2 \times 1/5 = 2/5\,,$$

$$\text{then, } 0.2_{\text{three}} > 0.2_{\text{five}}, \text{ because } 2/3 > 2/5\,.$$

Most of the students claimed first that the two numbers were not equal, and then they resorted to the above algorithmic approach to validate their initial response. In most cases, they drew place-value charts before they applied the algorithm. During the interviews a common response to justify their initial decision was:

S: From the beginning I knew that they couldn't be equal because they are in different bases ... and they are the same numbers..... I mean digits... [*she drew a place value chart*]... 'two times one-third' would be two-thirds, and now I have to compare it to 'two times one-fifth' which would be equal to two-fifths. So, 0.2 in base three is greater than 0.2 in base five.

An inadequate concept of place value surfaced during the interviews, although the students had applied a correct algorithmic conversion. An excerpt of a student's protocol follows:

I: What about if 0.2 is in base 3?

S: ..... [*drawing a column chart, and trying to figure out*]... Well, like in base 10..... but I just have a hard time thinking of it in base 3, because, first of all, I don't really know what the columns are called; and so I can't really say that this is 'two-tenths' anymore...... maybe I have to say it's 'two-thirds'...

I: Okay, and what makes you think of it as 'two-thirds'.

S: Wait, this column may be called the 'three' column, and then, this one is called the 'thirtieth', ... and the next one is called the 'three hundredth'. Well, if I use this same philosophy that I've just used for base 3, then I would have to say – 'one-fifth' ... 'one-fiftieth' ... 'one-five hundredth'....

Such students were trying to form an analogy with the place values in base ten. The sequence 1/5, 1/50, 1/500... was generated, probably in analogy to 1/10, 1/100, 1/1000... (Zazkis and Khoury, 1993). Nevertheless, this interpretation of place values didn't effect students' performance on Item 1, where only one digit in the fractional part of the number had to be considered.

3.3. *Non-conventional Conversion Strategies*

On the written assessment, 37 students responded by claiming that the two numbers were not equal, but either didn't succeed to perform the conversion or were not able to justify their decision. The most frequent strategies are discussed below.

1/3 or 0.3 Confusion. It was a popular error in assigning the place values to the digits to use the following reasoning:

$$0.2_{\text{three}} = 2 \times 0.3 = 0.6; \quad 0.2_{\text{five}} = 2 \times 0.5 = 1,$$

therefore, $0.2_{\text{three}} < 0.2_{\text{five}}$, since $0.6 < 1$.

Interpreting the value 1/3 of the first place to the right of the "decimal" point (in base 3) as 0.3, and respectively 1/5 (in base 5) as 0.5 may be explained as a confusion between similar-looking symbols (Markovits and Sowder, 1991). Since the symbols 1/10 and 0.1 are interchangeable, the implicit assumption of students applying the conversion above might have been that the symbols 1/3 and 0.3 are interchangeable as well.

Base as the Unit Whole Argument. Students using this argument assumed that the value of the base of the number system describes the size of the unit whole, and since 0.2 denotes a fraction, then $(0.2)_{\text{three}}$ meant to them a fractional part out of the unit whole and was treated as "two out of three" or "0.2 out of three". Excerpts of students' protocols, who used the above strategy are given:

S_1: The way I would approach this would be 'two out of three' and 'two out of five' ..... and 'two out of three' is bigger than 'two out of five' ..... you would rather have 'two parts out of three' than 'two parts out of five'...

S_2: I'm so used to base ten that understanding a different number concept... like if you use a square as your 'one' unit and you think of where 0.2 would be on there with a base 3 or a base 5... like you've got a different sized unit and so 0.2 might be more... it might be larger in base 3 than it would be in base 5... or vice versa.

I: So what does 0.2 in base 3 mean to you?

S_2: This would be 0.2 of 3 and this one is 0.2 of 5. That's because that's how I would understand it if it were in base 10. I'm trying to relate it back to something that I understand, and 0.2 in base 10 is so much of 10, and that's how I would go about it ...

Even though this strategy may "work", that is, lead students to a "correct answer" on Item 1, it uncovers a misconception. The place value base-systems are based on different regrouping rules, and not "different sized unit".

TABLE II

Frequency distribution of correct arguments used by pre-service teachers on item 2

School level of pre-service teacher	n	Invariance of fraction strategies			Total
		Division	Column chart	Qualitative invariance	
Elementary	100	1	1	24	26
Secondary	24	1	2	1	4
Total	124	2	3	25	30

Strategies on Item 2: "Is the number 'one-half' in base three equal to the number 'one-half' in base five?"

Students' responses on this item were classified as either: invariance of fraction strategies, or non-invariance of fraction strategies. We use the term "invariance" in a non-traditional way: Invariance here denotes that fractions do not change their values as a function of symbolic representation.

Computational arguments were less frequent on Item 2 due, mostly, to the nature of the task. Representing "one-half" in odd bases isn't a trivial mathematical task. It requires knowledge of repeating decimal fractions and geometric sequences. The reference to the odd bases was chosen in order to avoid conversion and to focus on the nature of the numbers discussed. But the tendency of some students to perform conversion and represent "one half" in various bases was very strong, that we suggested to some of those interviewed to consider one-half in bases 4 and 6. Students' responses to this suggestion are discussed further in the "analogical arguments" section.

3.4. *Invariance of Fraction Strategies*

As shown in Table II, only 26% of the elementary pre-service teachers and 17% of the secondary pre-service teachers claimed that the numbers "one-half" in base three and "one-half" in base five were equal. Three different invariance of fraction strategies were identified.

1. Division Strategy. The Division Strategy was a computational strategy used by two students. The strategy was used as follows:

$$(\text{onehalf})_{\text{three}} = (1/2)_{\text{three}} = 1_{\text{three}}/2_{\text{three}} = 1/2$$

$$(\text{onehalf})_{\text{five}} = (1/2)_{\text{five}} = 1_{\text{five}}/2_{\text{five}} = 1/2,$$

therefore, the numbers are equal.

The explicit assumption in this strategy is that a fraction is a result of a division process, in specific, that "one half" is a result of dividing one by two. The implicit assumption in using this strategy is that the symbols 1 and 2 have the same value in bases three, five and ten, which means, for example, that $1_{\text{three}} = 1_{\text{ten}}$.

2. Column Chart Strategy. The Column Chart Strategy was another infrequent ($n = 3$) algorithmic strategy used for Item 2. Students first drew the place-value column chart for each base, and then they placed 1/2 in the "ones" column in the place-value chart, and concluded that one-half in base three was equal to one-half in base five. An excerpt of a student's protocol using this strategy follows:

> S_1: Isn't that funny? I'm thinking of how this relates to base ten. So, how does 1/2 relate to base 10? Well, it's half of a one, and my ones are still the same in base 3 or base 5, they are still in this same column. So, why wouldn't 1/2 of one be the same in both bases? So, I'll say they are the same.

3. Qualitative Invariance Strategy. The Qualitative Invariance Strategy was the most frequent correct strategy used for Item 2. Students responded as such: "One-half in base three is equal to one-half in base five, because one-half is a half of a whole regardless of the different bases used." Of the invariance of fraction strategies, this strategy was the most frequently used, however, it was used by only 25 pre-service teachers out of a total of 124. The reasoning involved in this strategy is qualitative, rather than algorithmic or computational.

Examples of students' justifications are given below:

TABLE III

Frequency distribution of incorrect arguments used by pre-service teachers on Item 2

School level of pre-service teacher	n	Non-invariance of fraction strategies				Total
		Over-symbolization	Base as unit-whole	Qualitative non-invariance	Analogical arguments	
Elementary	100	17	13	39	5	74
Secondary	24	15	2	–	3	20
Total	124	32	15	39	8	94

S_1: .... Yes, one-half in base three and one-half in base five are equal; because you're just asking is one-half equal to one-half, and to me that's dealing with the concept of one-half of a whole, and not with the way you count it or write the number one-half...... I tried but I don't remember the way to write one-half in base three....

S_2: ...... it seems to me they should be the same regardless of the base, because they are one-half of one whole unit.

We observed that during the interviews, many of the pre-service elementary school teachers who had initially attempted various numerical manipulations to conclude inequality, tended to change their decisions, when the interviewer probed their reasoning further, and tended to use qualitative arguments to justify the equality. Students seemed to change their non-invariance strategies to a qualitative invariance strategy after the interviewer had asked them to compare the number 'one in base three' to the number 'one in base five'.

3.5. *Non-invariance of Fraction Strategies*

The majority of students (76%) claimed that the numbers "one-half" in base three "one-half" in base five were not equal. Four different non-invariance of fraction strategies were identified.

1. Overgeneralized Symbolization Strategy. The Overgeneralized Symbolization was an algorithmic strategy caused by representing the number "one-half" as the familiar decimal representation "0.5" regardless of the base, as follows:

$$(\text{onehalf})_{\text{three}} = (0.5)_{\text{three}} = 5 \times 1/3 = 5/3, \text{ and}$$
$$(\text{onehalf})_{\text{five}} = (0.5)_{\text{five}} = 5 \times 1/5 = 5/5 = 1,$$
$$\text{therefore } (\text{one-half})_{\text{three}} > (\text{onehalf})_{\text{five}}.$$

An excerpt of a student's protocol is given:

S: Oh boy!... So the question is: Is one-half in base three equal to one-half in base five? I changed one-half into a decimal. So, is .5 in base three equal to .5 in base five? I guess they're different.

I: Why?... Please explain what you're doing.

S: I wrote 5 times 1/3, for one-half in base three, but I don't really know what that means; and I did the same with 5 times 1/5, for one-half in base five.

Seventeen elementary education majors (17%, $n = 100$), and 15 mathematics education majors (63%, $n = 24$) used the above reasoning. This was the most frequent strategy used by the mathematics education majors.

The fact that the digit-symbol 5 is not used neither in base five nor in base three was ignored. The fact that later on, when performing the computational conversions, the numerical value of the fraction "one-half" appeared to be either one or greater than one, didn't seem to bother this group of pre-service teachers and didn't prevent them from completing the comparison. It seems that the link between the number "one-half" and its representation as a decimal fraction (0.5) was reflexive. This link didn't include an understanding of the representation system, and created an assumption that the same symbols will represent one-half in other bases as well. An excerpt of a student's protocol explains this choice:

I: Where did you get the .5 from?

S: Well it's from a half. I changed a half into a decimal, so it's .5 base 3.

I: Why?

S: Because it's on this side of the decimal.

I: How did you know that .5 is one-half in base 3?

S: Oh, I don't, but I don't. I guess, I'm just uh, I don't. I made an assumption.

2. Base as the Unit Whole Strategy. Similarly to the "base as unit whole" reasoning used on Item 1, the value of the base was also considered as the unit whole by some students in responding to Item 2. This led to interpreting "one half in base three" as "one half of three". Recording such an interpretation as $1/2 \times 3$, a student claimed that "because 5 is greater than 3, then 5 halves is more than 3 halves".

The Base as Unit Whole Strategy was also combined by some pre-service teachers with the Overgeneralized Symbolization. Such students first overgeneralized and symbolized "one-half" as "0.5" regardless of the base, and then they multiplied 0.5 by the value of the base, in the following way:

$$(\text{onehalf})_{\text{three}} = (0.5)_{\text{three}} = 0.5 \text{ of } 3 = 0.5 \times 3 = 1.5, \text{ and}$$
$$(\text{onehalf})_{\text{five}} = (0.5)_{\text{five}} = 0.5 \text{ of } 5 = 0.5 \times 5 = 2.5,$$
$$\text{therefore } (\text{one-half})_{\text{three}} < (\text{onehalf})_{\text{five}}.$$

In the following excerpt a student explains this strategy:

S: ..... *[S restated the I's question as follows...]* Is .5 in base three equal to .5 in base five? I'm using .5 in base three to visualize what one-half in base three would look like. Just looking at them, this one-half in base three would be bigger than one-half in base five.

I: What do you mean?

S: Well, judging from... [*figuring out*] what we just looked at, it's not going to be the same, because when you think of 'one-half' you have to think of it as 'one-half of three' and that is, [*S writes down*] '.5 × 3'. Also 'one-half of five' is '.5 × 5', and 1.5 is less than 2.5!...

There are different possibilities to explain the misconception that leads students to consider the base as the unit whole which is being operated on by the fraction one-half. Students may be considering the fraction one-half as an operator, and as such it needs to operate on an operand quantity. Since the name of the base is expressed subsequent to the name of the fraction, as in 'one-half in base three', this may be leading students to assume that the fraction is operating on the value of the base which in turn is considered as a unit whole. In any case, students seem to focus on the base of the numeration system as an operand quantity, and they seem to disregard the relation between the base of the numeration system and the assigned place values.

3. Qualitative Non-invariance Strategy. Students using the Qualitative Non-invariance Strategy claimed that the number one-half in base three had a different value than the number one-half in base five, and mostly they based their decision on the assumption that since the bases are different then the numbers should be different. This was the most frequent argument used by elementary education majors, however, this argument was not given by any of the mathematics education majors.

In the following excerpt a student explains her reasoning:

S: One-half in base three is less than one-half in base five, because the bases are different. Because in base three you have less numbers, so your one-half is going to be a different answer than in base five that has more numbers.....!

I: Do you mean by 'answer' the quantity one-half, or the way you write it in base three?

S: ....they're the same ....now, I'm confused....

4. Analogical Arguments. The analogical reasoning was applied in several ways. Some students based their reasoning on the previous item of the assessment. Therefore, whatever they concluded on Item 1, they applied for Item 2, providing the explanation "same as above". Other students attempted to derive a general rule, such as "$5 > 3$, therefore X in base five is greater than X in base 3". Other students based their conclusions on drawing an incorrect analogy with self-generated examples, illustrating it in the following way:

$21_{\text{three}} = 2 \times 3 + 1 = 7$; $21_{\text{five}} = 2 \times 5 + 1 = 11$

21_{three} is smaller than 21_{five}, therefore one-half in base three should be smaller than one-half in base five.

The use of analogical reasoning was identified by Piaget, Inhelder and Szeminska (1960) as one of the early phases of the conceptual development of fractions by children. It seems that when young adults are faced with a cognitive conflict they, at times, fall back into forming analogies as a mode of thought. But actually in the interview setting, the reasoning of forming analogies led some students to confident decisions. Jennifer, for example, had difficulty to compare "one-half in base 3" and "one-half in base 5" since she wanted to represent the number "one-half" in different bases. The interviewer then suggested to her to compare "one-half" in base 4 and "one-half in base 6". Following the conclusion of the numerical manipulations on bases 4 and 6, she was able to conclude about bases 3 and 5 as such:

Jennifer: Okay, 1/2 in base 4 would be over 4, 1/2 in base 6 should be over 6, 2 over 4, they're; both halves [*Here the student wrote 2/4 and 3/6*]. But in base 10, um if a unit's a unit is a unit but 1/2 of base 3 and 1/2 of base 5 is 1/2 of a unit, it's got to be the same thing. Simple as that, it's the same.

4. SUMMARY AND CONCLUSION

The present study investigated pre-service school teachers' concepts of invariance of fractional number under different symbolic representations. The focus of the study was on identifying and describing the arguments students use and the reasoning they provide regarding the invariance of fractions under numeration systems in different bases. An unfamiliar mathematical domain was chosen to avoid the possible direct application of any specific pre-existing algorithmic knowledge students may have.

The surprising finding in this study was that while all the mathematics education majors performed correctly on the first item, their percentage of correct performance on the second item was low (see Table I), and even lower than the percentage of correct performance of the elementary education majors. It seems like a belief in an algorithmic approach of conversion and hurrying up with applying computational skills was dominant among the mathematics education majors. It is possible that this tendency among the pre-service secondary school mathematics teachers to deal with mathematical issues and problems at an algorithmic level was strong enough to divert their attention from a detailed look on the nature of the questions asked and their respective responses.

The analysis of the strategies indicates that pre-service teachers' knowledge of place value and rational numbers is more syntactical than conceptual. Research studies reported about the possible effects of number size on the performance of students and their reasoning in multiplicative-structured situations (Bell et al., 1984; Noelting, 1980). However, the multiplicative relationships of twice as much, that is doubling, and of half as much, that is halving, are known to be easier concepts for most students to grasp than other multiplicative relationships of an integer other than two or a fraction other than 1/2 (Hart, 1981). The fact that the "simplest possible" fraction (one-half) was used in the present study emphasizes further the main findings which indicate that high percentages of pre-service school teachers have a disconnected knowledge of place value, decimals, and fractions, and that the majority of them use non-invariance of fraction strategies in unfamiliar problem-solving situations.

5. EPILOGUE

After completion of the interview, one of the students approached the interviewer and asked: "Now, please tell me the truth, do these numbers we were talking about *really exist*?"

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Department of Mathematical Sciences,
Northern Illinois University,
De Kalb, Illinois 60115, U.S.A.